THE COREFLECTIVE SUBCATEGORY OF SEQUENTIAL SPACES

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Certain theorems of recent interest [1, 2] concerning sequential spaces may be deduced from the fact that the category of sequential spaces, $\mathcal{S}$, is a coreflective subcategory of the category of topological spaces, $\mathfrak{T}$. A space is said to be sequential if it has the finest topology that permits the convergence of its convergent sequences.

THEOREM: $\mathcal{S}$ _is a coreflective subcategory of_ $\mathfrak{T}$.

Proof. We define a coreflector functor $R: \mathfrak{T} \to \mathcal{S}$ as follows:

For $X \in \mathfrak{T}_{ob}$, $R(X)$ is the space with the same underlying set as X and whose topology consists of those subsets O such that every sequence convergent in X to a point of O is eventually in O. [3] (Thus $R(X)$ has the finest topology that permits the convergence of all the sequences convergent in X.) For $f \in \mathfrak{T}(X, Y)$, $R(f)$ is that element of $\mathcal{S}(R(X), R(Y))$ that agrees with f on the underlying set. The coreflection map e_x, is the identity of the underlying set.

It may be verified by considering inverse images of open sets that $R(f)$ exists and it then follows easily that R is the desired coreflector functor.

By Theorem A of J. Kennison [4], it follows that $\mathcal{S}$ is closed under direct sums and quotient spaces, a result of S.P. Franklin [1].

COROLLARY 1. _If_ $\mathcal{C} \subseteq \mathcal{S}$ _and_ $\mathcal{S}'$ _is the full subcategory of_ $\mathfrak{T}$ _whose objects are all direct sums of sets of quotient objects of_ $\mathcal{C}$, _then_ $\mathcal{S}' \subseteq \mathcal{S}$.

COROLLARY 2. _If in addition_ $\mathcal{C}$ _contains_ $\omega + 1$ [1] _with order topology, then_ $\mathcal{S}' = \mathcal{S}$.

Proof. By Theorem A [4], $\mathcal{S}'$ is coreflective with 1-1, onto coreflection map. Let $X \in \mathcal{S}_{ob}$. If $s: \omega + 1 \to X$, then $s = e_x o R(s)$, where R is the coreflector to $\mathcal{S}'$ and e_x the coreflection map. Since a continuous map from $\omega + 1$ corresponds to a convergent sequence in image space, it follows that a sequence that is convergent to a point with the topology of X, converges to the same point with the topology of $R(X)$. Since X has the finest topology that permits the convergence of its convergent sequences and $R(X)$ has a topology at least as fine, $R(X)$ and X are homeomorphic. $\mathcal{S}'$ must be replete by the way it was defined and $\mathcal{S} = \mathcal{S}'$.

From Corollary 2, we obtain Franklin's characterizations of $\mathcal{S}$ [1]:

(1) Category of quotient spaces of metric spaces,

(2) Category of quotient spaces of first countable spaces,

(3) Category of quotient spaces of direct sums of copies $\omega + 1$.

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