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RINGS WITH ENOUGH INVERTIBLE IDEALS

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All rings are associative with identity element 1 and all modules are unital. A ring has enough invertible ideals if every ideal containing a regular element contains an invertible ideal. Lenagan [**8**, Theorem 3.3] has shown that right bounded hereditary Noetherian prime rings have enough invertible ideals. The proof is quite ingenious and involves the theory of cycles developed by Eisenbud and Robson in [**5**] and a theorem which shows that any ring S such that $R \subseteq S \subseteq Q$ satisfies the right restricted minimum condition, where Q is the classical quotient ring of R. In Section 1 we give an elementary proof of Lenagan's theorem based on another result of Eisenbud and Robson, namely every ideal of a hereditary Noetherian prime ring can be expressed as the product of an invertible ideal and an eventually idempotent ideal (see [**5**, Theorem 4.2]). We also take the opportunity to weaken the conditions on the ring R.

Section 2 is concerned with showing that if R is a prime Noetherian ring with enough invertible ideals then any locally Artinian R-module M is the direct sum of a completely faithful submodule C and a submodule U such that each element of U is annihilated by a non-zero ideal of R. This result generalises [**4**, Theorem 3.9].

1. Lenagan's theorem. Let R be a ring. An element c of R is *regular* if both $rc \neq 0$ and $cr \neq 0$ for every non-zero element r of R. Suppose that R is an *order* in a ring Q; that is, R is a subring of Q, each regular element of R is invertible in Q and each element of Q has the forms rc^{-1} and $d^{-1}s$ where $r, s, c, d, \in R$ and both c and d are regular. An ideal I of R will be called *invertible* provided there exists a sub-bimodule X of ${}_RQ_R$ such that $XI = IX = R$ and in this case we write I^{-1} for X. Note that if I is invertible then $1 \in II^{-1}$ implies

$$1 = \sum_{i=1}^{n} a_i r_i c_i^{-1}$$

for some positive integer n, $a_i \in I$, $r_i, c_i \in R$ with c_i regular $(1 \leqq i \leqq n)$. By [**6**, Lemma 4.2] it follows that I contains a regular element. We call an ideal I *integral* if it contains a regular element.

Throughout this section we shall suppose that R is an order in Q. If I is an integral ideal of R define

$$I^* = \{q \in Q : qI \leqq R\}.$$

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Suppose further that I is a projective right R-module. By the Dual Basis Lemma there exist an index set Λ, elements $a_\lambda \in I$ and R-homomorphisms $f_\lambda \in \operatorname{Hom}(I, R)$ $(\lambda \in \Lambda)$ such that

$$a = \sum a_\lambda f_\lambda(a) \quad (a \in I)$$

and for each a in I, $f_\lambda(a) = 0$ for all but possibly a finite collection of elements $\lambda \in \Lambda$. Since $IQ = Q$ it follows that for each λ in Λ f_λ can be lifted to an endomorphism of Q and hence there exists $q_\lambda \in I^*$ such that $f_\lambda(a) = q_\lambda a$ $(a \in I)$. In particular, if $c \in I$ and c is regular

$$c = \sum_{i=1}^{m} a_i q_i c$$

for some positive integer m, $a_i \in I$, $q_i \in I^*$ $(1 \leqq i \leqq m)$. Then

$$1 = \sum_{i=1}^{m} a_i q_i$$

and so

$$R \leqq II^* \quad \text{and} \quad I = \sum_{i=1}^{n} a_i R.$$

Moreover, $I = II^*I$ implies that II^* is an idempotent ideal of R. Note that $R \leqq I^*$ and hence $I \leqq II^*$. Conversely, if $R \leqq II^*$ then

$$1 = \sum_{i=1}^{m} a_i q_i$$

for some positive integer m and $a_i \in I$, $q_i \in I^*$ $(1 \leqq i \leqq m)$. Then

$$a = \sum_{i=1}^{m} a_i (q_i a) \quad (a \in I)$$

and I is a projective right R-module by the Dual Basis Lemma. We have proved:

LEMMA 1.1. *Let I be an integral ideal of R. Then I is a projective right R-module if and only if $R \leqq II^*$. In this case I is a finitely generated right ideal and I^*I is an idempotent ideal containing I.*

In particular Lemma 1.1 shows that invertible ideals are projective as right and left modules. Note also that if M is a maximal ideal of R then $M \leqq M^*M \leqq R$. Thus $M = M^*M$ or $M^*M = R$. It follows that if M is integral and projective as a right and left module then M is invertible or idempotent by the lemma. We mention one other consequence of Lemma 1.1 here. If I is an integral ideal of R and there exist ideals $A_1, \ldots, A_n$ such that $I = A_1 \ldots A_n$ and A_i is a projective right R-

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module ($1 \leqq i \leqq n$) then I is a projective right R-module. For

$$A_n^* \dots A_1^* I = A_n^* \dots (A_1^* A_1) \dots A_n \leqq A_n^* \dots (A_2^* A_2) \dots A_n \leqq R$$

which implies $A_n^* \dots A_1^* \leqq I^*$. Moreover

$$R \leqq A_1 A_1^* = A_1 R A_1^* \leqq A_1(A_2 A_2^*) A_1^* \leqq I(A_n^* \dots A_1^*) \leqq II^*.$$

By Lemma 1.1 I is a projective right R-module.

LEMMA 1.2. *Let R be a ring such that the integral prime ideals are finitely generated as right ideals. Let I be an integral ideal of R. Then there exists a finite collection of prime ideals P_i containing I ($1 \leqq i \leqq n$) such that $P_1 \dots P_n \leqq I$.*

Proof. Suppose not and let $\{I_\lambda : \lambda \in \Lambda\}$, Λ some index set, be a chain of integral ideals for each of which the result fails. Let I be the integral ideal $\bigcup_\Lambda I_\Lambda$. If

$$P_1 \dots P_n \leqq I \leqq \bigcap_{i=1}^{n} P_i$$

with P_i prime ($1 \leqq i \leqq n$) then $P_1 \dots P_n$ is a finitely generated right ideal and hence $P_1 \dots P_n \leqq I_\lambda$ for some λ in Λ, a contradiction. Thus Zorn's Lemma can be applied to give an ideal J maximal with respect to the property that there does not exist a finite collection of prime ideals P_i ($1 \leqq i \leqq n$) with

$$P_1 \dots P_n \leqq J \leqq \bigcap_{i=1}^{n} P_i.$$

Clearly J is not prime. It follows that there exist ideals A and B properly containing J such that $AB \leqq J$. By the choice of J there exist prime ideals Q_i ($1 \leqq i \leqq n$) such that

$$Q_1 \dots Q_k \leqq A \leqq \bigcap_{i=1}^{k} Q_i \quad \text{and} \quad Q_{k+1} \dots Q_m \leqq B \leqq \bigcap_{i=k+1}^{m} Q_i$$

for some $1 \leqq k < m$. Then

$$Q_1 \dots Q_m \leqq AB \leqq J \leqq A \cap B \leqq \bigcap_{i=1}^{m} Q_i,$$

a contradiction. The result follows.

COROLLARY 1.3. *Let R be a ring such that the integral prime ideals are finitely generated as right ideals. Then R satisfies the ascending chain condition on integral semiprime ideals.*

Proof. Let $X_1 \leqq X_2 \leqq \dots$ be an ascending chain of integral semiprime ideals of R and let X be the ideal $\bigcup_{i \geqq 1} X_i$. By the lemma there exists a

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finite collection of prime ideals P_i containing X $(1 \leqq i \leqq n)$ such that $P_1 \dots P_n \leqq X$. Since each P_i is a finitely generated right ideal it follows that $P_1 \dots P_n$ is a finitely generated right ideal and hence $P_1 \dots P_n \leqq X_m$ for some positive integer m. Hence $X^n \leqq P_1 \dots P_n \leqq X_m$ and $X \leqq X_m$ because X_m is semiprime. Thus $X_m = X_{m+1} = \dots$.

We next generalize [**5**, Theorem 4.2]. The proof is rather similar in parts but is included for completeness. An ideal I is called *eventually idempotent* if $I^k = I^{k+1}$ for some positive integer k.

THEOREM 1.4. *Let R be an order in a ring Q. Let I be an integral ideal of R such that the prime ideals containing I are invertible or maximal and projective as right and left modules. Then there exists an invertible ideal A and an eventually idempotent ideal B such that $I = AB$.*

Proof. By Lemma 1.1 any prime ideal containing I is a finitely generated right ideal. Thus by Corollary 1.3 R/I satisfies the ascending chain condition on semiprime ideals and there exists a finite collection of prime ideals P_i $(1 \leqq i \leqq n)$ such that $P_i \not\subseteq P_j$ $(i \neq j)$, $I \subseteq P_i$ $(1 \leqq i \leqq n)$ and $N^k \subseteq I$ for some positive integer k where $N = \bigcap_{i=1}^n P_i$ (Lemma 1.2). Clearly N is a semiprime ideal. Suppose the result is false for I and I is chosen so that N is as large as possible.

Suppose first that the intersection of any collection of the ideals P_i is not invertible. In particular this means that each ideal P_i is maximal $(1 \leqq i \leqq n)$. By the Chinese Remainder Theorem

$$R/N \cong (R/P_1) \oplus \dots \oplus (R/P_n).$$

Since P_i is a projective right R-module it follows that the right R-module R/P_i has projective dimension at most 1 $(1 \leqq i \leqq n)$ and hence the right R-module R/N has projective dimension at most 1. By Schanuel's Lemma N is a projective right R-module. Similarly N is a projective left R-module. By assumption N is not invertible. Suppose $N^*N \neq R$. If $N = N^*N$ then N is idempotent (Lemma 1.1) and hence $I = N$. Suppose $N < N^*N$. Again using the Chinese Remainder Theorem, if $X = N^*N$ then there exists an ideal Y such that $R = X + Y$ and $X \cap Y = N$. Moreover $N = NX$ and hence

$$XY \leqq X \cap Y = N = NX \leqq YX \leqq X \cap Y = N$$

so that $N = YX$ and $XY \leqq YX$. Since $N < Y < R$ it follows that Y is the intersection of a proper subset of the P_i $(1 \leqq i \leqq n)$ and, by the choice of I, $Y = AB$ where A is invertible and B eventually idempotent. Since $N < A$ and the intersection of any collection of the ideals P_i is not invertible we have $A = R$ and hence Y is eventually idempotent, say $Y^m = Y^{m+1}$. Then

$$N^m \geqq N^{m+1} = (YX)^{m+1} \geqq Y^{m+1}X^{m+1} = Y^mX \geqq (YX)^m = N^m,$$

giving $N^m = N^{m+1}$. Since $N^k \leqq I$ it follows that I is eventually idempotent.

Now suppose that $P_1 \cap \ldots \cap P_t$ is invertible where $1 \leqq t \leqq n$ and no intersection of $t + 1$ of the ideals P_i $(1 \leqq i \leqq n)$ is invertible. Let

$$C = P_1 \cap \ldots \cap P_t.$$

If D is the intersection of any collection of the ideals P_i $(t + 1 \leqq i \leqq n)$ then $C \cap D = CV$ where V is the ideal $C^{-1}(C \cap D)$. Then $CV \leqq D$ and $C \nleqq P_i$ $(t + 1 \leqq i \leqq n)$ together imply $V \leqq D$. Thus $C \cap D = CD$ and similarly $C \cap D = DC$. This shows in particular that for all $t + 1 \leqq i \leqq n$, P_i is not invertible and hence is maximal. Define

$$G = \bigcap_{i=t+1}^{n} P_i \quad \text{if } t < n$$

and $G = R$ if $t = n$. Then

$$N = CG = GC \text{ and } C + G = R.$$

It follows that $C^kG^k \leqq I$. Suppose $I \leqq C^{k+1}$. Then $C + G^k = R$ implies

$$C^k = C^{k+1} + C^kG^k \leqq C^{k+1}$$

and $C = R$, a contradiction. There exists a positive integer $s \leqq k$ such that $I \leqq C^s$, $I \nleqq C^{s+1}$. Consider the ideal $C^{-s}I$. Clearly

$$I \leqq C^{-s}I \text{ and } C^{k-s}G^k \leqq C^{-s}I.$$

If $C^{-s}I = R$ then $I = C^s$ and I is invertible. Otherwise there exist a positive integer v and prime ideals Q_i $(1 \leqq i \leqq v)$ such that if $N_1 = \bigcap_{i=1}^{v} Q_i$ then $C^{-s}I \leqq N_1$ and $N_1^q \leqq C^{-s}I$ for some $q \geqq 1$. Since $C^{k-s}G^k \leqq C^{-s}I$ it follows that $N \leqq N_1$. If $N = N_1$ then $C^{-s}I \leqq N \leqq C$ and hence $I \leqq C^{s+1}$, a contradiction. Thus, $N < N_1$ and by the choice of I, $C^{-s}I = EF$ for some invertible ideal E and eventually idempotent ideal F. Thus $I = (C^sE)F$ and C^sE is invertible, a contradiction.

We shall not require Theorem 1.4 in full in the sequel but only the following result which generalizes [**5**, Lemma 6.2] and which is proved in the course of proving Theorem 1.4.

COROLLARY 1.5. *Let I be an integral ideal of a ring R such that the prime ideals containing I are invertible or maximal and projective as right and left modules. Then there exists an invertible ideal A and an integral idempotent ideal B such that $AB = BA \leqq I$ and $A + B = R$.*

Note too that the proof of Theorem 1.4 shows that if R is a ring such that the integral prime ideals are invertible or maximal and projective as right and left modules and if R has the further property that integral maximal ideals commute then every integral ideal of R is projective as a

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right and left module. For in this situation any integral ideal $J = AI$ where A is an invertible ideal and I an idempotent ideal. There exists a semiprime ideal N such that $I \leqq N$ and $N^k \leqq I$ for some positive integer k. Moreover, $N = B \cap C = BC = CB$ where B is invertible and C a finite intersection of idempotent maximal ideals. As before C is a projective right R-module. Moreover, C is idempotent. Thus I idempotent implies

$$I = I^k \leqq N^k = (BC)^k = B^kC \leqq I$$

and hence $I = B^kC$. Thus $J = DC$ where $D = AB^k$ is invertible. Then

$$J^* = C^*D^{-1}$$

and

$$R = DRD^{-1} \leqq D(CC^*)D^{-1} = JJ^*$$

and it follows that J is a projective right R-module (Lemma 1.1). Similarly J is a projective left R-module.

A ring R will be called *right truncated* if for every element a in R the descending chain

$$aR \geqq a^2R \geqq a^3R \geqq \dots$$

terminates. Left perfect rings have descending chain condition on principal right ideals (see for example [**2**, p. 315. Theorem 28.4]) and hence are right truncated. On the other hand let K be a field of characteristic $p > 0$, G the Prüfer group of type p^∞ and R the group algebra KG. Then R is a commutative ring and its augmentation ideal A is the unique maximal ideal. The ideal A is nil and hence R is truncated. However R is not perfect for if G is generated by the elements $\{x_i : i \geqq 1\}$ where $x_1^p = 1$, $x_{i+1}^p = x_i$ $(i \geqq 1)$ then

$$(x_1 - 1)R > (x_1 - 1)(x_2 - 1)R$$
$$> (x_1 - 1)(x_2 - 1)(x_3 - 1)R > \dots .$$

This is so because

$$(x_1 - 1) \dots (x_n - 1)\{1 - (x_{n+1} - 1)r\} = 0$$

for some $n \geqq 1$ and r in R implies $(x_1 - 1) \dots (x_n - 1) = 0$ since $(x_{n+1} - 1)r \in A$ and so is nilpotent. If $(x_1 - 1) \dots (x_n - 1) = 0$ then

$$(x_n^{p^{n-1}} - 1)(x_n^{p^{n-2}} - 1) \dots (x_n - 1) = 0$$

and hence

$$1 + p + \dots + p^{n-1} \geqq p^n,$$

a contradiction.

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A ring R is *right bounded* provided every essential right ideal contains an integral ideal. Note that if R is an order in a ring Q then R satisfies the right Ore condition with respect to the regular elements of R and hence cR is an essential right ideal for any regular element c of R.

THEOREM 1.6. *Let R be an order in a ring Q such that every integral prime ideal is invertible or maximal and projective as a right and left R-module. Suppose further that R is right bounded and R/I is right truncated for every integral idempotent ideal I. Then R has enough invertible ideals.*

Proof. Let A be an integral ideal of R. Let c be a regular element in A. Let B be an integral ideal contained in cR. By Corollary 1.5 there exists an invertible ideal U and an integral idempotent ideal I such that $UI = IU \leqq B$. Consider the descending chain

$$cR + I \geqq c^2R + I \geqq \dots .$$

There exists a positive integer k such that $c^kR \leqq c^{k+1}R + I$ because R/I is right truncated. Now $B^{k+1} \leqq c^{k+1}R$ and hence

$$IU^{k+1} = (UI)^{k+1} \leqq B^{k+1} \leqq c^{k+1}R.$$

Now

$$c^kU^{k+1} \leqq (c^{k+1}R + I)U^{k+1} = c^{k+1}U^{k+1} + IU^{k+1} \leqq c^{k+1}R.$$

Thus $U^{k+1} \leqq cR \leqq A$ and U^{k+1} is an invertible ideal. This proves the theorem.

A ring R has the *right restricted minimum condition* provided the right R-module R/E is Artinian for any essential right ideal E of R. Theorem 1.6 generalizes the following result of Lenagan [**8**, Theorem 3.3].

COROLLARY 1.7. *Any right bounded hereditary Noetherian prime ring has enough invertible ideals.*

Proof. By [**6**, Theorems 4.1 and 4.4] R is an order in a simple Artinian ring. Also by a theorem of Webber [**12**] (or see [**4**, Theorem 1.3]) R satisfies the right restricted minimum condition so that every integral (i.e., non-zero) prime ideal is maximal and R/I is right truncated for every non-zero ideal I. Now apply the theorem.

To put Theorem 1.6 more into perspective we prove:

THEOREM 1.8. *Let R be a right Noetherian order in a simple Artinian ring such that every integral prime ideal is invertible or maximal and projective as a right and left R-module. Suppose further that R is right bounded and R/I is right truncated for every integral idempotent ideal I. Then R is right and left hereditary and left Noetherian.*

Proof. Suppose P is a prime ideal of R and R/P is right truncated. If $c \in R$ and $c + P$ is a regular element of R/P then R/P right truncated implies that $c + P$ is a unit in R/P. By [**6**, Theorem 3.9] R/P is a simple right Artinian ring.

Now suppose P is an invertible prime ideal. Let

$$X = \bigcap_{n=1}^{\infty} P^n.$$

Then X is a prime ideal of R. For let A and B be ideals of R and suppose $A \nleqq X$, $B \nleqq X$. There exist m, $n \geqq 0$ such that $A \leqq P^m$, $A \nleqq P^{m+1}$, $B \leqq P^n$, $B \nleqq P^{n+1}$, where we take $P^0 = R$. Then $P^{-m}A$ and BP^{-n} are ideals of R and $AB \leqq P^{m+n+1}$ implies

$$(P^{-m}A)\cdot(BP^{-n}) \leqq P.$$

But P is a prime ideal and so $P^{-m}A \leqq P$ (and $A \leqq P^{m+1}$) or $BP^{-n} \leqq P$ (and $B \leqq P^{n+1}$), giving a contradiction. Thus X is a prime ideal. Clearly P invertible implies $P > X$. If $X \neq 0$ then X is invertible and $X = PX$ gives $R = P$, a contradiction. Thus $X = 0$. By the proof of [**7**, Lemma 1] R/P is a simple right Artinian ring. Also by the proof of [**7**, Theorem] R is right hereditary.

Let E be an essential left ideal of R. Let c be a regular element in E [**6**, Theorem 3.9]. There exists an invertible ideal J such that $J \leqq cR$ (Theorem 1.6). Then $c^{-1}J \leqq R$ and hence $c^{-1} \in J^{-1}$. Thus $Jc^{-1} \leqq R$ and we conclude $J \leqq Rc \leqq E$. Thus R is left bounded. Since the prime ideals are finitely generated as left ideals and J contains a finite product of non-zero prime ideals (Lemmas 1.1 and 1.2) it follows that R/J is left Artinian and hence left Noetherian. Thus the fact that J is a finitely generated left ideal implies E is finitely generated. It follows that R is left Noetherian. By [**11**, Corollary 3] R is left hereditary.

2. Completely faithful modules. Let R be a ring. An R-module M is *faithful* provided $Mr \neq 0$ for every non-zero element r of R, otherwise it is *unfaithful*. An R-module M is *completely faithful* if X/Y is faithful for all submodules $X > Y$ of M. Clearly any submodule and any factor module of a completely faithful module are completely faithful.

LEMMA 2.1. *Let N be a submodule of a module M such that N and M/N are both completely faithful. Then M is completely faithful.*

Proof. Let $X \geqq Y$ be submodules of M such that $Xr \leqq Y$ for some non-zero element r in R. Then $(X \cap N)r \leqq (Y \cap N)$ and N completely faithful together imply

$$X \cap N = Y \cap N.$$

Similarly $(X + N)r \leqq Y + N$ and M/N completely faithful give $X + N = Y + N$. Then

$$Y = Y + (X \cap N) = Y.$$

It follows that M is completely faithful.

LEMMA 2.2. *For any module M there exists a unique maximal completely faithful submodule C which contains every completely faithful submodule of M.*

Proof. Suppose M contains non-zero completely faithful submodules, otherwise take $C = 0$. Let $\mathscr{S}$ denote the collection of completely faithful submodules of M. Define

$$C = \sum_{X \in \mathscr{S}} X.$$

It remains to prove that the submodule C is completely faithful. Let $A > B$ be submodules of C and suppose $Ar \leqq B$ for some element r of R. Let $a \in A$, $a \notin B$. Then there exist a positive integer n and completely faithful submodules X_i $(1 \leqq i \leqq n)$ of M such that $a \in X_1 + \ldots + X_n$. By Lemma 2.1 and induction on n the module $X_1 \oplus \ldots \oplus X_n$ is completely faithful and hence so is $X_1 + \ldots + X_n$. Thus $(aR)r \leqq (aR \cap B)$ implies $r = 0$. It follows that C is completely faithful.

Let M be a module. The unique maximal completely faithful submodule of M will be denoted by $C(M)$. Note that $C(M/C(M)) = 0$ by Lemma 2.1. Note further that if $M = \bigoplus_\Lambda M_\lambda$, for some index set Λ, then

$$C(M) = \bigoplus_\Lambda C(M_\lambda).$$

For, by Lemma 2.2 $C(M) \geqq \bigoplus_\Lambda C(M_\lambda)$; also if $\pi_\lambda : M \rightarrow M_\lambda$ is the canonical projection then $\pi_\lambda(C(M))$ is a completely faithful submodule of M_λ and hence

$$\pi_\lambda(C(M)) \leqq C(M_\lambda) \quad (\lambda \in \Lambda)$$

so that $C(M) \leqq \bigoplus_\Lambda C(M_\lambda)$. In addition if N is a submodule of M then

$$N \cap C(M) = C(N).$$

For, by Lemma 2.2,

$$N \cap C(M) \leqq C(N) \text{ and } N/(N \cap C(M)) \cong (N + C(M))/C(M)$$

implies

$$C(N/(N \cap C(M))) = 0.$$

If M is a module then it may well happen that $C(M) = 0$. Indeed if R is a ring then a necessary and sufficient condition for the existence of a

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non-zero completely faithful right R-module is that R be right primitive. For, if R is right primitive and V is a faithful irreducible right R-module then clearly V is completely faithful. Conversely, suppose M is a non-zero completely faithful right R-module. Let $m \in M$, $m \neq 0$. Then mR is completely faithful and any irreducible homomorphic image of mR is faithful. Thus R is right primitive.

A module M is *locally unfaithful* provided every finitely generated submodule is unfaithful. If R is a prime ring then an R-module M is locally unfaithful if and only if for any non-zero element m in M there exists a non-zero ideal I of R such that $mI = 0$.

Let R be a ring such that every non-zero ideal contains an invertible ideal. Then R is a prime ring. Conversely, if R is a prime Goldie ring with enough invertible ideals then every non-zero ideal of R contains an invertible ideal.

LEMMA 2.3. *Let R be a ring such that every non-zero ideal contains an invertible ideal. Let M be a cyclic R-module and N a submodule of M such that*

(i) *N is completely faithful and M/N unfaithful, or*

(ii) *N is unfaithful and M/N completely faithful.*

Then N is a direct summand of M.

The proof uses arguments similar to those used to prove [**4**, Theorem 3.9 and Lemma 3.10] but we include it for completeness.

Proof. Suppose M is a right R-module. Without loss of generality we can suppose $M = R/E$, $N = F/E$ where $E \leqq F$ are right ideals of R.

(i) There exists an invertible ideal I such that $I \leqq F$. Since F/E is completely faithful it follows that $F = FI + E$. Hence $I = FI + (E \cap I)$. Since I is invertible we have

$$R = II^{-1} = F + (E \cap I)I^{-1}.$$

Moreover, $EI \leqq E \cap I$ implies $E \leqq (E \cap I)I^{-1}$. Also

$$\{F \cap (E \cap I)I^{-1}\}I \leqq E$$

implies $F \cap (E \cap I)I^{-1} = E$ because F/E is completely faithful. Thus

$$R/E = (F/E) \oplus \{(E \cap I)I^{-1}/E\}.$$

(ii) There exists an invertible ideal J such that $FJ \leqq E$. Since R/F is completely faithful it follows that $R = F + J$. Now $(F \cap J)J^{-1}$ is a right ideal of R and

$$((F \cap J)J^{-1})J = F \cap J \leqq F.$$

Since R/F is completely faithful it follows that $(F \cap J)J^{-1} \leqq F$ and

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hence $F \cap J \leqq FJ \leqq E$. Thus

$$R/E = F/E \oplus (J + E)/E.$$

The next result concerns the exact sequence

(1) $0 \to A \to B \to C \to 0$

of right R-modules.

THEOREM 2.4. *Let R be an order in a ring Q such that every non-zero ideal contains an invertible ideal. Then the exact sequence* (1) *splits provided any one of the following statements holds:*

(i) *A is completely faithful and C locally unfaithful, or*

(ii) *A is unfaithful and C completely faithful, or*

(iii) *R is right Noetherian, A is locally unfaithful and C completely faithful.*

Proof. Without loss of generality we can suppose that A is a submodule of B. Let $b \in B$, $b \notin A$. Consider the cyclic module bR. In (i) $bR \cap A$ is a completely faithful submodule of bR and $bR/(bR \cap A) \cong (bR + A)/A$ is unfaithful. By Lemma 2.3

(2) $bR = (bR \cap A) \oplus D_b$

for some submodule D_b. In cases (ii) and (iii) $bR \cap A$ is an unfaithful submodule of bR (in (iii) because bR is a Noetherian module and hence $bR \cap A$ is finitely generated) and $bR/(bR \cap A) \cong (bR + A)/A$ is completely faithful. Again by Lemma 2.3 there exists a submodule D_b such that (2) holds.

Let $D = \sum_b D_b$. Note that in (i) D_b is unfaithful ($b \in B$) and so D is locally unfaithful. On the other hand in (ii) and (iii) D_b is completely faithful ($b \in B$) and hence so is D (Lemma 2.2). Clearly

$$B = A + D$$

and in all cases one of A, D is completely faithful and the other locally unfaithful. Thus $A \cap D = 0$ and we conclude $B = A \oplus D$.

COROLLARY 2.5. *Let R be a ring such that every non-zero ideal contains an invertible ideal. Let M be an R-module such that there exists a finite chain*

$$M = M_0 \geqq M_1 \geqq \ldots \geqq M_n = 0$$

of submodules M_i such that M_{i-1}/M_i is completely faithful or unfaithful $(1 \leqq i \leqq n)$. Then there exists an unfaithful submodule U of M such that $M = C(M) \oplus U$.

Proof. We prove the result by induction on n. The case $n = 1$ is clear. Let $N = M_1$. Then $N = C(N) \oplus V$ for some unfaithful submodule V

of N. If M/N is unfaithful apply (i) of the theorem to the module M/V to obtain

$$M/V = N/V \oplus W/V$$

for some submodule W of M such that $V \leqq W$ and W/V is unfaithful. Since R is prime it follows that W is unfaithful and $M = C(N) \oplus W$. Now suppose M/N is completely faithful. In this case apply (ii) of the theorem to $M/C(N)$ to obtain

$$M/C(N) = N/C(N) \oplus D/C(N)$$

for some submodule D of M containing $C(N)$. Since $D/C(N) \cong M/N$ it follows that $D/C(N)$, and hence D, is completely faithful (Lemma 2.1). Thus $M = D \oplus V$ and since V is unfaithful we have $D = C(M)$.

Corollary 2.5 generalizes [**4**, Theorem 3.9] as does the next result. A module M is *locally Artinian* provided every finitely generated submodule is Artinian. Clearly any infinite direct sum of irreducible modules is locally Artinian but not Artinian.

THEOREM 2.6. *Let R be a right Noetherian order in a simple Artinian ring such that R has enough invertible ideals and let M be a locally Artinian right R-module. Then there exists a locally unfaithful submodule N of M such that $M = C(M) \oplus N$.*

Proof. By Theorem 2.4(i) it is sufficient to prove that $M/C(M)$ is locally unfaithful. Let $m_1, \ldots, m_n$ be a finite collection of elements of M and consider the module

$$X = C(M) + m_1R + \ldots + m_nR.$$

Clearly $X/C(M)$ has finite composition length and $C(X/C(M)) = 0$. By Corollary 2.5 $X/C(M)$ is unfaithful. It follows that $M/C(M)$ is locally unfaithful and the result follows.

Note that in Theorem 2.6

$$N = \{m \in M : mI = 0 \text{ for some non-zero ideal } I \text{ of } R\}.$$

COROLLARY 2.7. *Let R be a prime Noetherian ring with enough invertible ideals and let M be a locally Artinian R-module. Then M is completely faithful if and only if the socle of M is completely faithful.*

Finally we mention some examples of primitive rings with enough invertible ideals. A ring R is called *hypercentral* provided whenever $I > J$ are ideals of R the ideal I/J of the ring R/J contains a non-zero central element of R/J. In particular every non-zero ideal of R contains a non-zero central element of R. Let R be an order in a ring Q such that R is prime and hypercentral; then every non-zero ideal of R contains an invertible ideal. This is because the ideal cR is invertible for any non-zero element c.

Example 2.8. Let A_n denote the nth Weyl algebra over a field F of characteristic 0 and D_n the division ring of fractions of A_n. Let t be any positive integer with $t \leqq n$. Then the polynomial ring $D_n[x_1, \ldots, x_t]$ is a primitive Noetherian hypercentral ring and so has enough invertible ideals.

Let $R \doteq D_n[x_1, \ldots, x_t]$. Then R is primitive by [**1**, Theorem 3] and Noetherian by the Hilbert Basis Theorem. That R is hypercentral follows at once from the next result.

LEMMA 2.9. *Let H be a hypercentral ring and S the polynomial ring $H[x]$. Then S is a hypercentral ring.*

Proof. Let $I > J$ be ideals of S. Let k be the least non-negative integer such that there is an element of degree k which lies in I but not J. Let I_k, J_k denote, respectively, the set of leading coefficients of elements of degree k in I, J together with the zero element in each case. Then $I_k \geqq J_k$ and I_k and J_k are ideals of H. Let

$$a = a_0 + a_1x + \ldots + a_kx^k \in I$$

but $a \notin J$ where $a_i \in H$ $(0 \leqq i \leqq k)$. Then $a_k \in I_k$, $a_k \notin J_k$, otherwise there exists $b \in J$ such that $a - b$ has degree $\leqq k$ and hence $a - b \in J$. Thus $I_k > J_k$. There exists $c_k \in I_k$ such that $c_k + J_k$ is a non-zero central element of the ring R/J_k. There exist $c_i \in H$ $(0 \leqq i \leqq k - 1)$ such that

$$c = c_0 + c_1x + \ldots + c_kx^k \in I.$$

If $h \in H$ then the leading coefficient of $ch - hc$ belongs to J_k and hence, by the choice of k, $ch - hc \in J$. It follows that $c + J$ is a non-zero central element of R/J. Hence R is a hypercentral ring.

Next we give a class of non-Noetherian examples.

Example 2.10. Let K be a field and G a torsion-free nilpotent group with centre Z such that G contains an Abelian subgroup A of rank not less than the cardinality of the group algebra KZ such that $A \cap Z = 1$. Let R be the group algebra KG. Then R is a primitive hypercentral right and left Ore domain. Moreover R is a non-Noetherian ring with enough invertible ideals.

The fact that R is primitive can be found in [**3**, Corollary 3.4]. That R is hypercentral is a consequence of [**10**, Theorem A]. The ring R is a right and left Ore domain by [**9**, Lemmas 13.1.6, 13.1.9 and 13.3.6].

An example of a group which satisfies the hypotheses of Example 2.10 can be obtained as follows. For each positive integer n define

$$H_n = \langle x_n, y_n, z_n; [x_n, z_n] = [y_n, z_n] = 1, [x_n, y_n] = z_n \rangle.$$

Let G be the direct product of the groups H_n $(n \geqq 1)$ and A the subgroup of G generated by the elements x_n $(n \geqq 1)$. Then G is torsion-free nilpotent of class 2, $A \cap Z = 1$ and the rank of A has the required property if K is a countable field.

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