ON AN INVERSION FORMULA

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1. Introduction. In this paper the author considers the problem of finding a formula of inversion for the integral transform defined by the equation

$$F(u)=\int_a^\infty f(r)J_u(kr)\frac{dr}{r} \tag{1}$$

where $a>0, k>0$ and $r^{-1}f(r)\in L(a,\infty)$. This transform appeared in connection with an earlier investigation [**4**] in which an attempt was made to devise an integral transform that could be adapted to the solution of certain boundary value problems involving the space form of the wave equation and the condition of radiation:

$$\lim_{r\to\infty} r^{1/2}[f'(r)-ikf(r)]=0. \tag{2}$$

The transform that is adapted to treat the kind of problem described is that defined by the equation

$$F_1(u)=\int_a^\infty f(r)H_u^{(1)}(kr)\frac{dr}{r} \tag{3}$$

where the kernel is a Hankel function of the first kind, the notation being that of Watson [**9**].

The integral formula derived in [**4**] is given by the equation

$$\begin{aligned} f(r)=\frac{1}{2i}\int_L \frac{uJ_{-u}(kr)F(u)\,du}{\sin u\pi}-\frac{1}{2i}\int_C \frac{uJ_u(kr)e^{-iu\pi}F(u)\,du}{\sin u\pi}\\ -\frac{1}{2}\int_C \frac{uH_u^{(1)}(kr)J_u(ka)F_1(u)\,du}{H_u^{(1)}(ka)} \end{aligned} \tag{4}$$

where L denotes the imaginary axis of the complex u-plane and C denotes a loop enclosing the positive real axis. The object in constructing the formula (4) was to generate the type of expansion which appears in the theory of diffraction and which involves the eigenfunctions $H_{u_n'}^{(1)}(kr)$ where $u_1', u_2', \ldots$ are the zeros of the function $H_u^{(1)}(ka)$ regarded as a function of the order u. This was achieved through the introduction of the two loop integrals into the formula (4). In proving the formula (4) it was assumed that, in addition to satisfying the radiation condition, the function $f(r)$ being expanded also satisfied the following integrability condition

$$r^{-1/2}[rf''(r)+f'(r)+k^2rf(r)]\in L(a,\infty). \tag{5}$$

Since

$$iH_u^{(1)}(kr)\sin u\pi=J_{-u}(kr)-J_u(kr)e^{-iu\pi}$$

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it follows that the transforms defined by (1) and (3) are related by the equation

$$iF_1(u)\sin u\pi = F(-u) - F(u)e^{-iu\pi}. \tag{6}$$

Since $F_1(u)$ can be expressed in terms of $F(u)$ by means of the preceding equation it follows that the equation (4) does provide a formula of inversion for the transform $F(u)$ but it is not in a very convenient form owing to the appearance of the loop integrals in (4).

In this paper an alternative formula of inversion is developed which contains an expansion different from the eigenfunction expansion mentioned above and which is simpler to apply in practice. The formula in question, which appears in the theorem stated in the next section of this paper, contains a series involving the functions $J_{u_n}(kr)$, where $u_1, u_2, \ldots$ are the zeros of the function $J_u(ka)$ regarded as a function of the order u, and it is proved without assuming that the function $f(r)$ to be expanded satisfies the condition (5) which is somewhat restrictive. The essential assumption is that $r^{-1}f(r) \in L(a, \infty)$. An extensive table of the zeros u_n has been compiled by S. Conde & S. L. Kalla [**1**] who have calculated the first ten such zeros for a large number of values of ka ranging from 0·001 to 10^6.

2. The integral theorem.

THEOREM. *Suppose that $f(r)$ is continuous for $r \geq a > 0$ and that $r^{-1}f(r) \in L(a, \infty)$. Let the function $F(u)$ be defined by equation* (1), *then, if $r > a$,*

$$f(r) = \frac{1}{2i}\int_L \frac{u\phi(u, r)F(u)\,du}{J_u(ka)} - \frac{1}{\pi}\sum_{u=u_n} \frac{uJ_u(kr)Y_u(ka)F(u)}{(\partial/\partial u)J_u(ka)} \tag{7}$$

where

$$\phi(u, r) = J_u(kr)Y_u(ka) - J_u(ka)Y_u(kr) \tag{8}$$

and L denotes the imaginary axis of the complex u-plane. The summation in (7) *is extended over all the negative zeros u_n of the function $J_u(ka)$, regarded as a function of its order u.*

When u is large the Bessel function $J_u(x)$ behaves like the function $(x/2)^u/\Gamma(1+u)$. By relating the Bessel functions appearing in (7) to their corresponding power functions, the proof of the above theorem can be reduced to that of the Mellin inversion theorem [**8**, p. 46].

We first form the equation

$$\int_{-iR}^{iR} \frac{u\phi(u, r)F(u)\,du}{J_u(ka)} = \int_a^\infty f(t)\frac{dt}{t}\int_{-iR}^{iR} \frac{u\phi(u, r)J_u(kt)\,du}{J_u(ka)} \tag{9}$$

where $R > 0$. The equation (9) follows after substituting the expression (1) for $F(u)$ and changing the order of integration. In order to justify the inversion of the order of integration it will be proved that the repeated integral is absolutely convergent. With this aim in view we appeal to the bound

$$|J_{is}(kt)| \leq (2/\pi kt)^{1/2}\cosh(s\pi/2). \tag{10}$$

This bound, which was derived in [**5**], applies on the imaginary axis where $u = is$, s real,

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and holds throughout the interval $a \leq t < \infty$. The factors remaining in the u-integral on the right hand side of (9) are independent of t and are bounded continuous functions of u in the relevant interval. Hence the modulus of the repeated integral does not exceed the quantity

$$C\int_a^\infty |f(t)|\, t^{-3/2}\, dt$$

where C is a constant. Since the above integral exists it follows that the repeated integral in question is absolutely convergent so that the validity of (9) is established.

It will now be proved that the u-integral appearing on the right hand side of (9) is a symmetric function of (r, t) in the sense that

$$\int_{-iR}^{iR} \frac{u\phi(u, r)J_u(kt)\, du}{J_u(ka)} = \int_{-iR}^{iR} \frac{u\phi(u, t)J_u(kr)\, du}{J_u(ka)}. \tag{11}$$

To obtain this result we note that the definition (8) of $\phi(u, r)$ implies that

$$\begin{aligned}\phi(u, r)J_u(kt) - \phi(u, t)J_u(kr) &= J_u(ka)[J_u(kr)Y_u(kt) - J_u(kt)Y_u(kr)]\\ &= -J_u(ka)[J_u(kr)J_{-u}(kt) - J_u(kt)J_{-u}(kr)]\operatorname{cosec} u\pi\end{aligned}$$

after using the identity

$$Y_u(x) = [J_u(x)\cos u\pi - J_{-u}(x)]\operatorname{cosec} u\pi. \tag{12}$$

It follows that

$$\int_{-iR}^{iR} \frac{u[\phi(u, r)J_u(kt) - \phi(u, t)J_u(kr)]\, du}{J_u(ka)} = -\int_{-iR}^{iR} \frac{u[J_u(kr)J_{-u}(kt) - J_u(kt)J_{-u}(kr)]\, du}{\sin u\pi} = 0$$

since the integrand is an odd function of the variable u.

The t-integration in (9) is now decomposed into the parts (a, r) and (r, ∞). By virtue of (11) the variables r, t may be interchanged in the u-integral that appears in the (a, r) part of the t-integral. This change yields the equation

$$\begin{aligned}\int_{-iR}^{iR} \frac{u\phi(u, r)F(u)\, du}{J_u(ka)} &= \int_a^r f(t)\frac{dt}{t}\int_{-iR}^{iR} \frac{u\phi(u, t)J_u(kr)\, du}{J_u(ka)}\\ &\quad + \int_r^\infty f(t)\frac{dt}{t}\int_{-iR}^{iR} \frac{u\phi(u, r)J_u(kt)\, du}{J_u(ka)}.\end{aligned} \tag{13}$$

It is now convenient to introduce the function $h(u, r, t)$ defined by the equation

$$\frac{\pi u\phi(u, r)J_u(kt)}{J_u(ka)} = \left(\frac{t}{r}\right)^u - \left(\frac{rt}{a^2}\right)^u + h(u, r, t). \tag{14}$$

The function $h(u, r, t)$ is defined in this way since it will appear shortly that when u is large the dominant term in the asymptotic expansion of the expression on the left hand side of (14) is the function $(t/r)^u - (rt/a^2)^u$. The procedure now is to insert the expression (14) into the integrals on the right hand side of (13). The resulting integrals involving the power terms $(t/r)^u$ and $(rt/a^2)^u$ will be evaluated by means of the Mellin inversion theorem, whilst those involving the function h can be evaluated with the aid of the calculus of residues. On using (14) to rewrite (13) we find the equation

$$\pi\int_{-iR}^{iR}\frac{u\phi(u,r)F(u)\,du}{J_u(ka)}=\int_a^r f(t)\frac{dt}{t}\int_{-iR}^{iR}\left[\left(\frac{r}{t}\right)^u-\left(\frac{rt}{a^2}\right)^u\right]du$$
$$+\int_r^\infty f(t)\frac{dt}{t}\int_{-iR}^{iR}\left[\left(\frac{t}{r}\right)^u-\left(\frac{rt}{a^2}\right)^u\right]du+I_1+I_2 \tag{15}$$

where

$$I_1=\int_a^r f(t)\frac{dt}{t}\int_{-iR}^{iR} h(u,t,r)\,du, \tag{16}$$

$$I_2=\int_r^\infty f(t)\frac{dt}{t}\int_{-iR}^{iR} h(u,r,t)\,du. \tag{17}$$

The two integrals on the right hand side of (15) involving the term (rt/a^2) may be combined to give a single integral over the interval $a\leq t<\infty$. The other two integrals, which involve the terms $(r/t)^u$ and $(t/r)^u$, respectively, may also be combined after changing the sign of the integration variable u in the first such integral. Since all four integrals are absolutely convergent, we find, on changing the orders of integration, that

$$\pi\int_{-iR}^{iR}\frac{u\phi(u,r)F(u)\,du}{J_u(ka)}=\int_{-iR}^{iR} r^{-u}\,du\int_a^\infty t^{u-1}f(t)\,dt$$
$$-\int_{-iR}^{iR}\left(\frac{a^2}{r}\right)^{-u}du\int_a^\infty t^{u-1}f(t)\,dt+I_1+I_2. \tag{18}$$

Since $r>a$ the first integral on the right hand side of equation (18) tends as $R\to\infty$ to $2i\pi f(r)$ by the Mellin inversion theorem [**8**, p. 46]. However, by the same theorem, the second integral tends to zero as $R\to\infty$ since $a^2/r<a$ therein, and

$$\lim_{R\to\infty}\int_{-iR}^{iR}\rho^{-u}\,du\int_a^\infty t^{u-1}f(t)\,dt=\begin{cases}0 & 0<\rho<a\\ 2i\pi f(\rho) & a<\rho\end{cases}$$

It follows that

$$\pi\int_L\frac{u\phi(u,r)F(u)\,du}{J_u(ka)}=2i\pi f(r)+\lim_{R\to\infty}(I_1+I_2). \tag{19}$$

The terms in (19) involving I_1 and I_2 may be determined by means of the calculus of residues and for this purpose it is necessary to determine the behaviour of the function h when u is large. This can be obtained from the equation

$$J_u(x)=\frac{(x/2)^u}{\Gamma(1+u)}\left[1-\frac{x^2}{4(1+u)}+O(u^{-2})\right]. \tag{20}$$

This relation holds whenever u is large and bounded away from the negative integers and applies uniformly in any bounded domain of values of x. On substituting the formula (12) for Y_u into the equation (8) we find that

$$\phi(u, r) = -[J_u(kr)J_{-u}(ka) - J_u(ka)J_{-u}(kr)]\operatorname{cosec} u\pi. \tag{21}$$

On inserting the expression (20) and appealing to the identity $\Gamma(1+u)\Gamma(1-u) = \pi u \operatorname{cosec} u\pi$ we find from (21) the equation

$$\pi u \phi(u, r) = \left(\frac{a}{r}\right)^u - \left(\frac{r}{a}\right)^u + \frac{k^2(r^2-a^2)}{4u}\left[\left(\frac{a}{r}\right)^u + \left(\frac{r}{a}\right)^u\right] + O\left[u^{-2}(r/a)^{|\mathrm{Re}(u)|}\right]. \tag{22}$$

In addition it follows from (20) that

$$\frac{J_u(kt)}{J_u(ka)} = \left(\frac{t}{a}\right)^u\left[1 - \frac{k^2(t^2-a^2)}{4u} - O(u^{-2})\right] \tag{23}$$

so that, on combining (21) and (22) we find that the function h defined by equation (14) possesses the asymptotic form

$$h(u, r, t) = \frac{k^2}{4u}(r^2-t^2)\left(\frac{t}{r}\right)^u + (r^2+t^2-2a^2)\left(\frac{rt}{a^2}\right)^u + O[u^{-2}(t/r)^u] + O[u^{-2}(rt/a^2)^u]. \tag{24}$$

We consider first the quantity I_1 which is defined by equation (16). Since the Bessel functions are entire functions of the order u the function h defined by (14) is an analytic function of u except for simple poles at the zeros u_n of the function $J_u(ka)$ regarded as a function of u. It is known [**3**] that these zeros are all real and simple and that the large such zeros are asymptotic to the large negative integers. Since $a \leq t \leq r$ for I_1 it follows from (24), with r, t interchanged, that the function $h(u, t, r)$ tends to zero as $|u| \to \infty$ in the half plane $\mathrm{Re}(u) \leq 0$. If we write $u = Re^{i\theta}$ the contour may be closed by adding the semicircle $C(R)$ having the segment $(-iR, iR)$ as its diameter and positioned to the left of this segment. If we take $R = m + \frac{1}{2}$, where m is a (large) positive integer, the semicircles will avoid the poles of the integrand. At a zero u_n the function $\phi(u, r)$ defined by (8) reduces to the product $J_u(kr)Y_u(ka)$ and on using (14) to calculate the residues at those zeros which are located inside the semicircle $C(R)$ we find the equation

$$\int_{-iR}^{iR} h(u, t, r)\, du = \int_{C(R)} h(u, t, r)\, du + 2i\pi^2 \sum_{u_n} \frac{uJ_u(kr)J_u(kt)Y_u(ka)}{(\partial/\partial u)J_u(ka)} \tag{25}$$

in which the summation extends over all the zeros u_n that lie in the interval $-R < u_n < 0$.

When (25) is substituted into (16) we obtain the equation

$$I_1 = \int_a^r f(t)\frac{dt}{t}\int_{C(R)} h(u, t, r)\, du + 2i\pi^2 \sum \frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_a^r f(t)J_u(kt)\frac{dt}{t}. \tag{26}$$

It will be shown that the repeated integral occurring on the right hand side of the preceding equation is $O(R^{-1/2})$ as $R \to \infty$. To do this the interval (a, r) appearing in this integral is decomposed into the parts $(a, r - R^{-1/2})$ and $(r - R^{-1/2}, r)$ which are considered separately.

For values of t in the first such interval we can obtain the bound

$$\left|\int_{C(R)}\left(\frac{r}{t}\right)^u\frac{du}{u}\right|\leqslant\int_{\pi/2}^{3\pi/2}\left(\frac{r}{t}\right)^{R\cos\theta}d\theta\leqslant\frac{M}{R\log(r/t)}$$

where M is a constant. Since the least value of $\log(r/t)$ in the stated interval is $-\log(1-r^{-1}R^{-1/2})>r^{-1}R^{-1/2}$ it follows that

$$\left|\int_{C(R)}\left(\frac{r}{t}\right)^u\frac{du}{u}\right|\leqslant\frac{Mr}{R^{1/2}}.$$

In the remaining segment $r-R^{-1/2}\leqslant t\leqslant r$ we see, since $\mathrm{Re}(u)\leqslant 0$, that

$$\left|\int_{C(R)}\left(\frac{r}{t}\right)^u\frac{du}{u}\right|\leqslant\int_{\pi/2}^{3\pi/2}d\theta=\pi.$$

Similar bounds apply to the contributions of the O-terms in (24) as well as that from the (rt/a^2) term. On forming the total contribution we find that it is $O(R^{-1/2})$ as stated. Hence the equation (26) implies that

$$I_1=O(R^{-1/2})+2i\pi^2\sum_{u=u_n}\frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_a^r f(t)J_u(kt)\frac{dt}{t}\tag{27}$$

where $-R<u_n<0$ in the summation.

We consider now the quantity I_2 which is defined by equation (17). The t-integration in (17) is decomposed into the three parts $(r, r+R^{-1/2})$, $(r+R^{-1/2}, t_0)$ and (t_0,∞) where t_0 is chosen large enough to ensure that

$$\int_{t_0}^{\infty}|f(t)|\frac{dt}{t}<\varepsilon.\tag{28}$$

Since $a\leqslant r\leqslant t$ in I_2 it follows from (24) that the function $h(u, r, t)$ tends to zero as $u\to\infty$ in the half plane $\mathrm{Re}(u)\leqslant 0$, so that the value of the u-integral in I_2 can be obtained as before by closing the contour on the left by means of the semicircle $C(R)$ and taking the residues at the poles. This procedure leads to the equation

$$I_2=I_0+O(R^{-1/2})+2i\pi^2\sum_{u=u_n}\frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_r^{t_0}f(t)J_u(kt)\frac{dt}{t}\tag{29}$$

where the summation is such that $-R<u_n<0$ and

$$I_0=\int_{t_0}^{\infty}f(t)\frac{dt}{t}\int_{-iR}^{iR}h(u, r, t)\,du.$$

It will now be shown that $I_0=O(\varepsilon)$. On setting $u=is$ in the above equation and inserting the value of the function $h(is, r, t)$ obtained from (14) we find the equation

$$I_0=\int_{t_0}^{\infty} f(t)\frac{dt}{t}\int_{-R}^{R} [(rt/a^2)^{is}-(t/r)^{is}]\,ds + \int_{t_0}^{\infty} f(t)\frac{dt}{t}\int_{-R}^{R} \frac{s\phi(is, r)J_{is}(kt)\,ds}{J_{is}(ka)}. \tag{30}$$

Now

$$\left|\int_{-R}^{R}\left(\frac{t}{r}\right)^{is} ds\right|=\left|\frac{2\sin[R\log(t/r)]}{\log(t/r)}\right|\leqslant\frac{2}{\log(t_0/r)}.$$

A similar bound applies to the (rt/a^2) integral in (30). Hence the magnitude of the first repeated integral in (30) is, by (28), $O(\varepsilon)$.

To estimate the value of the second repeated integral in (30) appeal is made to the following formula [**2,** p. 140]

$$J_{is}(kt)=(2\pi\rho)^{-1/2}\exp\left[\tfrac{1}{2}s\pi+i\rho-is\log\left(\frac{s+\rho}{kt}\right)-\frac{i\pi}{4}\right]\left[1-\frac{i}{8\rho}+\frac{5is^2}{24\rho^3}+O(s^{-2})\right] \tag{31}$$

where

$$\rho=(s^2+k^2t^2)^{1/2}. \tag{32}$$

The formula (31) applies as $s\to+\infty$ uniformly for $0\leqslant t<\infty$. If we set $t=a$ in (31) and simplify the resulting expression we obtain the expansion

$$J_{is}(ka)=(2\pi s)^{-1/2}\exp\left[\tfrac{1}{2}s\pi-is\log\frac{2s}{kae}-\frac{i\pi}{4}\right]\left[1+\frac{i}{12s}+\frac{ik^2a^2}{4s}+O(s^{-2})\right]. \tag{33}$$

On dividing (31) by (33) we obtain the formula

$$\frac{J_{is}(kt)}{J_{is}(ka)}=\left(\frac{s}{\rho}\right)^{1/2}\exp[i\Psi(s)][1+g(s)] \tag{34}$$

where

$$\Psi(s)=\rho-s\log\left(\frac{s+\rho}{kt}\right)+s\log\left(\frac{2s}{kae}\right) \tag{35}$$

$$g(s)=\frac{1}{8i\rho}+\frac{1}{12is}+\frac{k^2a^2}{4is}+\frac{5is^2}{24\rho^3}+O(s^{-2}). \tag{36}$$

The asymptotic behaviour of the factor $\phi(is, r)$ appearing in (30) is obtained from equation (22) after setting $u=is$ therein. This yields the equation

$$is\phi(is, r)=\left(\frac{a}{r}\right)^{is}-\left(\frac{r}{a}\right)^{is}+\frac{k^2(r^2-a^2)}{4is}\left[\left(\frac{a}{r}\right)^{is}+\left(\frac{r}{a}\right)^{is}\right]+O(s^{-2}). \tag{37}$$

Now let s_1, s_2 be large. It is shown in the appendix to this paper that

$$\int_{s_1}^{s_2}\frac{e^{ibs}J_{is}(kt)\,ds}{J_{is}(ka)}=i\left(\frac{s_1}{\rho_1}\right)^{1/2}\frac{e^{i\psi(s_1)}}{\psi'(s_1)}-i\left(\frac{s_2}{\rho_2}\right)^{1/2}\frac{e^{i\psi(s_2)}}{\psi'(s_2)}+O(s_1^{-1}) \tag{38}$$

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where $\psi(s)=\Psi(s)+bs$ and

$$\rho_1=(s_1^2+k^2t^2)^{1/2}, \qquad \rho_2=(s_2^2+k^2t^2)^{1/2}.$$

The formula (38) applies for s_1, s_2 arbitrarily large, $s_2>s_1$ and uniformly for $t\geqslant 3r$. It follows from (38) on setting $b=\log(a/r)$ and $s_2=R$ that

$$\left|\int_{s_1}^{R}\frac{(a/r)^{is}J_{is}(kt)\,ds}{J_{is}(ka)}\right|\leqslant M$$

where M is a constant. Similar bounds hold when the factor $(a/r)^{is}$ in the above integral is replaced by $(r/a)^{is}$ and for the corresponding integrals that contain the additional factor s^{-1}. Therefore on substituting the expression (37) we find that

$$\int_{t_0}^{\infty} f(t)\frac{dt}{t}\int_{s_1}^{R}\frac{s\phi(is,r)J_{is}(kt)\,ds}{J_{is}(ka)}=O(\varepsilon). \tag{39}$$

This applies for arbitrarily large R and ε is independent of R. In addition it follows with the aid of (10) and (28) that

$$\left|\int_{t_0}^{\infty} f(t)\frac{dt}{t}\int_{0}^{s_1}\frac{s\phi(is,r)J_{is}(kt)\,ds}{J_{is}(ka)}\right|\leqslant\varepsilon\int_0^{s_1}\left|\frac{s\phi(is,r)\cosh(s\pi/2)}{J_{is}(ka)}\right|ds. \tag{40}$$

On adding (39), (40) we find, if C_1 is a constant, the equation

$$\left|\int_{t_0}^{\infty} f(t)\frac{dt}{t}\int_{0}^{R}\frac{s\phi(is,r)J_{is}(kt)\,ds}{J_{is}(ka)}\right|\leqslant C_1\varepsilon.$$

A similar result applies to the corresponding integral in which the s-integration is taken over the interval $(-R,0)$ so that on combining these bounds it is seen that the magnitude of the second repeated integral appearing in equation (30) is $O(\varepsilon)$ as claimed and the quantity I_0 is itself $O(\varepsilon)$.

If we let $R\to\infty$ in equations (27) and (29) and add the resulting equations we find that

$$\lim_{R\to\infty}(I_1+I_2)=I_o+2i\pi^2\sum_{u_n}\frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_a^{t_0} f(t)J_u(kt)\frac{dt}{t} \tag{41}$$

where the summation now extends over all of the negative zeros of $J_u(ka)$. Finally it is shown in the appendix to this paper that, if C_2 is a further constant,

$$\sum_{u_n}\left|\frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_{t_0}^{\infty} f(t)J_u(kt)\frac{dt}{t}\right|\leqslant C_2\varepsilon^{1/2}. \tag{42}$$

Since ε is arbitrarily small it follows from (41), (42) that

$$\lim_{R\to\infty}(I_1+I_2)=2i\pi^2\sum_{u_n}\frac{uJ_u(kr)Y_u(ka)}{(\partial/\partial u)J_u(ka)}\int_a^{\infty} f(t)J_u(kt)\frac{dt}{t}. \tag{43}$$

The formula (7) of the theorem now follows after inserting the above result into equation (19).

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Appendix. The formula (38) used in the paper is proved by substituting the expression (34) into the integral occurring on the right hand side of equation (38). This leads to the equation

$$\int_{s_1}^{s_2} \frac{e^{ibs} J_{is}(kt)\, ds}{J_{is}(ka)} = \int_{s_1}^{s_2} (s/\rho)^{1/2} e^{i\psi}\, ds + \int_{s_1}^{s_2} (s/\rho)^{1/2} e^{i\psi} g(s)\, ds \tag{A.1}$$

where $\psi = \Psi + bs$ and Ψ is defined by equation (35). It can be shown by successive differentiation of (35) that $\psi''(s) = (\rho - s)/(\rho s)$ which is positive since $\rho > s$. It follows that the function $\psi'(s)$ is an increasing function of s and it can be shown that this function cannot be less than $\log 2$ whenever $s > 3kr$ and $t > 3r$. It can be shown by further differentiation that $\psi'''(s) = (s^3 - \rho^3)s^{-2}\rho^{-3}$ which is negative so that $\psi''(s)$ is a decreasing function of s. The monotone properties stated enable the various integrals that appear on the right hand side of equation (A.1) to be estimated by means of the second mean value theorem [**8**, p. 379]. With this aim in mind the first term on the right hand side of (A.1) is transformed by an integration by parts which yields the equation

$$\int_{s_1}^{s_2} (s/\rho)^{1/2} e^{i\psi}\, ds = i\left(\frac{s_1}{\rho_1}\right)^{1/2} \frac{e^{i\psi}(s_1)}{\psi'(s_1)} - i\left(\frac{s_2}{\rho_2}\right)^{1/2} \frac{e^{i\psi}(s_2)}{\psi'(s_2)} + i\int_{s_1}^{s_2} \left[\frac{k^2t^2}{2s^{1/2}\rho^{5/2}} - \left(\frac{s}{\rho}\right)^{1/2} \frac{\psi''(s)}{\psi'(s)}\right] \frac{e^{i\psi}(s)\, ds}{\psi'(s)}. \tag{A.2}$$

Now (s/ρ) is an increasing function of s whereas $\psi''(s)$ and $\psi'(s)^{-1}$ are decreasing functions of s, therefore after two applications of the mean value theorem we find the equation,

$$\int_{s_1}^{s_2} \left(\frac{s}{\rho}\right)^{1/2} \frac{\psi''(s) \cos \psi\, ds}{[\psi'(s)]^2} = \left(\frac{s_2}{\rho_2}\right)^{1/2} \int_{s_1'}^{s_2} \frac{\psi''(s) \cos \psi\, ds}{[\psi'(s)]^2} = \left(\frac{s_2}{\rho_2}\right)^{1/2} \frac{\psi''(s_1')}{[\psi'(s_1')]^3} [\sin \psi(s_2') - \sin \psi(s_1')] \tag{A.3}$$

where $s_1 < s_1' < s_2' < s_2$. Since $\psi''(s_1') \leqslant (s_1')^{-1} \leqslant s_1^{-1}$ the expression (A.3) is $O(s_1^{-1})$ uniformly for $t \geqslant 3r$. The remaining part of the integral on the right hand side of (A.2) can also be estimated by means of a single application of the mean value theorem, in which the factor $t^2 s^{-1/2} \rho^{-5/2} [\psi'(s)]^{-2}$ is extracted, to show that it too is $O(s_1^{-1})$ uniformly for $t \geqslant 3r$. Thus the equation (A.2) reveals that

$$\int_{s_1}^{s_2} \left(\frac{s}{\rho}\right)^{1/2} e^{i\psi}\, ds = i\left(\frac{s_1}{\rho_1}\right)^{1/2} \frac{e^{i\psi(s_1)}}{\psi'(s_1)} - i\left(\frac{s_2}{\rho_2}\right)^{1/2} \frac{e^{i\psi(s_2)}}{\psi'(s_2)} + O(s_1^{-1}). \tag{A.4}$$

Similar methods can be used to estimate the five integrals that appear when the formula (36) for $g(s)$ is inserted into the second integral on the right hand side of equation (A.1) and to show that this integral is also $O(s_1^{-1})$. On combining the equation (A.4) with the simplified form of equation (A.1) we obtain the formula (38) quoted earlier.

It remains to establish the inequality (42) and for this purpose it is necessary to estimate the values of the various Bessel functions appearing in this inequality. Each of

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these functions is evaluated at a negative zero u_n of the function $J_u(ka)$. The necessary bounds needed to estimate these quantities can be obtained by following a procedure similar to that adopted in [**5, 6**] where two related expansion theorems were studied, but under more restrictive conditions on the function being expanded. Since the method is the same as that used in [**5, 6**], except for minor changes, it is sufficient to state the final results. If u is a negative zero of $J_u(ka)$ it can be shown that

$$\int_a^\infty J_u(kt)^2\frac{dt}{t}\leq\frac{1}{|2u|}, \tag{A.5}$$

$$|uJ_u(kr)Y_u(ka)|\leq\frac{1}{\pi}\left(\frac{r}{a}\right)^{|u|}, \tag{A.6}$$

$$\left|\frac{\partial}{\partial u}J_u(ka)\right|\geq\frac{1}{2\sqrt{2}}\left(\frac{2}{ka}\right)^{|u|}\Gamma(|u|)^2. \tag{A.7}$$

A bound on the integral present in (42) may be found by applying the Schwarz inequality which shows that

$$\left|\int_{t_0}^\infty f(t)J_u(kt)\frac{dt}{t}\right|\leq\left\{\int_{t_0}^\infty |f(t)|^2\frac{dt}{t}\int_{t_0}^\infty J_u(kt)^2\frac{dt}{t}\right\}^{1/2}. \tag{A.8}$$

Now the function $f(t)$ is bounded so that $|f|\leq M^2$ where M is a constant. On applying (28) we find that

$$\int_{t_0}^\infty |f(t)|^2\frac{dt}{t}\leq M^2\int_{t_0}^\infty |f(t)|\frac{dt}{t}\leq M^2\varepsilon. \tag{A.9}$$

The Bessel function integral in (A.8) does not exceed that in (A.5) so that equation (A.8) states that

$$\left|\int_{t_0}^\infty f(t)J_u(kt)\frac{dt}{t}\right|\leq M\varepsilon^{1/2}|2u|^{-1/2}. \tag{A.10}$$

On combining (A.6), (A.7) and (A.10) we find that the summation in (42) does not exceed the quantity

$$\frac{2M\varepsilon^{1/2}}{\pi}\sum\frac{(kr/2)^{|u_n|}}{|u_n|^{1/2}\Gamma(|u_n|)} \tag{A.11}$$

where the summation is extended over all of the negative zeros u_n. Since these zeros are asymptotic to the large negative integers the series in (A.11) is convergent. Therefore the expression (A11) is itself $O(\varepsilon^{1/2})$ so that (42) is proved.

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