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MULTIPLE SCALAR TRANSPORT

BY
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1. We shall consider scalar transport phenomena in which the mass distribution of a large number of interacting mass particles varies as a result of multiple coalescence mechanisms. Four models will be considered; all the particles are of the same kind in the first three. In the first model an integer n ($n \geq 2$) is given and the coalescence mechanism is such that under suitable conditions n particles of masses $x_1, \ldots, x_n$ combine to form one particle of mass $\sum_1^n x_i$; we refer to this as the $n/1$ transport. In the second model a sequence $n_1, n_2, \ldots$ of integers is given, with $2 \leq n_1 < n_2 < \cdots$, and all the $n_1/1, n_2/1, \ldots$ transports are going on simultaneously. In the third model the coalescence mechanism is such that under suitable conditions n particles of masses $x_1, \ldots, x_n$ combine and also break up so as to give rise to m new particles of total mass $\sum_1^n x_i$; this may be referred to as the n/m transport. Finally, in the fourth model various 2/1 transports will be considered for the case of particles of several kinds.

These models are introduced partly for their own interest and partly in hope that they may be of some use in the physical sciences (especially in particle physics, colloid chemistry, and meteorology) and perhaps also in the biological and social sciences (breakup and formation of social groupings and aggregates, transport of wealth and power, concentration of biological characteristics and genetic transport).

Such applications would almost certainly call for a very considerable improvement in our models which are admittedly primitive. For that reason we do not take up here the mathematically rather sophisticated questions of the existence, uniqueness, and properties of the solutions of the various transport equations we derive. Instead, we just scratch the surface of the subject by formulating some problems and indicating some simple mathematics of the resulting equations.

For the background of the 2/1 transport for a single kind of masses the reader is referred to the recent exhaustive survey [1] which contains an up-to-date bibliography of more than two hundred titles. The types of transport which go beyond the 2/1 case appear to be new.

2. For our first model let $f(x, t)\,dx$ be the average number per unit volume of particles of mass (or volume) in the range x to $x+dx$ present at the time t. Let $n \geq 2$ be a fixed integer and suppose that the density function $f(x, t)$ varies in time due to n-tuple coalescences. That is, under some conditions n particles of masses

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$x_1, \ldots, x_n$ will unite to form a single particle of mass $\sum_1^n x_i$. Specifically, we suppose that

$$N(x_1, \ldots, x_n, t)\,dx_1, \ldots, dx_n\,dt \tag{1}$$

is the average number per unit volume of such multiple coalescences involving masses in the range x_i to x_i+dx_i $(i=1, \ldots, n)$ during the time interval t to $t+dt$. It is further assumed that

$$N(x_1, \ldots, x_n, t) = \varphi(x_1, \ldots, x_n) \prod_1 f(x_i, t) \tag{2}$$

where φ is a sufficiently well behaved nonnegative function, invariant under any permutation of its n arguments. It is also assumed that the particles do not break up and do not enter or leave the system. Under these conditions we have the mass conservation law expressed in the equation

$$\frac{\partial f(x, t)}{\partial t} = I[f(x, t)] - O[f(x, t)] \tag{3}$$

which we call a scalar transport equation; here I and O stand for 'in' and 'out' and I is the rate of formation of new particles of mass x out of smaller particles, while O is the rate of disappearance of masses x due to their coalescence with others. I and O may be expressed by N and (3) becomes then

$$\begin{aligned}\frac{\partial f(x, t)}{\partial t} = \frac{1}{n!} \underset{0 \le x_i, \sum_1^n x_i = x}{\int \cdots \int} N(x_1, \ldots, x_n, t)\,dx_2 \cdots dx_n \\ - \frac{1}{(n-1)!} \int_0^\infty \cdots \int_0^\infty N(x, x_2, \ldots, x_n, t)\,dx_2 \cdots dx_n.\end{aligned} \tag{4}$$

The symmetry reducing factorials must be inserted in (4) to prevent multiple counting of a single event. When finally (2) is used to substitute for N into (4) we obtain the transport equation for f:

$$\begin{aligned}\frac{\partial f(x, t)}{\partial t} = \frac{1}{n!} \int_0^x \int_0^{x-x_n} \cdots \int_0^{x-x_4-\cdots-x_n} \int_0^{x-x_3-\cdots-x_n} \\ \times \varphi\left(x - \sum_2^n x_i, x_2, \ldots, x_n\right) f\left(x - \sum_2^n x_i, t\right) \\ \times \prod_2^n f(x_i, t)\,dx_2 \cdots dx_n - \frac{1}{(n-1)!} f(x, t) \\ \times \int_0^\infty \cdots \int_0^\infty \varphi(x, x_2, \ldots, x_n) \prod_2^n f(x_i, t)\,dx_2 \cdots dx_n.\end{aligned} \tag{5}$$

Let $M(t)$ be the total mass per unit volume at the time t:

$$M(t) = \int_0^\infty x f(x, t)\,dx.$$

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We multiply both sides of (5) by x and integrate with respect to x from 0 to ∞; supposing that the order of integration with respect to x and differentiation with respect to t may be changed, we introduce new variables

$$y_i = x_i (i = 2, \ldots, n), \qquad y_1 = x - \sum_2^n x_i$$

and we find that $M'(t)=0$, that is, the total mass is conserved.

3. In this section the transport equation (5) will be solved for the very simple special case when $\varphi(x_1, \ldots, x_n)=c$ where c is a positive constant. We have then

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} = \frac{c}{n!}\int_0^x\int_0^{x-x_n}\cdots\int_0^{x-x_3-\cdots-x_n} f\left(x-\sum_2^n x_i, t\right)\prod_2^n f(x_i,t)\,dx_2\cdots dx_n \\ -\frac{c}{(n-1)!}f(x,t)\left[\int_0^\infty f(x,t)\,dx\right]^{n-1};\end{aligned} \tag{6}$$

$$f(x,0) = f(x) \tag{7}$$

is the known initial distribution. The multiple convolution integral in (6) suggests the use of Laplace transforms:

$$g(p,t) = \int_0^\infty e^{-px}f(x,t)\,dx, \qquad g(p) = g(p,0) = \int_0^\infty e^{-px}f(x)\,dx. \tag{8}$$

We put also

$$N(t) = \int_0^\infty f(x,t)\,dx, \qquad N(0) = N = \int_0^\infty f(x)\,dx \tag{9}$$

so that $N(t)=g(0,t)$ is the total number per unit volume of particles at the time t and N is the same initially. It will be supposed throughout that N is finite.

By the standard properties of Laplace transform we have the equation for g:

$$\frac{\partial g(p,t)}{\partial t} = \frac{c}{n!}g^n(p,t) - \frac{c}{(n-1)!}g(p,t)g^{n-1}(0,t). \tag{10}$$

Putting $p=0$ in the above we find that

$$\frac{d}{dt}N(t) = -\frac{c(n-1)}{n!}N^n(t)$$

so that

$$N(t) = \left[N^{1-n}+c\frac{(n-1)^2}{n!}t\right]^{1/1-n}.$$

Substituting this for $g(0,t)$ in (10) we have

$$\frac{\partial g(p,t)}{\partial t} = \frac{c}{n!}g^n(p,t) - \frac{c}{(n-1)!}\left[N^{1-n}+c\frac{(n-1)^2}{n!}t\right]^{-1}g(p,t) \tag{11}$$

which is an ordinary differential equation of the Bernoulli type and reduces to a

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linear equation on substituting $u=g^{1-n}$. Thus the solution $g(p, t)$ is obtained as

(12)
$$g(p, t) = \left[\left(N^{1-n}+c\frac{(n-1)^2}{n!}t\right)^{n/(n-1)}(N^n g^{1-n}(p)-N)+N^{1-n}+c\frac{(n-1)^2}{n!}t\right]^{1/(1-n)}$$

and inverting the first Laplace transform in (8) we obtain $f(x, t)$.

When the initial distribution is monodispersed with all particles of unit mass, we have $f(x)=N\,\delta(x-1)$ where $\delta(x)$ is the Dirac δ-function. By (8) $g(p)=Ne^{-p}$ and hence by (12)

$$g(p, t) = K[e^{p(n-1)} - L] \tag{13}$$

where

(14)
$$K = N^{-1/(n-1)}A^{-n/(n-1)^2}, \qquad L = 1-N^{-1}A^{-1/(n-1)}, \qquad A = N^{1-n}+c\frac{(n-1)^2}{n!}t.$$

Therefore, inverting the Laplace transform we have

$$f(x, t) = \frac{K}{2\pi i}\int_C e^{px}[e^{p(n-1)}-L]^{-1/(n-1)}\,dp$$

with a suitable contour C. This may be written as

$$f(x, t) = \frac{K}{2\pi i}\int_C e^{px-p}[1-Le^{-p(n-1)}]^{-1/(n-1)}\,dp$$

and, using the binomial theorem, as

$$f(x, t) = K\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{n-1}}{k}L^k\left[\frac{1}{2\pi i}\int_C e^{px-p-kp(n-1)}\,dp\right].$$

The expression in the brackets is again the Dirac δ-function:

$$\frac{1}{2\pi i}\int_C e^{px-p-kp(n-1)}\,dp = \delta(x-k(n-1)-1)$$

so that finally

$$f(x, t) = K\sum_{k=0}^{\infty}\binom{\frac{1}{n-1}+k-1}{k}L^k\,\delta[x-(k(n-1)+1)]. \tag{15}$$

It follows that if we start with unit masses and allow only n-tuple coalescences, we shall have later on only particles of mass $1, n, 2n-1, 3n-1, \ldots$; if $N_j(t)$ is the number of $(1+(n-1)j)$-tuples per unit volume at the time, then

$$N_j(t) = \binom{\frac{1}{n-1}+j-1}{j}KL^j$$

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where K and L are given by (14). It follows that $N_0(t)$ decreases steadily while $N_j(t)$ for $j>0$ rises from its initial value 0 at $t=0$ to a maximum M_j at some time t_j and then decreases to 0 asymptotically. The calculation of M_j and t_j is easy.

4. We consider next the transport equation (5) for the case of an unbounded kernel $\varphi(x_1, \ldots, x_n)=c\sum_1^n x_i$ (with $c>0$). We use again the Laplace transforms (8). Recalling that the total mass per unit volume

$$M = \int_0^\infty xf(x, t)\,dx$$

is constant, we find that

$$\frac{\partial g(p, t)}{\partial t} = \frac{c}{(n-1)!}\,\frac{\partial g(p, t)}{\partial p}\,[g^{n-1}(0, t)-g^{n-1}(p, t)]-\frac{cM}{(n-2)!}\,g(p, t)g^{n-2}(0, t). \tag{16}$$

With $p=0$ this yields for $g(0, t)=N(t)$ the equation

$$N'(t) = -\frac{cM}{(n-2)!}\,N^{n-1}(t) \tag{17}$$

so that

$$\begin{aligned} N(t) &= Ne^{-cMt} & n &= 2, \\ N(t) &= \left[\frac{cM}{(n-3)!}\,t+N^{2-n}\right] & n &\geq 3. \end{aligned} \tag{18}$$

We are interested here in the case $n\geq 3$; when $N(t)$ is substituted from (18) into (16) the resulting partial differential equation does not appear to be explicitly solvable in a closed form. However, it is possible to handle the moments of f: let

$$M_k(t) = \int_0^\infty x^k f(x, t)\,dx$$

so that $M_0(t)=N(t)$ and $M_1(t)=M$ is the constant total mass per unit volume. In terms of the Laplace transforms we have

$$M_k(t) = (-1)^k\,\frac{\partial^k g(p, t)}{\partial p^k}\bigg|_{p=0}.$$

If (16) is differentiated twice with respect to p and then we set $p=0$, we obtain

$$M_2'(t) = \frac{3cM}{(n-2)!}\,M_2(t)T^{-1}+\frac{cM^3}{(n-3)!}\,T^{(n-3)/(2-n)}-\frac{cM}{(n-2)!}\,T^{(n-1)/(2-n)} \tag{19}$$

where for brevity

$$T = \frac{cM}{(n-3)!}\,t+N^{2-n}.$$

(19) may be solved for $M_2(t)$ and we find that

$$M_2(t) = (n-2)M^2T^{4/(n-2)}+KT^{3/(n-2)}+T^{2/(n-2)}$$

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where

$$K = N^3M_2(0) - (n-2)M^2N^{-1} - N.$$

Similarly, the differential equations for higher moments are obtained by differentiating (16) with respect to p a number of times, and putting $p=0$.

5. Here we consider the case of the unbounded kernel $\varphi(x_1, \ldots, x_n) = cx_1 \cdots x_n$ with positive constant c. The use of Laplace transforms (8) leads now to

$$\frac{\partial g(p,t)}{\partial t} = (-1)^n \frac{c}{n!}\left[\frac{\partial g(p,t)}{\partial p}\right]^n + \frac{cM^{n-1}}{(n-1)!}\frac{\partial g(p,t)}{\partial p}. \tag{20}$$

Since this is a partial differential equation without either the dependent variable g or the independent variables p and t, we use the Lagrange-Charpit method of elimination, [2], to solve it. Write (20) as $g_t = F(g_p)$ so that $F(x) = Ax^n + Bx$, where $A = (-1)^n c/n!$ and $B = cM^{n-1}/(n-1)!$. Then $g(p,t)$ is obtained by eliminating $\psi(p,t)$ out of the system

$$\begin{aligned} g(p,t) &= p\psi(p,t) + tF[\psi(p,t)] + \phi[\psi(p,t)] \\ 0 &= p + tF'[\psi(p,t)] + \phi'[\psi(p,t)]. \end{aligned} \tag{21}$$

Here ϕ is a suitable function which ensures the correct initial condition. The initial value $g(p) = g(p, 0)$ is known from (8); put $t=0$ in (21) and let $\psi(p) = \psi(p, 0)$. Then

$$g(p) = p\psi(p) + \phi[\psi(p)], \qquad 0 = p + \phi'[\psi(p)].$$

Differentiating the first equation with respect to p we have $\psi(p) = g'(p)$ so that

$$\phi(g'(p)) = g(p) - pg'(p)$$

which gives us ϕ. Now the second equation in (21) is solved for $\psi(p,t)$ and this is substituted into the first equation, giving us $g(p,t)$. Finally f is obtained as the inverse Laplace transform.

Suppose, for instance, that we have initially monodispersed particles with $f(x,0) = N\delta(x-1)$. Then $g(p) = Ne^{-p}$ and $g'(p) = -Ne^{-p}$; hence $\phi'(u) = \log(-u/N)$ and $\psi(p,t)$ is therefore a solution of

$$p + t[nA\psi^{n-1}(p,t) + B] + \log[-\psi(p,t)/N] = 0.$$

As in the previous section we can use (21) after differentiating it and putting $p=0$, to compute the moments. In particular, the 0-th moment

$$N(t) = \int_0^\infty f(x,t)\,dx = g(0,t)$$

is obtained as

$$N(t) = (1-n)At\psi^n(t) - \psi(t)$$

where

$$t[nA\psi^{n-1}(t) + B] + \log[-\psi(t)/N] = 0.$$

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6. All the examples so far are special cases of $n/1$ transport. We consider now the case of $(n_1+n_2+\cdots)/1$ transport. That is, there is a sequence $n_1, n_2, \ldots$ of integers, with $2\leq n_1<n_2<\cdots$, and the n_1-tuple, n_2-tuple, ... coalescence mechanisms are all going on simultaneously. Rewriting (3) as

$$\frac{\partial f(x,t)}{\partial t}=I_n[f(x,t)]-O_n[f(x,t)]$$

we have for our transport equation

$$\frac{\partial f(x,t)}{\partial t}=\sum_{\kappa}\{I_{nk}[f(x,t)]-O_{n_k}[f(x,t)]\}. \tag{22}$$

Suppose again that all the kernels are constant: $\varphi_{n_k}(x_1,\ldots,x_{n_k})=c_{n_k}$, and introduce the transport function

$$F(z)=\sum_2^\infty \frac{c_n z^n}{n!}$$

where $c_n=c_{n_k}$ if n is in the sequence $n_1, n_2, \ldots$ and $c_n=0$ otherwise. Then, taking Laplace transforms (8) we obtain

$$\frac{\partial g(p,t)}{\partial t}=F[g(p,t)]-g(p,t)F'[g(p,t)]. \tag{23}$$

In particular, for the 0-th moment $N(t)=g(0,t)$ we have the differential equation

$$N'(t)=F[N(t)]-N(t)F'[N(t)]. \tag{24}$$

Differentiating (23) successively with respect to p and putting $p=0$ we obtain the relations for higher moments:

$$M_2'(t)=F''[N(t)]M^2,\qquad M_3'=-M^3F'''[N(t)]-3MF''[N(t)]M_2(t)$$

and so on, so that these moments may be evaluated once $N(t)$ is known.

As an example we take the $(2+3)/1$ transport with the transport function

$$F(z)=c_2z^2/2+c_3z^3/6;$$

then (24) is

$$N'(t)=-\frac{c_2}{2}N^2(t)-\frac{c_3}{3}N^3(t)$$

so that $N(t)$ is determined by the relation

$$t=\frac{2}{c_2}\left[\frac{1}{N(t)}-\frac{1}{N}\right]+\frac{4c_3}{3c_2^2}\log\frac{N(t)[3c_2+2c_3N(t)]}{N(3c_2+2c_3N)}.$$

It follows that for large t we have asymptotically

$$N(t)\sim 2/(c_2t);$$

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thus the number $N(t)$ of particles per unit volume depends for large t essentially only on the $2/1$ mechanism constant c_2.

7. For the case of $2/2$ transport, with all particles of one kind, we have to consider a coalescence mechanism in which masses x and y undergo a sequence of breakups and recombinations so as to end up with two masses again, say u and v. Conservation of mass gives us $x+y=u+v$; hence, supposing linear dependence of u and v on x and y, we have

$$u = ax+by, \qquad v = (1-a)x+(1-b)y, \qquad 0 \le a, b \le 1.$$

Further, since all particles are of one kind the masses x and y are indistinguishable and therefore $a=b$. Hence

$$u = a(x+y), \qquad v = (1-a)(x+y)$$

and on the grounds of symmetry we may assume without loss of generality that $0 \le a \le 1/2$.

Thus the $2/2$ transport depends on the coalescence mechanism in which a fixed number a is given, with $0 \le a \le 1/2$, and when two masses x and y coalesce the result is again two particles, of mass $a(x+y)$ and $(1-a)(x+y)$. The limiting case $a=0$ is the $2/1$ transport: x and y coalesce to form a single mass $x+y$; the other limiting case $a=1/2$ is the averaging transport: x and y coalesce to form two particles of the mean mass $(x+y)/2$.

To obtain the $2/2$ transport equation we observe that a particle of mass x can be created in two possible ways: either as a result of a coalescence of masses u and v with $u+v=x/a$, or as a result of similar coalescence but with $u+v=x/(1-a)$. However, a particle of mass x can disappear in only one way: by coalescing with another particle. Keeping the meaning of $f(x, t)$ and $\varphi(x, y)$ the same as before and recalling the definition (1), we obtain as the $2/2$ transport equation

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} &= \frac{1}{2a}\int_0^{x/a} f(y,t) f\left(\frac{x}{a}-y, t\right)\varphi\left(y, \frac{x}{a}-y\right) dy \\ &\quad + \frac{1}{2(1-a)}\int_0^{x} f(y,t) f\left(\frac{x}{1-a}-y, t\right) \\ &\quad \times \varphi\left(y, \frac{x}{1-a}-y\right) dy - f(x,t)\int_0^\infty f(y,t)\varphi(x,y)\, dy.\end{aligned} \tag{25}$$

In the limiting case $a \to 0$ this becomes the $2/1$ transport equation provided that we interpret

$$\lim_{a\to 0} f\left(\frac{x}{a}-y, t\right)\varphi\left(y, \frac{x}{a}-y\right)\Big/ a \tag{26}$$

as 0. It will be noticed that in this transport not only the total mass per unit volume, $\int_0^\infty xf(x, t)\, dx$, but also the total number of particles per unit volume, $\int_0^\infty f(x, t)\, dx$, remains constant; simple computation with (25) shows that this is so.

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For the n/m transport we have the coalescence mechanism in which n particles of masses $x_1, \ldots, x_n$ meet, break up and coalesce, resulting in m particles of masses $y_1, \ldots, y_m$. Assuming as before mass conservation and the linearity of y's in terms of x's, we suppose that there are m fixed constants $a_1, \ldots, a_m$ satisfying

$$0 \leq a_j, \qquad \sum_1^m a_j = 1 \tag{27}$$

and we have then

$$y_j = a_j \sum_1^n x_i, \qquad j = 1, \ldots, m. \tag{28}$$

To obtain the transport equation for this case let us write (5) as

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{n!} I(x) - \frac{1}{(n-1)!} O(x),$$

then the present transport equation is

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{n!} \sum_1^m a_j^{-1} I(x/a_j) - \frac{1}{(n-1)!} O(x). \tag{29}$$

8. Finally, we consider coalescence among particles of several kinds. Let there be k kinds of particles, with the corresponding density functions $f_i(x, t)$ $(i=1, \ldots, k)$. The transport will be of the 2/1 type (higher transports might also be considered but at the cost of considerably greater complexity in formulas). Let $F(i, j)$ be an integer-valued function defined for $1 \leq i, j \leq k$ and satisfying the condition $1 \leq F(i, j) \leq k$; we introduce the coalescence symbols ε_{ij}^s for $1 \leq i, j, s \leq k$ by

$$\begin{aligned} \varepsilon_{ij}^s &= 1 \quad \text{if } F(i, j) = s \\ \varepsilon_{ij}^s &= 0 \quad \text{if } F(i, j) \neq s; \end{aligned}$$

they are to have the symmetry property $\varepsilon_{ij}^s = \varepsilon_{ji}^s$. We postulate the coalescence mechanism under which a particle of mass x and the i-th kind and a particle of mass y and the j-th kind will give rise, on coalescing, to a particle of mass $x+y$ and the s-th kind, if and only if $\varepsilon_{ij}^s = 1$, otherwise there is no coalescence. In place of the kernel $\varphi(x, y)$ we have now $k(k+1)/2$ kernels $\varphi_{ij}(x, y)$ with the symmetry properties

$$\varphi_{ij}(x, y) = \varphi_{ji}(x, y) = \varphi_{ij}(y, x)$$

though actually $\varphi_{ij}(x, y)$ needs to be given only for the case when s exists such that $\varepsilon_{ij}^s = 1$.

Proceeding as before we obtain the transport equations as

$$\begin{aligned} \frac{\partial f_s(x, t)}{\partial t} &= \tfrac{1}{2} \sum_{i,j} \varepsilon_{ij}^s \int_0^x f_i(y, t) f_j(x-y, t) \varphi_{ij}(y, x-y)\, dy \\ &\quad - \sum_{i,j} \varepsilon_{is}^j f_s(x, t) \int_0^\infty f_i(y, t) \varphi_{is}(x, y)\, dy, \qquad s = 1, \ldots, k. \end{aligned} \tag{30}$$

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Let

$$M_s(t) = \int_0^\infty x f_s(x, t)\, dx$$

be the total mass per unit volume of the particles of the s-th kind at the time t. Multiplying (30) by x, integrating with respect to x from 0 to ∞, and interchanging the order of integration and differentiation, we obtain, on introducing new variables y and $x-y$,

$$\begin{aligned} M_s'(t) = \sum_{i,j} \varepsilon_{ij}^s \int_0^\infty \int_0^\infty x f_i(y, t) f_j(x, t) \varphi_{ij}(x, y)\, dy\, dx \\ -\sum_{i,j} \varepsilon_{is}^j \int_0^\infty \int_0^\infty x f_s(x, t) f_i(y, t) \varphi_{is}(x, y)\, dy\, dx \end{aligned} \tag{31}$$

from which there results the overall mass conservation

$$\sum_1^k M_s'(t) = 0.$$

On the other hand, each quantity $M_s(t)$ separately is in general not constant and the behaviour of the quantities $M_s(t)$ is here of some interest.

As an example, we consider the case of two kinds of particles with constant positive kernels

$$\varphi_{11}(x, y) = c_{11}, \qquad \varphi_{12}(x, y) = c_{12}, \qquad \varphi_{22}(x, y) = c_{22}$$

where the first kind dominates the second kind in the sense that

$$\varepsilon_{11}^1 - \varepsilon_{22}^2 = \varepsilon_{12}^1 = 1$$

and all other coalescence symbols are 0. We let $N_s(t)$ be the total numbers per unit volume:

$$N_s(t) = \int_0^\infty f_s(x, t)\, dx, \qquad s = 1, 2.$$

The transport equations (30) are

$$\begin{aligned} \frac{\partial f_1(x, t)}{\partial t} &= \frac{c_{11}}{2} \int_0^x f_1(y, t) f_1(x-y, t)\, dy + c_{12} \int_0^x f_1(y, t) f_2(x-y, t)\, dy \\ &\quad - c_{11} f_1(x, t) \int_0^\infty f_1(y, t)\, dy - c_{12} f_1(x, t) \int_0^\infty f_2(y, t)\, dy, \\ \frac{\partial f_2(x, t)}{\partial t} &= \frac{c_{22}}{2} \int_0^x f_2(y, t) f_2(x-y, t)\, dy \\ &\quad - c_{22} f_2(x, t) \int_0^\infty f_2(y, t)\, dy - c_{12} f_2(x, t) \int_0^\infty f_1(y, t)\, dy. \end{aligned} \tag{32}$$

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Introducing the Laplace transforms

$$g_s(p,t)=\int_0^\infty e^{-px}f_s(x,t)\,dx, \qquad s=1,2,$$

we have $N_s(t)=g_s(0,t)$ and (32) become

(33)
$$\frac{\partial g_1(p,t)}{\partial t}=\frac{c_{11}}{2}g_1^2(p,t)+c_{12}g_1(p,t)g_2(p,t)-c_{11}g_1(p,t)g_1(0,t)-c_{12}g_1(p,t)g_2(0,t),$$

$$\frac{\partial g_2(p,t)}{\partial t}=\frac{c_{22}}{2}g_2^2(p,t)-c_{22}g_2(p,t)g_2(0,t)-c_{12}g_2(p,t)g_1(0,t).$$

Letting $p=0$ we have

$$N_1'(t)=-\frac{c_{11}}{2}N_1^2(t), \qquad N_2'(t)=-\frac{c_{22}}{2}N_2^2(t)-c_{12}N_1(t)N_2(t). \tag{34}$$

If $N_1=N_1(0)$ and $N_2=N_2(0)$ are the known initial values then

$$N_1(t)=\frac{2N_1}{N_1c_{11}t+2}$$

is the solution of the first equation in (34). When the above is substituted for $N_1(t)$ into the second equation of (34) we have a Bernoulli type differential equation for $N_2(t)$ and there are two cases to consider depending on whether $2c_{12}=c_{11}$ or $2c_{12}\neq c_{11}$. The solution $N_2(t)$ is

$$N_2(t)=\left(\frac{N_1c_{11}}{2}t+1\right)^{-1}\left[N_2^{-1}+\frac{c_{22}}{N_1c_{11}}\log\left(\frac{N_1c_{11}}{2}t+1\right)\right]^{-1} \quad \text{if } 2c_{12}=c_{11}$$

$$N_2(t)=\left[\frac{N_1(c_{11}-2c_{12})-c_{22}N_2}{N_1N_2(c_{11}-2c_{12})}\left(\frac{N_1c_{11}}{2}t+1\right)^{2c_{12}/c_{11}}\right.$$
$$\left.+\frac{c_{22}}{N_1(c_1-2c_{12})}\left(\frac{N_1c_{11}}{2}t+1\right)\right]^{-1} \quad \text{if } 2c_{12}\neq c_{11}.$$

The mass transport equations (31) are here

$$M_1'(t)=c_{12}M_2(t)N_1(t), \qquad M_2'(t)=-c_{12}M_2(t)N\ (t);$$

if $M_1=M_1(0)$ and $M_2=M_2(0)$ are the known initial values then we have

$$M_1(t)=M_1+M_2\left[1-\left(\frac{N_1c_{11}}{2}t+1\right)^{-2c_{12}/c_{11}}\right]$$

$$M_2(t)=M_2\left(\frac{N_1c_{11}}{2}t+1\right)^{-2c_{12}/c_{11}}.$$

It is also possible here to solve the transport equations (33) themselves but the solutions involve non-elementary integrals of the binomial type.

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