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A CHARACTERIZATION OF SATURATED C^*-ALGEBRAIC BUNDLES OVER FINITE GROUPS

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Abstract

Let A be a unital C^*-algebra. Let (B, E) be a pair consisting of a unital C^*-algebra B containing A as a C^*-subalgebra with a unit that is also the unit of B, and a conditional expectation E from B onto A that is of index-finite type and of depth 2. Let B_1 be the C^*-basic construction induced by (B, E). In this paper, we shall show that any such pair (B, E) satisfying the conditions that $A' \cap B = \mathbb{C}1$ and that $A' \cap B_1$ is commutative is constructed by a saturated C^*-algebraic bundle over a finite group. Furthermore, we shall give a necessary and sufficient condition for B to be described as a twisted crossed product of A by its twisted action of a finite group under the condition that $A' \cap B_1$ is commutative.

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1. Introduction

The Jones index theory on type II_1 factors [11] caused a revolutionary change to the theory of operator algebras. The paragroup theory due to Ocneanu [18, 19] and the classification results for subfactors due to Popa [25, 26] should be mentioned in particular. The Jones index theory was extended to unital C^*-algebras by Watatani [35] and many interesting results of C^*-index theory can be found in the work of Izumi [8].

In this paper, we consider a condition for an inclusion of unital C^*-algebras, denoted by $\mathcal{L}^0$ in Section 2, which is stronger than the depth 2 requirement and which characterizes subfactors arising from crossed products by outer actions of finite groups [16]. It is known that the condition $\mathcal{L}^0$ does not characterize crossed product inclusions of C^*-algebras because of the presence of a K-theoretical obstruction. We show that the condition $\mathcal{L}^0$ is still useful for characterizing inclusions arising from saturated C^*-algebraic bundles over finite groups instead of those arising from crossed products. In Section 3, we prove that an inclusion induced by a saturated C^*-algebraic bundle over a finite group is of index-finite type and depth 2. In Section 4, we construct a saturated C^*-algebraic bundle over a finite group from an inclusion of C^*-algebras satisfying the condition. In Section 8, we shall give a necessary and sufficient condition for a saturated C^*-algebraic bundle over a finite group to be described as a twisted crossed product with the finite group. We shall also prove that any C^*-algebraic bundle over a finite group can be constructed as a corner of a crossed product inclusion of C^*-algebras after stabilization.

When $i = 1, 2$, let p_i be a projection in a C^*-algebra C. We write $p_1 \sim p_2$ in C if p_1 is Murray–von Neumann equivalent to p_2 in C.

2. Three sets and their equivalence relations

Let A be a unital C^*-algebra. Let (B, E) be a pair consisting of a unital C^*-algebra B including A with a common unit, and a conditional expectation E from B onto A that is of index-finite type and of depth 2. Let $\mathcal{L}$ be the set of all such pairs. Let $(B, E), (D, F) \in \mathcal{L}$. We say that (B, E) is *equivalent* to (D, F), we write $(B, E) \sim (D, F)$, if there exists an isomorphism π of B onto D such that $E = F \circ \pi$. We denote by $[B, E]$ the equivalence class of (B, E) in $\mathcal{L}$ and by $\mathcal{L}/\sim$ the set of all equivalence classes of elements in $\mathcal{L}$. Let $\mathcal{L}^0$ be the set of all elements $(B, E) \in \mathcal{L}$ such that $A' \cap B = \mathbb{C}1$ and $A' \cap B_1$ is commutative, where B_1 is the C^*-basic construction induced by (B, E). Let $\mathcal{L}^0/\sim$ be the set of all equivalence classes of elements in $\mathcal{L}^0$.

Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a C^*-algebraic bundle over a finite group G such that $B_e = C$, where e is the unit element in G and C is a unital C^*-algebra. We say that $\mathcal{B}$ is *saturated* if $\overline{B_t B_s} = B_{ts}$ for all $t, s \in G$ (see Fell and Doran [6]). Let $\bigoplus_{t \in G} B_t$ be the graded C^*-algebra induced by $\mathcal{B} = \{B_t\}_{t \in G}$, which is defined in Exel [5], where we regard B_t as a closed subspace of $\bigoplus_{s \in G} B_s$ for all $t \in G$. Let $\mathcal{B} = \{B_t\}_{t \in G}$ and $\mathcal{D} = \{D_h\}_{h \in H}$ be C^*-algebraic bundles over the finite groups G and H, whose fibers at the unit element are equal to C. We say that $\mathcal{B}$ is *equivalent* to $\mathcal{D}$, we write $\mathcal{B} \sim \mathcal{D}$, if there exists an isomorphism λ of G onto H satisfying the condition that, for all $t \in G$, there exists a linear isomorphism π_t of B_t onto $D_{\lambda(t)}$ such that $\pi_{ts}(xy) = \pi_t(x)\pi_s(y)$ and $\pi_{t^{-1}}(x^*) = \pi_t(x)^*$, for all $x \in B_t$ and $y \in B_s$, and $\pi_e = \mathrm{id}$ on $B_e = D_e = C$. Let $\mathcal{M}$ be the set of all saturated C^*-algebraic bundles over finite groups whose fiber at the unit element is equal to the unital C^*-algebra A. We can easily see that the above relation is an equivalence relation in $\mathcal{M}$ by routine computations. We denote by $[\mathcal{B}]$ the equivalence class of $\mathcal{B}$ in $\mathcal{M}$ and by $\mathcal{M}/\sim$ the set of all equivalence classes of $\mathcal{B}$ in $\mathcal{M}$.

Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a countably infinite-dimensional Hilbert space and denote by C^s a stable C^*-algebra $C \otimes \mathbb{K}$ for each C^*-algebra C. Let $\mathcal{N}$ be the set of all finite group actions on A^s. We define an equivalence relation in $\mathcal{N}$ as follows: for all $(G, \beta), (H, \gamma) \in \mathcal{N}$ we say that (G, β) is *equivalent* to (H, γ), we write $(G, \beta) \sim (H, \gamma)$, if there exists an isomorphism λ of G onto H satisfying the condition that (G, β) is exterior equivalent to $(G, \gamma_{\lambda(\cdot)})$. We denote by $[G, \beta]$ the equivalence class of (G, β) in $\mathcal{N}$ and denote by $\mathcal{N}/\sim$ the set of all equivalence classes of (G, β) in $\mathcal{N}$.

For each C^*-algebra C, let $M(C)$ be its multiplier algebra. For any automorphism α of C, we can extend α to a strictly continuous automorphism of $M(C)$ following the results of Busby [4] or Jensen and Thomsen [9]. We denote it by the same symbol α.

In this paper, we shall use the phrase 'Hilbert C^*-bimodule' in the sense of Kajiwara and Watatani [13].

3. Construction of a map from $\mathcal{M}/\sim$ to $\mathcal{L}/\sim$

Let $\mathcal{B} = \{B_t\}_{t\in G} \in \mathcal{M}$ and $B = \bigoplus_{t\in G} B_t$. Let E be the conditional expectation from B onto A defined by $E(x) = x_e$ for all $x = \sum_{t\in G} x_t \in B$. We call it the *canonical* conditional expectation from B onto A. Since $\mathcal{B}$ is saturated, $\overline{B_t B_t^*} = A$ for all $t \in G$. Since A is unital, there exists a finite set $\{x_i^t\}_{i=1}^{n_t} \subset B_t$ such that $\sum_{i=1}^{n_t} x_i^t x_i^{t*} = 1$ for all $t \in G$.

LEMMA 3.1. *The set $\{(x_i^t, x_i^{t*}) \mid i = 1, 2, \ldots, n_t, t \in G\}$ is a quasi-basis for E.*

PROOF. For all $a = \sum_{s\in G} a_s \in B$,

$$\sum_{i=1}^{n_t} E(ax_i^t)x_i^{t*} = \sum_{i=1}^{n_t} a_{t^{-1}} x_i^t x_i^{t*} = a_{t^{-1}}$$

for all $t \in G$. Thus $\sum_{t\in G} \sum_{i=1}^{n_t} E(ax_i^t)x_i^{t*} = \sum_{t\in G} a_{t^{-1}} = a$. Hence the lemma is proved. $\square$

We denote by $|G|$ the order of G.

COROLLARY 3.2. *With the above notation,* Index $E = |G|$.

PROOF. By Lemma 3.1, Index $E = \sum_{t\in G} \sum_{i=1}^{n_t} x_i^t x_i^{t*} = |G|$. $\square$

Let e_A be the Jones projection and let B_1 be the C^*-basic construction induced by E. Let E_1 be the dual conditional expectation of E from B_1 onto B.

LEMMA 3.3. *Let $e_t = \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*}$ for all $t \in G$. Then e_t is a projection in $A' \cap B_1$. Furthermore, e_t is independent of the choice of $\{x_i^t\}_{i=1}^{n_t}$.*

PROOF. It is clear that e_t is a self-adjoint element in B_1. Since $x_i^{t*} x_j^t \in A$, by Watatani [35, Lemma 2.1.1] it commutes with e_A. Thus

$$e_t^2 = \sum_{i,j=1}^{n_t} x_i^t e_A x_i^{t*} x_j^t e_A x_j^{t*} = \sum_{i,j=1}^{n_t} x_i^t x_i^{t*} x_j^t e_A x_j^{t*} = \sum_{j=1}^{n_t} x_j^t e_A x_j^{t*} = e_t.$$

Hence, e_t is a projection in B_1. Furthermore, for all $a \in A$,

$$ae_t = a \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*} = \sum_{i,j=1}^{n_t} x_j^t x_j^{t*} a x_i^t e_A x_i^{t*} = \sum_{i,j=1}^{n_t} x_j^t e_A x_j^{t*} a x_i^t x_i^{t*} = e_t a$$

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since $x_j^{t*}ax_i^t \in A$. Hence $e_t \in A' \cap B_1$. Now let $\{y_j^t\}_{j=1}^{m_t} \subset B_t$ be another finite set satisfying the condition that $\sum_{j=1}^{m_t} y_j^t y_j^{t*} = 1$. Then

$$e_t = \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*} = \sum_{i=1}^{n_t}\sum_{j=1}^{m_t} x_i^t e_A x_i^{t*} y_j^t y_j^{t*} = \sum_{i=1}^{n_t}\sum_{j=1}^{m_t} x_i^t x_i^{t*} y_j^t e_A y_j^{t*} = \sum_{j=1}^{m_t} y_j^t e_A y_j^{t*}$$

since $x_i^{t*}y_j^t \in A$. Therefore, the lemma is proved. □

REMARK 3.4. By easy computations, we can see that $e_t e_s = 0$ for all $t, s \in G$ where $t \neq s$ and $\sum_{t\in G} e_t = 1$.

Let e_t be as above and let $C^*\langle B, e_t\rangle$ be a C^*-subalgebra of B_1 generated by B and e_t for all $t \in G$.

LEMMA 3.5. *With the above notation, $C^*\langle B, e_t\rangle = B_1$ for all $t \in G$.*

PROOF. Since $x_i^{t^{-1}} x_j^t \in A$,

$$\sum_{i=1}^{n_{t^{-1}}} x_i^{t^{-1}} e_t x_i^{t^{-1}*} = \sum_{i=1}^{n_{t^{-1}}}\sum_{j=1}^{n_t} x_i^{t^{-1}} x_j^t e_A x_j^{t*} x_i^{t^{-1}*} = e_A \sum_{i=1}^{n_{t^{-1}}}\sum_{j=1}^{n_t} x_i^{t^{-1}} x_j^t x_j^{t*} x_i^{t^{-1}*} = e_A.$$

Since $C^*\langle B, e_t\rangle$ is a C^*-subalgebra of B_1, the lemma is proved. □

The following lemma shows that for all $t \in G$ there exists an automorphism $\alpha_t^{\mathcal{B}}$ of B_1 such that $\alpha_t^{\mathcal{B}}(b) = b$ for all $b \in B$ and $\alpha_t^{\mathcal{B}}(e_A) = e_{t^{-1}}$.

LEMMA 3.6. *For all $t \in G$, there exists a unique automorphism $\alpha_t^{\mathcal{B}}$ of B_1 such that $\alpha_t^{\mathcal{B}}(b) = b$ for all $b \in B$ and $\alpha_t^{\mathcal{B}}(e_A) = e_{t^{-1}}$.*

PROOF. We shall show that e_t is a Jones projection for all $t \in G$. By Lemma 3.3, e_t is a projection in $A' \cap B_1$. For all $b = \sum_{s\in G} b_s \in B$,

$$\begin{aligned} e_t b e_t &= \sum_{i,j}^{n_t} x_i^t e_A x_i^{t*} b x_j^t e_A x_j^{t*} = \sum_{i,j}^{n_t} x_i^t E(x_i^{t*} b x_j^t) e_A x_j^{t*} \\ &= \sum_{i,j}^{n_t} x_i^t x_i^{t*} b_e x_j^t e_A x_j^{t*} = \sum_{j}^{n_t} b_e x_j^t e_A x_j^{t*} = E(b) e_t. \end{aligned}$$

Since $E_1(e_t) = \sum_i^{n_t} x_i^t E_1(e_A) x_i^{t*} = (\text{Index } E)^{-1}$, if $a \in A$ and $ae_t = 0$, then we have $a = (\text{Index } E) E_1(ae_t) = 0$. Thus, the map $A \ni a \mapsto ae_t \in B_1$ is injective. Since $C^*\langle B, e_t\rangle = B_1$ by Lemma 3.5, there exists a unique automorphism $\alpha_t^{\mathcal{B}}$ of B_1 such that $\alpha_t^{\mathcal{B}}(b) = b$ for all $b \in B$ and $\alpha_t^{\mathcal{B}}(e_A) = e_{t^{-1}}$ by [35, Proposition 2.2.1]. □

For a C^*-algebra C, we denote by $\text{Aut}(C)$ the group of all automorphisms of C.

LEMMA 3.7. *With the above notation, the map $t \in G \mapsto \alpha_t^{\mathcal{B}} \in \text{Aut}(B_1)$ is an action of G on B_1.*

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PROOF. We note that $e_e = e_A$. Hence $\alpha_e^{\mathcal{B}} = \mathrm{id}$ on B_1. For all $t, s \in G$,

$$(\alpha_t^{\mathcal{B}} \circ \alpha_s^{\mathcal{B}})(e_A) = \alpha_t^{\mathcal{B}}\left(\sum_{i=1}^{n_{s^{-1}}} x_i^{s^{-1}} e_A x_i^{s^{-1}*}\right) = \sum_{i=1}^{n_{s^{-1}}} \sum_{j=1}^{n_{t^{-1}}} x_i^{s^{-1}} x_j^{t^{-1}} e_A x_j^{t^{-1}*} x_i^{s^{-1}*}.$$

Since $x_i^{s^{-1}} x_j^{t^{-1}} \in B_{s^{-1}t^{-1}}$ and

$$\sum_{i=1}^{n_{s^{-1}}} \sum_{j=1}^{n_{t^{-1}}} x_i^{s^{-1}} x_j^{t^{-1}} x_j^{t^{-1}*} x_i^{s^{-1}*} = 1,$$

by Lemma 3.3,

$$\sum_{i=1}^{n_{s^{-1}}} \sum_{j=1}^{n_{t^{-1}}} x_i^{s^{-1}} x_j^{t^{-1}} e_A x_j^{t^{-1}*} x_i^{s^{-1}*} = e_{s^{-1}t^{-1}}.$$

Thus,

$$(\alpha_t^{\mathcal{B}} \circ \alpha_s^{\mathcal{B}})(e_A) = e_{(ts)^{-1}} = \alpha_{ts}^{\mathcal{B}}(e_A).$$

Hence the map $t \mapsto \alpha_t^{\mathcal{B}}$ is an action of G on B_1. □

We call $(G, \alpha^{\mathcal{B}})$ the *action on B_1 induced by $\mathcal{B}$*.

LEMMA 3.8. *With the above notation, the inclusion $A \subset B$ is of depth* 2.

PROOF. We shall prove this lemma in the same way as Osaka and Teruya [20, Lemma 3.4]. We have only to show that $(A' \cap B_1)e_2(A' \cap B_1)$ contains the unit, where e_2 is the Jones projection induced by E_1. By Lemma 3.3, it follows that $e_t e_2 e_t \in (A' \cap B_1)e_2(A' \cap B_1)$ for all $t \in G$. Also, for all $t \in G$,

$$\begin{aligned} e_t e_2 e_t &= \sum_{i,j=1}^{n_t} x_i^t e_A x_i^{t*} e_2 x_j^t e_A x_j^{t*} = \sum_{i,j=1}^{n_t} x_i^t e_A e_2 x_i^{t*} x_j^t e_A x_j^{t*} \\ &= \sum_{i,j=1}^{n_t} x_i^t e_A e_2 e_A x_i^{t*} x_j^t x_j^{t*} \\ &= \sum_{i,j=1}^{n_t} \frac{1}{\mathrm{Index}\ E} x_i^t e_A x_i^{t*} x_j^t x_j^{t*} = \frac{1}{\mathrm{Index}\ E} \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*}, \end{aligned}$$

since $e_2 b = b e_2$ for all $b \in B$, we have that $B_t^* B_t \subset A$ and $e_A e_2 e_A = (\mathrm{Index}\ E)^{-1} e_A$ by [35, Lemma 2.3.5]. Thus,

$$\sum_{t \in G} e_t e_2 e_t = \frac{1}{\mathrm{Index}\ E} \sum_{t \in G} \sum_{i=1}^{n_t} x_i^t e_A x_i^{t*} = \frac{1}{\mathrm{Index}\ E} \in \mathbb{C}1.$$

Therefore, the lemma is proved. □

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We denote by $(B, E)_{\mathcal{B}}$ the element in $\mathcal{L}$ induced by $\mathcal{B}$ described above. Also, we denote its equivalence class by $[B, E]_{\mathcal{B}}$.

Let $\mathcal{B} = \{B_t\}_{t\in G}$ and $\mathcal{D} = \{D_h\}_{h\in H}$ be elements in $\mathcal{M}$. Let $B = \bigoplus_{t\in G} B_t$ and $D = \bigoplus_{h\in H} D_h$. Furthermore, let E and F be the canonical conditional expectations from B and D onto A, respectively.

LEMMA 3.9. *With the above notation, let $(B, E)_{\mathcal{B}}$ and $(D, F)_{\mathcal{D}}$ be the elements in $\mathcal{L}$ induced by $\mathcal{B}$ and $\mathcal{D}$ in $\mathcal{M}$. If $\mathcal{B} \sim \mathcal{D}$ in $\mathcal{M}$, then $(B, E)_{\mathcal{B}} \sim (D, F)_{\mathcal{D}}$ in $\mathcal{L}$.*

PROOF. Since $\mathcal{B} \sim \mathcal{D}$ in $\mathcal{M}$, there exists an isomorphism λ of G onto H satisfying the condition that, for all $t \in G$, there exists a linear isomorphism π_t of B_t onto $D_{\lambda(t)}$ such that $\pi_{ts}(xy) = \pi_t(x)\pi_s(y)$, $\pi_{t^{-1}}(x^*) = \pi_t(x)^*$ and $\pi_e = \mathrm{id}$ on $B_e = D_e = A$. Let π be the map from B to D defined by $\pi(x) = \sum_{t\in G} \pi_t(x_t)$ for all $x = \bigoplus_{t\in G} x_t$ in B. By easy computations, π is an isomorphism of B onto D and $E = F \circ \pi$. Therefore, $(B, E)_{\mathcal{B}} \sim (D, F)_{\mathcal{D}}$ in $\mathcal{L}$. $\square$

By Lemma 3.9, we can define a map $\mathcal{H}$ from $\mathcal{M}/\!\sim$ to $\mathcal{L}/\!\sim$ by $\mathcal{H}([\mathcal{B}]) = [B, E]_{\mathcal{B}}$ for all $\mathcal{B} = \{B_t\}_{t\in G} \in \mathcal{M}$.

4. Construction of a map from $\mathcal{L}^0/\!\sim$ to $\mathcal{N}/\!\sim$

Suppose that $(B, E) \in \mathcal{L}^0$. Since $A' \cap B_1$ is commutative, there exists $n \in \mathbb{N}$ such that $A' \cap B_1 \cong \mathbb{C}^n$. Let $\{e_t\}_{t\in G}$ be the set of minimal projections in $A' \cap B_1$, where G is a finite set with n elements containing the distinguished point e with $e_e = e_A$ and where e_A is the Jones projection induced by E. Since $A \subset B$ is of depth 2, it follows that $A' \cap B_2 \cong M_n(\mathbb{C})$, where B_2 is the C^*-basic construction induced by $B \subset B_1$. Thus, for all $t \in G$ there exists a unitary element $u_t \in A' \cap B_2$ with $u_t e_A u_t^* = e_t$, where $u_e = 1$. For all $a \in A$,

$$e_t a e_t = u_t e_A u_t^* a u_t e_A u_t^* = E(a) e_t,$$

since $u_t \in A' \cap B_2$. If $ae_t = 0$, then

$$0 = ae_t = au_t e_A u_t^* = u_t a e_A u_t^*.$$

Hence $ae_A = 0$. Thus $a = 0$. By [35, Proposition 2.2.1], there exists a unique isomorphism α_t of B_1 onto $C^*\langle B, e_t\rangle$ such that $\alpha_t(b) = b$ for all $b \in B$ and $\alpha_t(e_A) = e_t$. Hence

$$\mathbb{C}^n \cong A' \cap B_1 \supset \alpha_t(A' \cap B_1) = A' \cap C^*\langle B, e_t\rangle \cong \mathbb{C}^n.$$

Thus, $A' \cap B_1 = A' \cap C^*\langle B, e_t\rangle$. As $e_A \in C^*\langle B, e_t\rangle$, we see that $\alpha_t \in \mathrm{Aut}(B_1)$ for all $t \in G$. As $\alpha_t(A' \cap B_1) = A' \cap B_1$ for all $t \in G$, we see that $(\alpha_t \circ \alpha_s)(e_A) \in A' \cap B_1$ for all $s, t \in G$ and $(\alpha_t \circ \alpha_s)(e_A)$ is a minimal projection in $A' \cap B_1$. Hence there exists $r \in G$ where $(\alpha_t \circ \alpha_s)(e_A) = \alpha_r(e_A)$. Therefore, we can define a multiplication

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in G such that $\alpha_t \circ \alpha_s = \alpha_{ts}$. In the same way as above, we can define the inverse of each element in G. Thus G is a group with the unit element e.

LEMMA 4.1. *With the above notation, α_t is outer for all $t \in G \setminus \{e\}$.*

PROOF. We suppose that there is an element $t \in G \setminus \{e\}$ such that α_t is inner. Then there is a unitary element $u_t \in B_1$ where $\alpha_t = \mathrm{Ad}(u_t)$. Hence, for all $b \in B$, we have that $b = u_t b u_t^*$. Thus $u_t \in B' \cap B_1$. Since $A' \cap B = \mathbb{C}1$, so is $B' \cap B_1$ by [35, the proof of Proposition 2.7.3]. It follows that $\alpha_t = \mathrm{id}$ and that $e_A = e_t$. This is a contradiction since $t \neq e$. □

Since B is a fixed point C^*-subalgebra for (G, α), we can identify B_2 with $B_1 \rtimes_\alpha G$. Since $e_A \otimes 1$ is a full projection in $M(B_1^s)$, by Brown [2, Lemma 2.5], there exists an isometry $w \in M(B_1^s)$ such that $ww^* = e_A \otimes 1$. Noting further that $M(B_1^s) \subset M(B_2^s)$, by easy computations, $\mathrm{Ad}(w)$ is an isomorphism of B_2^s onto B^s such that $\mathrm{Ad}(w)(B_1^s) = A^s$, where we identify A with Ae_A and B with $e_A B e_2 e_A$. Let

$$\beta_t = \mathrm{Ad}(w) \circ (\alpha_t \otimes \mathrm{id}) \circ \mathrm{Ad}(w^*).$$

By the definition of β_t, (G, β) is an action of G on A^s. We call (G, α) and (G, β) the *actions of G on B_1 and A^s induced by* (B, E) and we denote them by $(G, \alpha)_{(B,E)}$ and $(G, \beta)_{(B,E)}$.

LEMMA 4.2. *With the above notation, $[G, \beta]_{(B,E)}$ is independent of the choice of the isometry $w \in M(B_1^s)$.*

PROOF. Let z be another isometry in $M(B_1^s)$ where $zz^* = e_A \otimes 1$. Let $(G, \gamma)_{(B,E)}$ be the action of G on A^s as above. Then $\gamma_t = \mathrm{Ad}(zw^*\beta_t(wz^*)) \circ \beta_t$ and $zw^*\beta_t(wz^*)$ is a unitary element in $M(Ae_A \otimes \mathbb{K})$. Since we identify A with Ae_A, we have that $(G, \beta)_{(B,E)}$ is exterior equivalent to $(G, \gamma)_{(B,E)}$. Therefore, the lemma is proved. □

Let (D, F) be another element in $\mathcal{L}^0$ and let $(H, \delta)_{(D,F)}$ be the action on D_1 induced by (D, F), where D_1 is the C^*-basic construction induced by (D, F).

LEMMA 4.3. *With the above notation, if $(B, E) \sim (D, F)$ in $\mathcal{L}^0$, then there exist an isomorphism π_1 of B_1 onto D_1 and an isomorphism λ of G onto H such that $\delta_{\lambda(t)} = \pi_1 \circ \beta_t \circ \pi_1^{-1}$ for all $t \in G$, where B_1 and D_1 are the C^*-basic constructions induced by (B, E) and (D, F).*

PROOF. Since $(B, E) \sim (D, F)$ in $\mathcal{L}^0$, there exists an isomorphism π of B onto D such that $E = F \circ \pi$. Let e_A and f_A be the Jones projections induced by E and F. Then there exists an isomorphism π_1 of B_1 onto D_1 determined by $\pi_1(e_A) = f_A$ and $\pi_1(x) = \pi(x)$ for all $x \in B$. Hence π_1 is an isomorphism of $A' \cap B_1$ onto $A' \cap D_1$. Thus, by the definitions of $(G, \alpha)_{(B,E)}$ and $(H, \delta)_{(D,F)}$, there exists an isomorphism λ of G onto H such that $\delta_{\lambda(t)} = \pi_1 \circ \beta_t \circ \pi_1^{-1}$ for all $t \in G$. Therefore, the lemma is proved. □

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COROLLARY 4.4. *Let* $(B, E), (D, F) \in \mathcal{L}^0$ *and let* $(G, \beta)_{(B,E)}$ *and* $(H, \gamma)_{(D,F)}$ *be the actions on* A^s *induced by* (B, E) *and* (D, F), *respectively. If* $(B, E) \sim (D, F)$ *in* $\mathcal{L}^0$, *then* $(G, \beta)_{(B,E)} \sim (H, \gamma)_{(D,F)}$ *in* $\mathcal{N}$.

PROOF. This is immediate by Lemmas 4.2 and 4.3. □

By the above corollary, we can define a map $\mathcal{F}$ from $\mathcal{L}^0/\sim$ to $\mathcal{N}/\sim$ by

$$\mathcal{F}([B, E]) = [G, \beta]_{(B,E)},$$

where $[G, \beta]_{(B,E)}$ is the equivalence class of the action of G on A^s induced by (B, E). Since $(B, E) \in \mathcal{L}^0$, there exists a quasi-basis $\{(u_i, u_i^*)\}_{i=1}^m$ for E. For each right Hilbert A-module X, let $\mathbb{K}_A(X)$ be the C^*-algebra generated by the right rank-one operators on X. Since we regard B as a right Hilbert A-module by using E, we can construct a C^*-algebra $\mathbb{K}_A(B)$ which is isomorphic to B_1. We identify $\mathbb{K}_A(B)$ with B_1. Let $x_i^t = e_t(u_i)$ for all $t \in G$ and $i = 1, 2, \ldots, m$.

LEMMA 4.5. *With the above notation,* $e_t = \sum_{i=1}^m x_i^t e_A x_i^{t*}$ *for all* $t \in G$.

PROOF. For all $b \in B$, we have that $e_t(b) = \sum_{i=1}^m u_i E(u_i^* e_t(b))$ since $e_t(b) \in B$. Since e_t is a projection in $\mathbb{K}_A(B)$,

$$\begin{aligned} e_t(b) = e_t^2(b) = e_t\left(\sum_{i=1}^m u_i E(u_i^* e_t(b))\right) &= \sum_{i=1}^m e_t(u_i) E(u_i^* e_t(b)) \\ &= \sum_{i=1}^m e_t(u_i) \langle u_i, e_t(b)\rangle_A. \end{aligned}$$

On the other hand, if we regard $\sum_{i=1}^m x_i^t e_A x_i^{t*}$ as an element in $\mathbb{K}_A(B)$, we have the following equations:

$$\begin{aligned} \sum_{i=1}^m x_i^t e_A x_i^{t*}(b) = \sum_{i=1}^m x_i^t E(x_i^{t*} b) &= \sum_{i=1}^m e_t(u_i) \langle e_t(u_i), b\rangle_A \\ &= \sum_{i=1}^m e_t(u_i) \langle u_i, e_t(b)\rangle_A. \end{aligned}$$

Therefore, the lemma is proved. □

5. Construction of a map from $\mathcal{N}/\sim$ to $\mathcal{M}/\sim$

In this section we shall construct a map $\mathcal{G}$ from $\mathcal{N}/\sim$ to $\mathcal{M}/\sim$. Let (G, β) be an action of a finite group G on A^s. Let e_{00} be a rank-one projection in $\mathbb{K}$ and put $p = 1 \otimes e_{00}$. Let X_{β_t} be the Banach space $pA^s\beta_t(p)$ for all $t \in G$. We define a product $\cdot : X_{\beta_t} \times X_{\beta_s} \longrightarrow X_{\beta_{ts}}$ and an involution $\sharp : X_{\beta_t} \longrightarrow X_{\beta_{t^{-1}}}$ as follows. For all $x \in X_{\beta_t}$ and $y \in X_{\beta_s}$ we define $x \cdot y = x\beta_t(y)$ and $x^\sharp = \beta_{t^{-1}}(x^*)$. By routine computations,

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we see that $\mathcal{B}_{(G,\beta)} = \{X_{\beta_t}\}_{t\in G}$ is a saturated C^*-algebraic bundle with a product $\cdot$ and an involution $\sharp$, and that $X_{\beta_e} = A$, where we identify $pA^s p$ with A. We call this $\mathcal{B}_{(G,\beta)}$ *the saturated C^*-algebraic bundle over G induced by (G, β)*. Let (H, γ) be another action of a finite group H on A^s and let $\mathcal{B}_{(H,\gamma)}$ be the saturated C^*-algebraic bundle over a finite group induced by (H, γ).

LEMMA 5.1. *With the above notation, if $(G, \beta) \sim (H, \gamma)$ in $\mathcal{N}$, then $\mathcal{B}_{(G,\beta)} \sim \mathcal{B}_{(H,\gamma)}$ in $\mathcal{M}$.*

PROOF. Since $(G, \beta) \sim (H, \gamma)$ in $\mathcal{N}$, we identify G with H. Then there exists a unitary element $v_t \in M(A^s)$ for all $t \in G$ satisfying the conditions that $\gamma_t = \mathrm{Ad}(v_t) \circ \beta_t$ and that $v_{ts} = v_t\beta_t(v_s)$ for all $t, s \in G$. For all $t \in G$, let π_t be the map from X_{β_t} to X_{γ_t} defined by $\pi_t(pa\beta_t(p)) = pa\beta_t(p)v_t^* = pav_t^*\gamma_t(p)$ for all $a \in A^s$. Then clearly π_t is a linear isomorphism of X_{β_t} onto X_{γ_t}. For all $a, b \in A^s$ and $t, s \in G$,

$$\begin{aligned}\pi_t(pa\beta_t(p)) \cdot \pi_s(pb\beta_s(p)) &= pav_t^*\gamma_t(p)\gamma_t(b)\gamma_t(v_s^*)\gamma_{ts}(p)\\ &= pa\beta_t(p)\beta_t(b)\beta_t(v_s^*)v_t^*\gamma_{ts}(p) = pa\beta_t(p)\beta_t(b)v_{ts}^*\gamma_{ts}(p)\\ &= \pi_{ts}(pa\beta_t(p)\beta_t(b)\beta_{ts}(p)) = \pi_{ts}(pa\beta_t(p) \cdot pb\beta_s(p)),\end{aligned}$$

since $\beta_t = \mathrm{Ad}(v_t^*) \circ \gamma_t$ and $v_{ts}^* = \beta_t(v_s^*)v_t^*$ for all $t, s \in G$. Also,

$$\begin{aligned}\pi_t(pa\beta_t(p))^\sharp &= (pav_t^*\gamma_t(p))^\sharp = p\gamma_{t^{-1}}(v_t)\gamma_{t^{-1}}(a^*)\gamma_{t^{-1}}(p)\\ &= p\beta_{t^{-1}}(a^*)\beta_{t^{-1}}(p)v_{t^{-1}}^* = \pi_{t^{-1}}(p\beta_{t^{-1}}(a^*)\beta_{t^{-1}}(p)) = \pi_{t^{-1}}((pa\beta_t(p))^\sharp),\end{aligned}$$

since $\gamma_{t^{-1}} = \mathrm{Ad}(v_{t^{-1}}) \circ \beta_{t^{-1}}$ and $1 = v_{t^{-1}}\beta_{t^{-1}}(v_t)$. Furthermore, $\pi_e = \mathrm{id}$ on the space $X_{\beta_e} = X_{\gamma_e} = pA^s p$. Therefore, the lemma is proved. □

By Lemma 5.1, we can define a map $\mathcal{G}$ from $\mathcal{N}/\sim$ to $\mathcal{M}/\sim$ by $\mathcal{G}([G, \beta]) = [\mathcal{B}_{(G,\beta)}]$ for all $(G, \beta) \in \mathcal{N}$.

6. Composition of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$

In this section we shall show that $\mathcal{H} \circ \mathcal{G} \circ \mathcal{F} = \mathrm{id}$ on $\mathcal{L}^0/\sim$. Let $(B, E) \in \mathcal{L}^0$ and let $(G, \alpha)_{(B,E)}$ and $(G, \beta)_{(B,E)}$ be the actions of G on B_1 and A^s induced by (B, E), respectively. We construct the saturated C^*-algebraic bundle $\{Y_{\alpha_t}\}_{t\in G}$ over G induced by $(G, \alpha)_{(B,E)}$ in a similar way to as in Section 5.

LEMMA 6.1. *Let $(B, E) \in \mathcal{L}^0$ and let $(G, \alpha)_{(B,E)}$ and $(G, \beta)_{(B,E)}$ be the actions of G on B_1 and A^s induced by (B, E), respectively. Let $\{X_{\beta_t}\}_{t\in G}$ and $\{Y_{\alpha_t}\}_{t\in G}$ be the saturated C^*-algebraic bundles over G induced by $(G, \beta)_{(B,E)}$ and $(G, \alpha)_{(B,E)}$. Then $\{X_{\beta_t}\}_{t\in G} \sim \{Y_{\alpha_t}\}_{t\in G}$ in $\mathcal{M}$.*

PROOF. We recall that $X_{\beta_t} = pA^s\beta_t(p)$ and $Y_{\alpha_t} = e_A B_1\alpha_t(e_A)$ for all $t \in G$, where $p = 1 \otimes e_{00}$ and e_{00} is a rank-one projection in $\mathbb{K}$, and we identify $pA^s p$ and Ae_A with A in the usual way. For all $t \in G$, let π_t be the map from $pA^s\beta_t(p)$ to $e_A B_1\alpha_t(e_A)$ defined, for all $a \in A^s$, by

$$\pi_t(pa\beta_t(p)) = (e_A \otimes e_{00})(\psi_e \otimes \mathrm{id})(a)w(\alpha_t \otimes \mathrm{id})(w^*)(\alpha_t(e_A) \otimes e_{00}),$$

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where ψ_e is an isomorphism of A onto Ae_A defined by $\psi_e(x) = xe_A$ for all $x \in A$ and we identify $e_A B_1 \alpha_t(e_A)$ with $(e_A \otimes e_{00}) B_1^s (\alpha_t \otimes \mathrm{id})(e_A \otimes e_{00})$. Then, from the above definition, π_t is a linear isomorphism of $pA^s\beta_t(p)$ onto $e_A B_1 \alpha_t(e_A)$. By routine computations, $\pi_{ts}(x \cdot y) = \pi_t(x) \cdot \pi_s(y)$ and $\pi_{t^{-1}}(x^\sharp) = \pi_t(x)^\sharp$ for all $x \in pA^s\beta_t(p)$ and $y \in pA^s\beta_s(p)$. Furthermore, $\pi_e = \mathrm{id}$ on A. Therefore, the lemma is proved. □

THEOREM 6.2. *$\mathcal{H} \circ \mathcal{G} \circ \mathcal{F} = \mathrm{id}$ on $\mathcal{L}^0/\sim$.*

PROOF. Let $(B, E) \in \mathcal{L}^0$. From Lemma 6.1, $(\mathcal{G} \circ \mathcal{F})([B, E]) = [\{Y_{\alpha_t}\}_{t \in G}]$. Let $D = \bigoplus_{t \in G} Y_{\alpha_t}$ and let F be the canonical conditional expectation from D onto $Y_{\alpha_e} = Ae_A$, where we identify Ae_A with A. We shall prove that $(B, E) \sim (D, F)$ in $\mathcal{L}^0$. Let π be a linear map from B to D defined by $\pi(x) = \bigoplus_{t \in G} e_A x \alpha_t(e_A)$ for all $x \in B$. Since $\bigoplus_{t \in G} \alpha_t(e_A) = 1$, by easy computations $\pi(x) \cdot \pi(y) = \pi(xy)$ for all $x, y \in B$. Also, by easy computations, $\pi(x)^\sharp = \pi(x^*)$ for all $x \in B$ and $F(\pi(x)) = e_A x e_A = E(x) e_A$. This means that $E = F \circ \pi$, since we identify Ae_A with A. For an $x \in B$, we suppose that $\pi(x) = 0$. Then $0 = \bigoplus_{t \in G} e_A x e_t = e_A x$. Hence $x = 0$. Thus π is injective. Furthermore, for all $t \in G$, let $x_t, y_t \in B$. Then, by Lemma 4.5,

$$\bigoplus_{t \in G} e_A x_t e_A y_t \alpha_t(e_A) = \bigoplus_{t \in G} \sum_{i=1}^m e_A x_t e_A y_t x_i^t e_A x_i^{t*} = \bigoplus_{t \in G} \sum_{i=1}^m e_A E(x_t) E(y_t x_i^t) x_i^{t*}.$$

Let $b = \sum_{i=1}^m \sum_{t \in G} E(x_t) E(y_t x_i^t) x_i^{t*} \in B$. Then, by the above equation,

$$\begin{aligned}\pi(b) &= \bigoplus_{s \in G} e_A b \alpha_s(e_A) = \bigoplus_{s \in G} \sum_{i=1}^m \sum_{t \in G} e_A E(x_t) E(y_t x_i^t) x_i^{t*} \alpha_s(e_A) \\ &= \bigoplus_{s \in G} e_A x_s e_A y_s \alpha_s(e_A),\end{aligned}$$

since

$$\alpha_t(e_A)\alpha_s(e_A) = \begin{cases} 0 & \text{if } s \neq t, \\ \alpha_s(e_A) & \text{if } s = t. \end{cases}$$

Hence π is surjective. Therefore, the theorem is proved. □

REMARK 6.3. Let $\mathcal{B} = \{B_t\}_{t \in G}$ be an element in $\mathcal{M}$ and let $(G, \alpha^{\mathcal{B}})$ be the action on B_1 induced by $\mathcal{B}$. In the same way as in Section 4, we can construct the action (G, β) on A^s induced by $(G, \alpha^{\mathcal{B}})$ and define a map $\mathcal{K}$ from $\mathcal{M}/\sim$ to $\mathcal{N}/\sim$. Furthermore, by routine computations, $\mathcal{K} \circ \mathcal{G} = \mathrm{id}$ on $\mathcal{N}/\sim$ and $\mathcal{G} \circ \mathcal{K} = \mathrm{id}$ on $\mathcal{M}/\sim$.

7. Images

In this section, we shall compute $\mathcal{F}(\mathcal{L}^0/\sim)$ and $(\mathcal{G} \circ \mathcal{F})(\mathcal{L}^0/\sim)$.

LEMMA 7.1. *Let $(B, E) \in \mathcal{L}^0$ and let $(G, \beta)_{(B,E)}$ be the action of G on A^s induced by (B, E). Then β_t is outer for all $t \in G \setminus \{e\}$.*

PROOF. Since $A' \cap B = \mathbb{C}1$, we have that $A' \cap A = \mathbb{C}1$. We suppose that there exists a $t \in G \setminus \{e\}$ such that β_t is inner in $M(A^s)$. Let $(G, \alpha)_{(B,E)}$ be the action of G on B_1 induced by (B, E). Then, by the definition of β_t and routine computations, there exists a unitary element $v_t \in M(A^s)$ such that $\alpha_t \otimes \mathrm{id} = \mathrm{Ad}(v_t)$. Hence by Phillips and Raeburn [24, Lemma 2.3], α_t is inner in B_1. This is a contradiction by Lemma 4.1. □

For all Hilbert A–A-bimodules X and Y, let ${}_A\mathrm{Hom}_A(X, Y)$ be the space of all A–A-homomorphisms of X to Y. If $X = Y$, we denote this space by ${}_A\mathrm{End}_A(X)$.

LEMMA 7.2. *Let $(G, \beta) \in \mathcal{N}$. We suppose that β_t is outer for all $t \in G \setminus \{e\}$ and that $A' \cap A = \mathbb{C}1$. Then the saturated C^*-algebraic bundle over G, $\mathcal{B}_{(G,E)} = \{X_{\beta_t}\}_{t\in G}$ induced by (G, β), has the following property:*

$${}_A\mathrm{Hom}_A(X_{\beta_t}, X_{\beta_s}) = \begin{cases} \mathbb{C}\,\mathrm{id} & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

PROOF. First, we note that X_{β_t} is an A–A-equivalence bimodule for all $t \in G$. We shall show that, for all $t, s \in G$ where $t \neq s$, $X_{\beta_t} \ncong X_{\beta_s}$ as A–A-equivalence bimodules. We suppose that there exist $t, s \in G$ such that $t \neq s$ and $X_{\beta_t} \cong X_{\beta_s}$ as A–A-equivalence bimodules. Then, by Brown *et al.* [3, Corollary 3.5], there is a unitary element $v \in M(A^s)$ such that $\beta_t = \mathrm{Ad}(v) \circ \beta_s$. This is a contradiction. Since $A' \cap A = \mathbb{C}1$, we have that X_{β_t} is irreducible for all $t \in G$. Indeed, by [13, the remark after Lemma 1.10 and Corollary 1.28],

$${}_A\mathrm{End}_A(X_{\beta_t}) \cong (A \otimes e_{00})' \cap \mathbb{K}_A(X_{\beta_t}) = (A \otimes e_{00})' \cap (A \otimes e_{00}),$$

where we identify A with $A \otimes e_{00}$ and $\beta_t(p)A^s\beta_t(p)$, and e_{00} is a rank-one projection in $\mathbb{K}$ with $p = 1 \otimes e_{00}$. Thus, ${}_A\mathrm{End}_A(X_{\beta_t}) = \mathbb{C}\,\mathrm{id}$. Therefore, the lemma is proved. □

Let $\mathcal{N}^0$ be the set of all actions $(G, \beta) \in \mathcal{N}$ satisfying the condition that β_t is outer for all $t \in G \setminus \{e\}$.

Let $\mathcal{B} = \{B_t\}_{t\in G} \in \mathcal{M}$. Then we can regard B_t as a Hilbert A–A-bimodule for all $t \in G$.

LEMMA 7.3. *Let $\mathcal{B} = \{B_t\}_{t\in G}$ be an element in $\mathcal{M}$ satisfying the condition that*

$${}_A\mathrm{Hom}_A(B_t, B_s) = \begin{cases} \mathbb{C}\,\mathrm{id} & \text{if } t = s, \\ 0 & \text{if } t \neq s, \end{cases}$$

and let $(B, E)_{\mathcal{B}}$ be the element in $\mathcal{L}$ induced by $\mathcal{B}$. Then $A' \cap B = \mathbb{C}1$ and $A' \cap B_1$ is commutative.

PROOF. First, we show that $A' \cap B_1$ is commutative. Note that $A' \cap B_1 \cong {}_A\mathrm{End}_A(B)$. Since $B = \bigoplus_{t\in G} B_t$, by the assumption of the lemma, ${}_A\mathrm{End}_A(B) \cong \mathbb{C}^n$, where $n = |G|$. Next, we shall show that $A' \cap B = \mathbb{C}1$. Let $x = \bigoplus_{t\in G} x_t \in A' \cap B$. Then $x_t a = a x_t$ for all $t \in G$ and $a \in A$. For all $t \in G \setminus \{e\}$, let $T_{x_{t^{-1}}}$ be the

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homomorphism of B_t to $B_e(=A)$ defined by $T_{x_{t^{-1}}}y = x_{t^{-1}}y$ for all $y \in B_t$. Since $x_{t^{-1}}y \in B_e$ and $x_{t^{-1}}a = ax_{t^{-1}}$ for all $a \in A$, we have that $T_{x_{t^{-1}}} \in {}_A\mathrm{Hom}_A(B_t, B_e)$. Hence $T_{x_{t^{-1}}} = 0$. Thus $x_{t^{-1}}x_{t^{-1}}^* = 0$, that is, $x_{t^{-1}} = 0$. Let T_{x_e} be an endomorphism defined as above. Then since $ax_e = x_e a$ for all $a \in A$, we have that $T_{x_e} \in {}_A\mathrm{End}_A(B_e)$. Hence there exists $\lambda \in \mathbb{C}$ such that $T_{x_e} = \lambda\,\mathrm{id}$. Thus $T_{x_e}(1) = \lambda 1$, that is, $x_e = \lambda 1$. Therefore, $A' \cap B = \mathbb{C}1$. □

Let $\mathcal{M}^0$ be the set of all $\mathcal{B}$ in $\mathcal{M}$ satisfying the assumption of Lemma 7.3.

PROPOSITION 7.4. *$\mathcal{F}(\mathcal{L}^0/\sim) = \mathcal{N}^0/\sim$ and $(\mathcal{G} \circ \mathcal{F})(\mathcal{L}^0/\sim) = \mathcal{M}^0/\sim$.*

PROOF. This is immediate by Lemmas 7.1–7.3 and Theorem 6.2. □

8. Twisted actions of finite groups

Let $(B, E) \in \mathcal{L}^0$. As mentioned in Section 4, there exists a group G such that $A' \cap B_1 = \bigoplus_{t \in G} \mathbb{C}e_t$, where $\{e_t\}_{t \in G}$ is a family of mutually orthogonal minimal projections in $A' \cap B_1$. Let e be the unit element in G and e_A the Jones projection induced by E. Then, $e_e = e_A$. Also, there is an action β of G on B_1 defined by

$$\beta_t(e_A) = e_t, \quad \beta_t(b) = b$$

for all $t \in G$ and $b \in B$.

In this section, we shall give a necessary and sufficient condition for B to be described as a twisted crossed product of A and a twisted action of G on A under the above condition.

Suppose that $e_A \sim \beta_t(e_A)$ in B_1 for all $t \in G$. Then there exists a partial isometry w_t in B_1 such that $w_t^* w_t = e_A$ and $w_t w_t^* = e_t$ $(= \beta_t(e_A))$ for all $t \in G$. Since $w_t \in B_1$, we can write $w_t = \sum_{i=1}^{m_t} x_i^t e_A y_i^t$, where $x_i^t, y_i^t \in B$ for $i = 1, 2, \ldots, m_t$. Now we put $u_t^* = \sum_{i=1}^{m_t} x_i^t E(y_i^t) \in B$ for all $t \in G$. Then $w_t e_A = u_t^* e_A$ for all $t \in G$. If $t = e$, then $w_e^* w_e = w_e w_e^* = e_A$. Hence we may assume that $w_e = e_A$. Thus $u_e = 1$.

LEMMA 8.1. *For all $t \in G$, the element u_t is unitary in B.*

PROOF. For all $t \in G$,

$$u_t^* u_t = nE_1(u_t^* e_A u_t) = nE_1(w_t e_A w_t^*) = nE_1(e_t) = 1,$$

as $E_1(e_t) = 1/n$. As $e_A u_t u_t^* e_A = e_A w_t^* w_t e_A = e_A$, we see that $E(u_t u_t^*) e_A = e_A$. Thus $E(u_t u_t^*) = 1$. Hence $u_t u_t^* = 1$, since E is faithful and u_t is an isometry. □

LEMMA 8.2. *For all $s, t \in G$, $\beta_s(e_t) = e_{st}$.*

PROOF. For all $s, t \in G$,

$$\beta_s(e_t) = (\beta_s \circ \beta_t)(e_A) = \beta_{st}(e_A) = e_{st}.$$ □

LEMMA 8.3. *$u_s u_t u_{st}^* \in A$ for all $s, t \in G$. In particular, $u_{t^{-1}} u_t \in A$ for all $t \in G$.*

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PROOF. For all $s, t \in G$,

$$\begin{aligned} u_{st}^* e_A u_{st} &= e_{st} = \beta_s(e_t) = \beta_s(u_t^* e_A u_t) = u_t^* \beta_s(e_A) u_t \\ &= u_t^* e_s u_t = u_t^* u_s^* e_A u_s u_t, \end{aligned}$$

by Lemma 8.2. Hence $u_s u_t u_{st}^* e_A = e_A u_s u_t u_{st}^*$. Thus $u_s u_t u_{st}^* \in A$. Also, since $u_e = 1$, we have that $u_{t^{-1}} u_t \in A$. □

For all $s, t \in G$, let $u(s, t) = u_s u_t u_{st}^*$. Then $u(s, t)$ is a unitary element in A by Lemmas 8.1 and 8.3.

LEMMA 8.4. *For all* $t \in G$, $u_t A u_t^* = A$.

PROOF. For all $t \in G$ and $a \in A$,

$$u_t a u_t^* e_A = u_t a e_t u_t^* = u_t e_t a u_t^* = e_A u_t a u_t^*,$$

since $e_t \in A' \cap B_1$. Thus $u_t a u_t^* \in A$. Hence $u_t A u_t^* \subseteq A$. On the other hand, since $u_{t^{-1}} u_t \in A$ by Lemma 8.3,

$$\begin{aligned} u_t e_A u_t^* &= u_t u_t^* u_{t^{-1}}^* u_{t^{-1}} u_t e_A u_t^* = u_t u_t^* u_{t^{-1}}^* e_A u_{t^{-1}} u_t u_t^* \\ &= u_{t^{-1}}^* e_A u_{t^{-1}} = e_{t^{-1}}, \end{aligned}$$

and

$$u_t^* a u_t e_A = u_t^* a e_{t^{-1}} u_t = u_t^* e_{t^{-1}} a u_t = e_A u_t^* a u_t.$$

Hence $u_t^* a u_t \in A$. Thus, the lemma is proved. □

For all $t \in G$, let $\alpha_t = \mathrm{Ad}(u_t)|_A$, the restriction of $\mathrm{Ad}(u_t)$ to A. By Lemma 8.4, α_t is an automorphism of A for all $t \in G$. We shall show that (α, u) is a twisted action of G on A, as defined by Packer and Raeburn [22] and Quigg [27].

LEMMA 8.5. (α, u) *is a twisted action of* G *on* A.

PROOF. Clearly $u(e, t) = u(t, e) = 1$ for all $t \in G$ and $\alpha_s \circ \alpha_t = \mathrm{Ad}(u(s, t)) \circ \alpha_{st}$ for all $s, t \in G$. We have only to show that $\alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$ for all $s, t, r \in G$. Indeed,

$$\begin{aligned} \alpha_r(u(s, t))u(r, st) &= \mathrm{Ad}(u_r)(u_s u_t u_{st}^*) u_r u_{st} u_{rst}^* = u_r u_s u_t u_{rst}^* \\ &= (u_r u_s u_{rs}^*)(u_{rs} u_t u_{rst}^*) = u(r, s)u(rs, t). \end{aligned}$$ □

Following Quigg [27], we define the reduced twisted crossed product $A \rtimes_{\alpha,u} G$ associated with a twisted action (α, u) of G on A. We may assume that A acts on a Hilbert space $\mathcal{H}$ faithfully and nondegenerately. Also, we may assume that there is a unitary map $v : G \longrightarrow \mathbb{B}(\mathcal{H})$ such that

$$\alpha_t = \mathrm{Ad}(v_t), \qquad v_s v_t = u(s, t) v_{st}$$

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for all $s, t \in G$, where $\mathbb{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Then, given any element $x \in A \rtimes_{\alpha,u} G$, we can write $x = \sum_{t\in G} a_t v_t$, where $a_t \in A$ for all $t \in G$. Let F be the canonical conditional expectation from $A \rtimes_{\alpha,u} G$ onto A defined by $F(x) = a_e$, where $x = \sum_{t\in G} a_t v_t$, with $a_t \in A$ for all $t \in G$. We shall show that $(A \rtimes_{\alpha,u} G, F) \sim (B, E)$. In order to do so, we need the following lemmas.

LEMMA 8.6. *$E(u_t) = 0$, and α_t is an outer automorphism of A for all $t \in G \setminus \{e\}$.*

PROOF. We note that, for all $t \in G \setminus \{e\}$,

$$e_A w_t e_A = e_A w_t w_t^* w_t e_A = e_A e_t w_t e_A = 0.$$

On the other hand,

$$e_A w_t e_A = \sum_{i=1}^{m_t} e_A x_i^t e_A y_i^t e_A = \sum_{i=1}^{m_t} E(x_i^t)E(y_i^t)e_A.$$

Hence $\sum_{i=1}^{m_t} E(x_i^t)E(y_i^t) = 0$. Thus $E(u_t) = 0$, since $E(u_t) = \sum_{i=1}^{m_t} E(y_i^t)^* E(x_i^t)^*$. Next, suppose that α_t is an inner automorphism of A, that is, there is a unitary element z in A such that $\alpha_t(a) = zaz^*$ for all $a \in A$. Then $u_t^* za = u_t^* \alpha_t(a) z = a u_t^* z$ for all $a \in A$ and hence $u_t^* z \in A' \cap B = \mathbb{C}1$, that is, $u_t = \lambda z$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Therefore,

$$e_t = u_t e_A u_t^* = z e_A z^* = e_A.$$

This is a contradiction. □

For a twisted action (α, u) of G, we call the action a *twisted outer action* if α_t is outer for all $t \in G \setminus \{e\}$.

LEMMA 8.7. *$\{(u_s^*, u_s)\}_{s\in G}$ and $\{(u_s, u_s^*)\}_{s\in G}$ are quasi-bases for E.*

PROOF. Since

$$\sum_{s\in G} u_s^* e_A u_s = \sum_{s\in G} w_s e_A w_s^* = \sum_{s\in G} e_s = 1,$$

$\{(u_s^*, u_s)\}_{s\in G}$ is a quasi-basis for E. Since $u_{s^{-1}} u_s \in A$ by Lemma 8.3,

$$u_s e_A u_s^* = u_s u_{s^{-1}} u_{s^{-1}}^* e_A u_{s^{-1}} u_{s^{-1}}^* u_s^* = u_s u_{s^{-1}} e_{s^{-1}} u_{s^{-1}}^* u_s^* = e_{s^{-1}}.$$

Therefore, $\sum_{s\in G} u_s e_A u_s^* = \sum_{s\in G} e_{s^{-1}} = 1$ and hence $\{(u_s, u_s^*)\}_{s\in G}$ is a quasi-basis for E. □

PROPOSITION 8.8. $(B, E) \sim (A \rtimes_{\alpha,u} G, F)$.

PROOF. We shall show that there exists an isomorphism π of B onto $A \rtimes_{\alpha,u} G$ such that $E = F \circ \pi$. For all $x \in B$, we can write $x = \sum_{t\in G} E(x u_t^*) u_t$ by Lemma 8.7. We define a map π from B to $A \rtimes_{\alpha,u} G$ by $\pi(x) = \sum_{t\in G} E(x u_t^*) v_t$. It is clear that π is linear. Using the equalities

$$\alpha_t = \mathrm{Ad}(u_t) = \mathrm{Ad}(v_t), \quad u(s, t) = u_s u_t u_{st} = v_s v_t v_{st} \tag{8.1}$$

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we can see that $\pi(x)\pi(y) = \pi(xy)$ and $\pi(x^*) = \pi(x)^*$ for all $x, y \in B$ by Lemma 8.7. We now claim that π is bijective. For all $z \in A \rtimes_{\alpha,u} G$, we write $z = \sum_{t\in G} z_t v_t$, where $z_t \in A$ for all $t \in G$. Let $x = \sum_{t\in G} z_t u_t$. Then, by the above equations (8.1) and Lemma 8.6, $\pi(x) = z$. Next, we suppose that $\pi(x) = 0$ for an element $x \in B$, that is, $\sum_{t\in G} E(xu_t^*)v_t = 0$. For all $s \in G$,

$$\begin{aligned} 0 &= F\left(\sum_{t\in G} E(xu_t^*)v_t v_{s^{-1}}\right) = F\left(\sum_{t\in G} E(xu_t^*)u(t, s^{-1})v_{ts^{-1}}\right) \\ &= F\left(\sum_{t\in G} E(xu_t^* u(t, s^{-1}))v_{ts^{-1}}\right) = E(xu_s^* u(s, s^{-1})) = E(xu_s^*)u(s, s^{-1}), \end{aligned}$$

since $u(t, s^{-1}) \in A$. Hence $E(xu_s^*) = 0$ for all $s \in G$. Thus $x = 0$ by Lemma 8.7. It follows that π is bijective. Furthermore, for all $x = \sum_{t\in G} E(xu_t^*)u_t \in B$,

$$(F \circ \pi)(x) = F\left(\sum_{t\in G} E(xu_t^*)v_t\right) = E(xu_e^*) = E(x).$$

This concludes the proof. □

THEOREM 8.9. *Let $A \subset B$ be an irreducible inclusion of unital C^*-algebras, E be a conditional expectation from B onto A which is of index-finite type and of depth 2, and B_1 be the C^*-basic construction induced by the inclusion $A \subset B$. Suppose that there exists an action β of a finite group G on B_1 such that B is the fixed point algebra of B_1 by β. Then the following conditions are equivalent:*

(1) *$\beta_t(e_A) \sim e_A$ in B_1 for all $t \in G$;*
(2) *there exists a twisted outer action (α, u) of G on A with the property that $(B, E) \sim (A \rtimes_{\alpha,u} G, F)$, where F is the canonical conditional expectation from $A \rtimes_{\alpha,u} G$ onto A.*

PROOF. That (1) implies (2) is immediate by Proposition 8.8. That (2) implies (1) is clear. □

PROPOSITION 8.10. *With the above notation and assumptions, the following conditions are equivalent:*

(1) *there exist a C^*-subalgebra P of A with a common unit and a conditional expectation H from A onto P of index-finite type such that $(B, E) \sim (P_1, H_1)$, where P_1 is the C^*-basic construction induced by the inclusion $P \subset A$ and H_1 is the dual conditional expectation of H;*
(2) *there exists an outer action α of G on A such that $(B, E) \sim (A \rtimes_\alpha G, F)$, where F is the canonical conditional expectation from $A \rtimes_\alpha G$ onto A.*

PROOF. First we show that (1) implies (2): Since $A \subset B$ is of depth 2, so is $P \subset A$. Thus $P' \cap A = \mathbb{C}1$ and $P' \cap B_1 \cong M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the $(n \times n)$-matrix algebra over $\mathbb{C}$ and n is the order of G. As mentioned at the beginning of this section, $\{e_t\}_{t\in G}$ is a family of mutually orthogonal minimal projections in $A'\cap B_1$. Since $A'\cap B_1\subset P'\cap B_1$, we have that $\{e_t\}_{t\in G}$ is also a family of mutually orthogonal minimal projections in $P'\cap B_1$ (which is $M_n(\mathbb{C})$) and hence $e_t\sim e_A$ in $P'\cap B_1$ for all $t\in G$. Hence, for all $t\in G$, there exists a partial isometry $w_t\in P'\cap B_1$ such that $w_t^*w_t=e_A$ and $w_tw_t^*=e_t$. Since $w_t\in B_1$, we can write $w_t=\sum_{i=1}^{m_t}x_i^te_Ay_i^t$, where $x_i^t, y_i^t\in B$. Put $u_t^*=\sum_{i=1}^{m_t}x_i^tE(y_i^t)$ for all $t\in G$. Then, by Lemma 8.1, u_t is a unitary element in B and $w_te_A=u_t^*e_A$ for all $t\in G$. For all $x\in P$,

$$u_t^*xe_A=u_t^*e_Ax=w_te_Ax=xw_te_A=xu_t^*e_A.$$

Hence $u_t^*x=xu_t^*$ for all $x\in P$. Thus $u_t\in P'\cap B$ for all $t\in G$. Furthermore, since $u_su_tu_{st}^*\in A$ for all $s,t\in G$ by Lemma 8.3, $u_su_tu_{st}^*\in P'\cap A=\mathbb{C}1$ for all $s,t\in G$. Let $\alpha_t=\mathrm{Ad}(u_t)$ for all $t\in G$. Then α is an action of G on A by the proofs of Lemmas 8.4 and 8.5. Therefore, we can complete the proof in the same way as in the proof that (1) implies (2) in Theorem 8.9.

Now we show that (2) implies (1): Let P be the fixed point algebra $A^{(G,\alpha)}$. Then the conclusion is obvious. □

REMARK 8.11. When A is a factor, there always exists a tunnel construction $P\subset A$. Therefore, given any twisted crossed product $A\rtimes_{\alpha,u}G$ by a twisted outer action of a finite group G on a factor A and the canonical conditional expectation F from $A\rtimes_{\alpha,u}G$ onto A, we have $(A\rtimes_{\alpha,u}G, F)\sim(A\rtimes_\beta G, E)$ for some outer action β of G on A and the canonical conditional expectation E from $A\rtimes_\beta G$ onto A. This means that any 2-cocycle is a coboundary, as was observed in [10, 32, 34].

9. Applications

9.1. The case of a finite abelian group. Let G be a finite abelian group and α be an outer action of G on a simple unital C^*-algebra B. Let A be the fixed point algebra $B^{(G,\alpha)}$ and E be the canonical conditional expectation from B onto A defined by $E(x)=(1/n)\sum_{s\in G}\alpha_s(x)$ for all $x\in B$, where n is the order of G. Then it is well known that Index $E=n$ and the C^*-basic construction B_1 induced by the inclusion $A\subset B$ is the crossed product $B\rtimes_\alpha G$, where B is the fixed point algebra by the dual action $\hat{\alpha}$ of the dual group $\hat{G}$ of G on $B\rtimes_\alpha G$. We can see that, in general, B cannot be described as a twisted crossed product $A\rtimes_{\beta,u}\hat{G}$ for any twisted action of the dual group $\hat{G}$ on A. For instance, let θ be an irrational number in $(0,1)$ and A_θ be the corresponding irrational rotation C^*-algebra. Let σ be the involutive automorphism of A_θ determined by $\sigma(u)=u^*$ and $\sigma(v)=v^*$, where u and v are unitary generators in A_θ. Suppose that C_θ is the fixed point algebra $A_\theta^{(\mathbb{Z}/2\mathbb{Z},\sigma)}=\{x\in A_\theta\mid\sigma(x)=x\}$. In [15], it was proved that the inclusion $C_\theta\subset A_\theta$ cannot be described as a twisted crossed product $C_\theta\subset C_\theta\rtimes_{\alpha,u}\mathbb{Z}/2\mathbb{Z}$ for any twisted action (α,u) of $\mathbb{Z}/2\mathbb{Z}$ on C_θ. However, we can obtain the following corollary from Theorem 8.9 immediately.

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COROLLARY 9.1. *Let $\alpha : G \to \mathrm{Aut}(B)$ be an outer action of a finite group G on a simple unital C^*-algebra B and A be the fixed point algebra $B^{(G,\alpha)}$. Define the canonical conditional expectation E from B to A by $E(b) = (1/|G|)\sum_{s\in G}\alpha_s(b)$ for all $b \in B$. If G is a finite abelian group, then $(B, E) \sim (A \rtimes_{\beta,u} \hat{G}, F)$ for some twisted outer action (β, u) of the dual group $\hat{G}$ of G on A if and only if there exist unitary elements $\{u_\gamma\}_{\gamma\in\hat{G}}$ in B such that $\alpha_s(u_\gamma) = \langle\gamma, s\rangle u_\gamma$ for all $s \in G$ and $\gamma \in \hat{G}$, where $\langle\gamma, s\rangle$ is the dual pairing of $\gamma \in \hat{G}$ and $s \in G$.*

PROOF. In the crossed product $B_1 = B \rtimes_\alpha G$, let $\{u_s\}_{s\in G}$ be the canonical unitary elements such that $u_s u_t = u_{st}$ and $u_s b u_s^* = \alpha_s(b)$ for all $b \in B$ and $s, t \in G$. The dual action $\hat{\alpha}$ of the dual group $\hat{G}$ on B_1 is defined by $\hat{\alpha}_\gamma(b) = b$ and $\hat{\alpha}_\gamma(u_s) = \langle\gamma, s\rangle^{-1} u_s$ for $\gamma \in \hat{G}$, $s \in G$ and $b \in B$. Let e_A be the Jones projection for the inclusion $A \subset B$ and let $e_\gamma = \hat{\alpha}_{\gamma^{-1}}(e_A)$ for $\gamma \in \hat{G}$. Since $e_A = (1/n)\sum_{s\in G} u_s$, for $\gamma, \delta \in \hat{G}$, where n is the order of G,

$$
\begin{aligned}
e_\gamma e_\delta &= \frac{1}{n^2}\sum_{s,t\in G}\hat{\alpha}_{\gamma^{-1}}(u_s)\hat{\alpha}_{\delta^{-1}}(u_t) = \frac{1}{n^2}\sum_{s,t\in G}\langle\gamma^{-1}, s\rangle\langle\delta^{-1}, t\rangle u_{st} \\
&= \frac{1}{n^2}\sum_{s,t\in G}\langle\gamma^{-1}, st^{-1}\rangle\langle\delta^{-1}, t\rangle u_s \\
&= \frac{1}{n^2}\sum_{s\in G}\langle\gamma^{-1}, s\rangle\left(\sum_{t\in G}\langle\gamma^{-1}, t^{-1}\rangle\langle\delta^{-1}, t\rangle\right)u_s \\
&= \frac{1}{n^2}\sum_{s\in G}\langle\gamma^{-1}, s\rangle\left(\sum_{t\in G}\langle\gamma\delta^{-1}, t\rangle\right)u_s \\
&= \begin{cases} e_\gamma & \text{if } \delta = \gamma, \\ 0 & \text{if } \delta \neq \gamma. \end{cases}
\end{aligned}
$$

Therefore, $\{e_\gamma\}_{\gamma\in\hat{G}}$ is a family of mutually orthogonal projections in $A' \cap B_1$ and it is obvious that $\sum_{\gamma\in\hat{G}} e_\gamma = 1$, since the dimension of $A' \cap B_1$ is n, the order of $\hat{G}$. Suppose that B can be described by $A \rtimes_{\beta,u} \hat{G}$ for some twisted action (β, u) of the dual group $\hat{G}$ of G on A. Then e_γ is equivalent to e_A by Theorem 8.9. As mentioned in the beginning of Section 8, there is a unitary element u_γ in B such that $u_\gamma e_A u_\gamma^* = e_\gamma = (1/n)\sum_{s\in G}\langle\gamma^{-1}, s\rangle u_s$. On the other hand,

$$
u_\gamma e_A u_\gamma^* = \frac{1}{n}\sum_{s\in G} u_\gamma u_s u_\gamma^* = \frac{1}{n}\sum_{s\in G} u_\gamma \alpha_s(u_\gamma^*) u_s.
$$

So $\alpha_s(u_\gamma) = \langle\gamma, s\rangle u_\gamma$. Conversely, if there exist unitary elements $\{u_\gamma\}_{\gamma\in\hat{G}}$ in B such that $\alpha_s(u_\gamma) = \langle\gamma, s\rangle u_\gamma$ for all $s \in G$, then it is easy to check that $e_\gamma = u_\gamma e_A u_\gamma^*$. Hence, the corollary is proved by Theorem 8.9. □

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EXAMPLE 9.2. Let A_θ be the irrational rotation C^*-algebra generated by two unitary elements u, v satisfying the condition that $uv = e^{2\pi i\theta} vu$. Define an automorphism σ on A_θ by $\sigma(u) = e^{(2\pi/n)i} u$ and $\sigma(v) = v$. Then $\sigma^n = \mathrm{id}$ and hence we can define an action α of $\mathbb{Z}/n\mathbb{Z}$ on A_θ by $\alpha_k = \sigma^k$. It is easy to see that the fixed point algebra $A_\theta^{(\mathbb{Z}/n\mathbb{Z},\alpha)}$ is the irrational rotational C^*-algebra $A_{n\theta}$ generated by u^n and v. Since $\alpha_m(u^k) = e^{(2\pi mk/n)i} u^k$ for $k = 1, 2, \ldots, n-1$, there exists a twisted action (β, w) of $\mathbb{Z}/n\mathbb{Z}$ on $A_{n\theta}$ such that A_θ can be described as the twisted crossed product $A_{n\theta} \rtimes_{\beta,w} \mathbb{Z}/n\mathbb{Z}$ by Corollary 9.1.

9.2. Reduced inclusions by projections. Let $A \subset B$ be an inclusion of simple unital C^*-algebras and E be a conditional expectation from B onto A of index-finite type. Let $\{(v_i, v_i^*)\}_{i=1}^n$ be a quasi-basis for E and p be a projection in A. Since A is a simple unital C^*-algebra, there exist elements $\{y_j\}_{j=1}^m$ in A such that $\sum_{j=1}^m y_j p y_j^* = 1$. Then the set $\{(pv_i y_j p, py_j^* v_i^* p)\}_{1\le i\le n, 1\le j\le m}$ is a quasi-basis for a conditional expectation $E_p = E|_{pBp}$ from pBp onto pAp. Moreover, $(\mathrm{Index}\, E)p = \mathrm{Index}\, E_p$ (see [20, Proposition 4.1]). So the cardinality of the quasi-basis for $pAp \subset pBp$ usually increases more than the cardinality of the quasi-basis for $A \subset B$. We note that if B can be described as a twisted crossed product of some twisted action of some finite group on A, then there is a quasi-basis such that the cardinality of the quasi-basis is the order of the finite group.

THEOREM 9.3. *Let A be a simple unital C^*-algebra and α be an outer action of a finite group G on A. Suppose that B is the crossed product $A \rtimes_\alpha G$ and that p is a projection in A. Then pBp can be described as a twisted crossed product $pAp \rtimes_{\beta,u} G$ by a twisted action (β, u) of G on pAp if $\alpha_s(p)$ is equivalent to p in A for all $s \in G$.*

PROOF. In the crossed product $B = A \rtimes_\alpha G$, let $\{u_s\}_{s\in G}$ be the canonical unitary elements such that $u_s u_t = u_{st}$ and $u_s a u_s^* = \alpha_s(a)$ for all $s, t \in G$ and $a \in A$. Let e_A be the Jones projection for the canonical conditional expectation E from B onto A and B_1 be the C^*-basic construction induced by E. Put $e_s = u_s e_A u_s^*$. Then e_s commutes with any element in A and $A' \cap B_1 = \bigoplus_{s\in G} \mathbb{C} e_s \cong \mathbb{C}^n$, where n is the order of G. Let p be a projection in A. Then it is obvious that $e_A p (= p e_A)$ is the Jones projection for the conditional expectation E_p from pBp onto pAp defined by $E_p(pbp) = pE(b)p$ for $b \in B$ and that pB_1p is the C^*-basic construction for E_p. Suppose that A acts on some Hilbert space $\mathcal{H}$ faithfully and nondegenerately. Then $A' \ni x \mapsto xp \in pA'$ is injective since A is simple. So we can see that $A' \cap B_k$ and $p(A' \cap B_k)p = pA'p \cap pB_kp$ are isomorphic for all $k \in \mathbb{N}$, where B_k is the nth C^*-basic construction induced by $A \subset B$. Therefore, the derived tower for $pAp \subset pBp$ is the same as that for $A \subset B$, and hence $pAp \subset pBp$ is of depth 2 and $pA'p \cap pB_1p$ is commutative. Suppose that, for all $s \in G$, $\alpha_s(p) \sim p$ in A. Then, for all $s \in G$, there is a partial isometry v_s in A such that $v_s v_s^* = p$ and $v_s^* v_s = \alpha_s(p)$. Put $w_s = v_s u_s v_{s^{-1}}^* \in pBp$. Since $v_t e_s = e_s v_t$ and $e_s u_s = u_s e_A$ for $s, t \in G$, we

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have that

$$\begin{aligned} w_s e_A p w_s^* &= v_s u_s v_{s-1}^* e_A p v_{s-1} u_s^* v_s^* = e_s v_s u_s v_{s-1}^* v_{s-1} u_s^* v_s^* \\ &= e_s v_s u_s \alpha_{s-1}(p) u_s^* v_s^* = e_s p. \end{aligned}$$

Therefore, $e_s p$ is equivalent to $e_A p$ in $pB_1 p$. So, by Theorem 8.9, pBp can be described as a twisted crossed product $pAp \rtimes_{\beta,u} G$. □

COROLLARY 9.4. *Let α be an outer action of a finite group G on a simple unital C^*-algebra A. If A is a uniformly hyperfinite algebra or an irrational rotation C^*-algebra, then, for all projection p in A, the set $p(A \rtimes_\alpha G)p$ can be described as a twisted crossed product $pAp \rtimes_{\beta,u} G$ by a twisted action (β, u) of G on pAp.*

PROOF. If A is a uniformly hyperfinite algebra or an irrational rotation C^*-algebra, then A has a unique tracial state τ. By this uniqueness, $\tau \circ \alpha = \tau$ for all automorphisms α of A. So it is obvious that $\alpha_t(p) \sim p$ in A for any projection $p \in A$ and $t \in G$. Hence, the corollary is proved by Theorem 9.3. □

REMARK 9.5. We denote by n the cardinality of a quasi-basis for $A \subset B$; then $\mathrm{tsr}(B) \le \mathrm{tsr}(A) + n - 1$, where $\mathrm{tsr}(C)$ is the topological stable rank of C^*-algebra C [12, Theorem 2.1]. If A is a uniformly hyperfinite algebra or an irrational rotation C^*-algebra, then, for any outer action α of a finite group G on A and any projection p in A, the topological stable rank of $p(A \rtimes_\alpha G)p$ is at most the order of G, since $p(A \rtimes_\alpha G)p$ can be described as a twisted crossed product $pAp \rtimes_{\beta,u} G$ by some twisted action (β, u) of G on pAp by the previous corollary. On the other hand, let $\{e_{ij}\}_{i,j=1}^n$ be matrix units in A such that $q = \sum_{i=1}^n e_{ii}$ is a projection in A. Then, using [31, Theorem 6.1],

$$\begin{aligned} \mathrm{tsr}(A \rtimes_\alpha G) &\le \mathrm{tsr}(q(A \rtimes_\alpha G)q) = \mathrm{tsr}(M_n(e_{11}(A \rtimes_\alpha G)e_{11})) \\ &\le \left\lceil \frac{\mathrm{tsr}(e_{11}(A \rtimes_\alpha G)e_{11}) - 1}{n} \right\rceil + 1 \le \left\lceil \frac{|G| - 1}{n} \right\rceil + 1, \end{aligned}$$

where $\lceil t \rceil$ denotes the least integer that is greater than or equal to t. There exists a sequence of mutually orthogonal equivalent projections $\lceil p_i \rceil_{i=1}^n$ in A such that $n > |G|$. So we can show that $\mathrm{tsr}(A \rtimes_\alpha G) \le 2$. Osaka and the second author of this paper proved this in a more general setting as follows. If A is a simple unital C^*-algebra with the SP property, that is, for which all nonzero hereditary C^*-subalgebras contain a nonzero projection, and such that $\mathrm{tsr}(A) = 1$, and α is an action of a finite group G on A, then $\mathrm{tsr}(A \rtimes_\alpha G) \le 2$ (see [21]).

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