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A NEW PROOF OF A WATSON'S FORMULA

BY
KRZYSZTOF STEMPAK

ABSTRACT. A new proof of a product formula for Laguerre polynomials, due originally to Watson, is given. Considering the commutative Banach algebra of radial functions on the Heisenberg groups $\mathbf{H}_n$, $n \geqq 2$, we observe that Watson's formula holds for $z = 1, 2, 3, \ldots$. Then, applying a complex function theory argument, we establish the validity of this formula for other complex values of z, i.e. for $\operatorname{Re} z > -1/2$.

In 1939 Watson, (cf. [6]), established a product formula for Laguerre polynomials which may be rewritten in the following form: for $\operatorname{Re} z > -1/2$, $x, y \in \mathbf{R}$, $k = 0, 1, 2, 3, \ldots$, we have

$$(1) \qquad L_k^z(x^2)L_k^z(y^2) = \frac{2^{z-1/2}\Gamma(z+k+1)}{k!\pi^{1/2}} \int_0^\pi e^{xy\cos\theta} \frac{J_{z-1/2}(xy\sin\theta)}{(xy\sin\theta)^{z-1/2}} \times L_k^z(x^2+y^2-2xy\cos\theta)\sin^{2z}\theta d\theta.$$

Here, L_k^z, $k = 0, 1, 2, \ldots$, denote the Laguerre polynomials of order z

$$(2) \qquad L_k^z(t) = \sum_{j=0}^{k} \binom{k+z}{k-j} \frac{(-t)^j}{j!}$$

and J_ν means the Bessel function defined for $\operatorname{Re} \nu > -1/2$ by

$$(3) \qquad J_\nu(t) = (t/2)^\nu \Gamma(\nu + 1/2)^{-1}\Gamma(1/2)^{-1} \int_0^\pi e^{it\cos\theta} \sin^{2\nu}\theta d\theta.$$

The aim of this note is to prove (1) using the Heisenberg group approach and a theorem of Carlson, (cf. [5]).

First we establish (1) for $z = 1, 2, 3, \ldots$. To do this fix a positive integer $n \geqq 2$ and consider the $2n + 1$ dimensional Heisenberg group $\mathbf{H}_n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ with the multiplication given by

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$$(\bar{x}_1, \bar{y}_1, t_1)(\bar{x}_2, \bar{y}_2, t_2) = (\bar{x}_1 + \bar{x}_2, \bar{y}_1 + \bar{y}_2, t_1 + t_2 + \bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)$$

A function $\tilde{f}$ on $\mathbf{H}_n$ is said to be radial if $\tilde{f}(\bar{x}, \bar{y}, t) = f(\|(\bar{x}, \bar{y})\|, t)$ for a function f on $X = \mathbf{R}_+ \times \mathbf{R}$, where

$$\|(\bar{x}, \bar{y})\| = \left(\sum_{i=1}^{n} (x_i^2 + y_i^2)\right)^{1/2}, \bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n).$$

The function f is then called the radial part of $\tilde{f}$. It is known, (cf. [1], [3]), that the space $L_r^1(\mathbf{H}_n)$ of all integrable, radial functions on $\mathbf{H}_n$ forms a commutative (under the convolution on $\mathbf{H}_n$) Banach algebra and for any $k = 0, 1, 2, \dots,$ and $\lambda \in \mathbf{R}, \lambda \neq 0$, the mapping

$$L_r^1(\mathbf{H}_n) \ni \tilde{f} \to \alpha(\tilde{f}) = \int_{\mathbf{H}_n} \tilde{f}\tilde{\phi}_{\lambda,k} d\bar{x}d\bar{y}dt, \tag{4}$$

where

$$\tilde{\phi}_{\lambda,k}(\bar{x}, \bar{y}, t) = \binom{n+k-1}{k}^{-1} e^{i\lambda t} e^{-(|\lambda|/2)\|(\bar{x},\bar{y})\|^2} L_k^{n-1}(|\lambda| \,\|(\bar{x}, \bar{y})\|^2)$$

gives a multiplicative functional on $L_r(\mathbf{H}_n)$.

Now, for $\tilde{f}, \tilde{g} \in L_r^1(\mathbf{H}_n)$ with the radial parts f, g respectively, denote by $f * g$ the radial part of the convolution $\tilde{f} * \tilde{g}$ on $\mathbf{H}_n$ given by

$$\tilde{f}(\bar{x}_1, \bar{y}_1, \bar{t}_1) = \int_{\mathbf{R}^{2n+1}} \tilde{f}((\bar{x}_1, \bar{y}_1, t_1)(-\bar{x}, -\bar{y}, -t))\tilde{g}(\bar{x}, \bar{y}, t)d\bar{x}d\bar{y}dt.$$

(We abuse the notation slightly by using the same symbol $*$ to denote the convolution in $\mathbf{H}_n$ and this given by (5)).

Using the polar coordinates in $\mathbf{H}_n$

$$x_1 = r \prod_{j=1}^{2n-1} \sin \phi_j,\ x_i = r \cos \phi_{i-1} \prod_{j=1}^{2n-1} \sin \phi_j,\ 2 \leqq i \leqq n,\ i \neq n,$$

$$y_i = r \cos \phi_{n+i-1} \prod_{j=n+1}^{2n-1} \sin \phi_j,\ 1 \leqq i \leqq n,\ i \neq n-1,$$

$$x_n = r \cos \phi_{2n-2} \sin \phi_{2n-1},\ y_{n-1} = r \cos \phi_{n-1} \prod_{j=n}^{2n-1} \sin \phi_j,$$

where $r > 0$, $0 < \phi_1 < 2\pi$, $0 < \phi_j < \pi$, $j = 2, 3, \dots, 2n-1$, one can verify, (cf. also [4]), that

$$f * g(x, t) = \int_X T^{y,u} f(x, t) g(y, u) d\mu_n(y, u). \tag{5}$$

Here $d\mu_n(y, u) = \sigma(n)y^{2n-1}dydu$ where $\sigma(n)$ is a constant that comes from polar coordinates and the generalized translations $T^{y,u}$, $y \geqq 0$, $u \in \mathbf{R}$, are given by

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$$(6)\qquad T^{y,u}f(x,t) = \frac{n-1}{\pi}\int_0^\pi\int_0^\pi \times f((x^2+y^2-2xy\cos\theta)^{1/2}, t-u+xy\cos\phi\sin\theta)\sin^{2n-3}\phi\sin^{2n-2}\theta d\phi d\theta.$$

Equivalently,

$$(6)'\qquad T^{y,u}f(x,t) = \frac{n-1}{\pi}\int_{|x-y|}^{x+y}\int_{-1}^{1} \times f(v, t-u+2s\cdot\Delta(x,y,v))W_{x,y}(v,s)d\mu_n(v,s)$$

where the function $W_{x,y}$, defined on $[\,|x-y|, x+y] \times [-1,1]$, is

$$W_{x,y}(v,s) = 2^{2n-3}\frac{n-1}{\pi}\frac{\Delta(x,y,v)^{2n-3}}{(xyv)^{2n-2}}(1-s^2)^{n-2},$$

and $\Delta(x,y,v)$, for $v \in [\,|x-y|, x+y]$, means the area of a triangle with sides x, y, v.

It is clear that the mapping $L_r^1(\mathbf{H}_n) \ni \tilde{f} \to \psi(\tilde{f}) = f \in L^1(\mu_n)$, where $L^1(\mu_n) = L^1(X, d\mu_n)$, establishes an isometric isomorphism between the Banach algebras $L_r^1(\mathbf{H}_n)$ and $L^1(\mu_n)$, if the multiplication in the second space is given by (5). The fact that $L^1(\mu_n)$ with (5) as the multiplication forms a commutative Banach algebra may also be easily verified using

$$(7)\qquad W_{x,y}(v,s)d\mu_n(v,s)d\mu_n(x,t) = W_{v,y}(x,s)d\mu_n(x,t)d\mu_n(v,s)$$

and

$$(8)\qquad \int_X W_{x,y}(v,s)d\mu_n(v,s) = 1.$$

Moreover, (7) implies

$$(9)\qquad \langle T^{y,u}f, g\rangle = \langle f, T^{y,-u}g\rangle$$

say, for $f \in L^1(\mu_n)$ and $g \in L^\infty(\mu_n)$, where

$$\langle f, g\rangle = \int_X fg d\mu_n.$$

Put

$$\phi_{\lambda,k}(x,t) = \binom{n+k-1}{k}^{-1} e^{i\lambda t}\exp\left(-\frac{|\lambda|}{2}x^2\right)L_k^{n-1}(|\lambda|x^2).$$

Since (4) gives a multiplicative functional on $L_r^1(\mathbf{H}_n)$ then also $\beta(f) = \alpha(\psi^{-1}(f))$ establishes a multiplicative functional on $L^1(\mu_n)$, that is

$$(10)\qquad \beta(f * g) = \beta(f)\beta(g)$$

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for all $f, g \in L^1(\mu_n)$. But, clearly, $\beta(f) = \langle f, \phi_{\lambda,k}\rangle$ and so, combining (9), (10) and the fact that $\phi_{\lambda,k}$ is continuous we get

$$\phi_{\lambda,k}(x, t)\phi_{\lambda,k}(y, u) = T^{y,-u}\phi_{\lambda,k}(x, t) \tag{11}$$

for every (x, t), $(y, u) \in X$. Now, taking $\lambda = 1$ in (11) and using the identity

$$\int_0^\pi \cos(a \cos \phi)(\sin \phi)^{2n-3}\, d\phi = \pi^{1/2}2^{n-3/2}\Gamma(n - 1)J_{n-3/2}(a)a^{3/2-n},$$

we easily get (1) for $z = n - 1$ and therefore (1) is showed for $z = 1, 2, 3, \ldots$.

To get other values of z we use Carlson's theorem, (cf. [5], p. 186): If an analytic function $f(z)$ on the half-space $\operatorname{Re} z > -\delta$, $\delta > 0$, satisfies $|f(z)| \leqq C \exp(k|z|)$ for $\operatorname{Re} z \geqq 0$, with $k < \pi$, $C > 0$ and $f(n) = 0$ for $n = 1, 2, 3, \ldots$, then $f = 0$ identically. So, fix $x, y \geqq 0$ and $k \in \mathbf{N}$. Then both sides of (1) are analytic functions of z, $\operatorname{Re} z > -1/2$. To obtain the required growth conditions note, that in fact, both sides of (1) are polynomially bounded on $\operatorname{Re} z > 0$. Indeed, using (2) we get

$$|L_k^z(t)| \leqq C(1 + |z|)^k, \tag{12}$$

with a constant $C > 0$ independent of $t \in [a, b]$, for any bounded interval $[a, b]$. Therefore,

$$|L^z(x^2)L^z(y^2)| \leqq C(1 + |z|)^{2k}.$$

On the other hand, using (3), we obtain

$$|(2/t)^{z-1/2}J_{z-1/2}(t)| \leqq |\Gamma(z)|^{-1}\Gamma(\operatorname{Re} z)\Gamma(\operatorname{Re} z + 1/2)^{-1}, \tag{13}$$

for $\operatorname{Re} z > 0$ and $t \geqq 0$. Since, by (12),

$$|L_k^z(x^2 + y^2 - 2xy \cos \theta)| \leqq C(1 + |z|)^k, \quad 0 < \theta < \pi, \tag{14}$$

and also

$$\int_0^\pi \sin^{2\operatorname{Re} z} \theta d\theta = \Gamma(1/2)\Gamma(\operatorname{Re} z + 1/2)\Gamma(\operatorname{Re} z + 1)^{-1}, \quad \operatorname{Re} z > 0,$$

we estimate $R(z)$, the right side of (1), by

$$\begin{aligned} |R(z)| &\leqq C\frac{|\Gamma(z + k + 1)|}{|\Gamma(z)|}\Gamma(\operatorname{Re} z + 1)^{-1}\Gamma(\operatorname{Re} z)(1 + |z|)^k \\ &\leqq C(1 + |z|)^{2k} \end{aligned}$$

where $\operatorname{Re} z > 0$. This, in virtue of Carlson's theorem, concludes the proof of (1).

REMARK. Recently, (cf. [2]), C. Markett has given another, analytic proof of (1). His approach is based upon an investigation of the Laguerre translation operator $T_t^\alpha(f, x)$, $\alpha \geqq -1/2$, which for a sufficiently smooth function f is defined as the solution $u(x, t)$, symmetric in $x, t \geqq 0$, of the associated Cauchy problem

$$(D_x^\alpha - D_t^\alpha)u(x, t) = 0, \qquad 0 < t \leqq x$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = \frac{\partial}{\partial t} u(x, t)_{|t=0} = 0, \quad x > 0.$$

Here the singular differential operator D^α is given by

$$D_x^\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x}\frac{d}{dx} - x^2.$$

The crucial point of this proof consists in finding an explicit form of the corresponding Riemann function. Note, that Markett's approach does not distinguish between the classical formula of Watson (Re $z > -1/2$) and the limiting case $z = -1/2$ (cf. [2] for more details). Note also that our underlying differential operator is the second order partial differential operator

$$L = \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \quad x > 0, t \in \mathbf{R},$$

(cf. [4] for details) and the generalized translations associated with L are positive operators (which is obviously not true for the Laguerre translation operators $T_t^\alpha(f, x)$ mentioned above).

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCLAW
PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND (CURRENT ADDRESS)
AND
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602, USA