STRONGLY REGULAR EXTENSIONS OF RINGS

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As defined by Arens and Kaplansky [2] a ring A is strongly regular (s.r.) in case to each \( a \in A \) there corresponds \( x = x_a \in A \) depending on \( a \) such that \( a^2 x = a \). In the present article a ring A is defined to be a s.r. extension of a subring B in case each \( a \in A \) satisfies \( a^2 x - a \in B \) with \( x = x_a \in A \). S.r. rings are, then, s.r. extensions of each subring. A ring A which is a s.r. extension of the center has been called a \( \xi \)-ring (see Utumi [13], Drazin [3], Martindale [11], and their bibliographies).

Arens and Kaplansky showed that a s.r. ring is a subdirect sum of division rings. Since any s.r. ring is semisimple, a later result stating that any semisimple \( \xi \)-ring is a subdirect sum of division rings (see [11]) contains this result. In §2 of the present article, a further generalization is obtained: (1) If a semisimple ring A is a s.r. extension of a commutative subring B, then A is a subdirect sum of division rings. For the proof, the reduction to the case A is primitive is immediate, but at this stage an innovation is made. Instead of specializing B, as has been done in the previous work along these lines, a structure theorem (Theorem 2.1) for a primitive s.r. extension A of an arbitrary ring B is obtained first of all: (2) If A is a primitive ring, not a division ring, and if A/B is s.r., then B is dense in the finite topology on A. Of course, (1) is an immediate consequence, but more can be squeezed out of (2). For example, (2) shows that in order that a primitive ring A be a s.r. extension of a subring B, it is necessary that B be a primitive ring, or an integral domain. (A bit of duality can be introduced here, since in §1 it is shown that a directly...
irreducible s.r. extension of an integral domain is necessarily an integral
domain.) A fairly easy consequence of this is that in order that a semisimple
ring $A$ be a s.r. extension of a subring $B$, it is necessary that $B$ be a subdirect
sum of primitive rings and integral domains (Corollary 2.2 ff.).

Another consequence of (2) is that any s.r. extension of a division ring is
a subdirect sum of division rings. This fact is also implied by the theorem of
Arens and Kaplansky inasmuch as a s.r. extension of a s.r. ring is a s.r. ring.
However, this and the above results are all obtained in a new way independently
of the previous results for $\tau$-rings and s.r. rings.

The structure of $A$ is not known in the general case when $A$ is a s.r. ex-
tension of a commutative subring $B$. However the centralizer of $B$ in $A$ is a
$\xi$-ring, so that some information on the structure of $A$ is available.

In § 3 the results on s.r. extensions are applied in extending the results of
Nakayama [12] on the commutativity of rings, continuing a program which I
began in [4]. Any future improvements in the theory of s.r. extensions will
net corresponding improvements in this direction also.

A simple computation shows that a ring is regular ($axa = a$) in the sense
of von Neumann if and only if every principal one-sided ideal has an idempotent
generator. Arens and Kaplansky introduced the notion of strong regularity
($a^2 x = a$), whereby not only are these idempotent generators demanded but
also nilpotent elements are banished. Here, and more generally in $\xi$-rings, the
emphasis has shifted from the manufacture of idempotents to the disposition
of the nilpotent elements of index two: they must all lie in the center. In § 5
the position in a primitive ring $A$ of the subring $T(A)$ generated by the
nilpotent elements of index two is investigated. One finds in important special
cases (e.g., if $A$ is an algebraic algebra, or if $A$ has a minimal left ideal) that
the subring $T(A)$, and also the subring $E(A)$ generated by the idempotents
of $A$, is dense in $A$, if $A$ is not division. This clearly illustrates my allusion
above to the extent to which the structure of an s.r. extension $A/B$ is influenced
by the fact that $B$ contains the subring $T(A)$.

1. Directly irreducible strongly regular extensions. If $A$ is a $\xi$-ring with
center $Z$, and if $a, x \in A$ satisfy $a^2 x = a \in Z$ then [11, Theorem 1] states that
$ax = xa$. The verbatim proof (accredited to Herstein) given there establishes
this implication for $CN$-rings$^3$, a fact which is stated in the proposition below. By reproducing its proof here, I have been able to make this section, and the section following, relatively self-contained.

**Proposition 1.1.** In any $CN$-ring, any two elements $a$ and $x$ which satisfy $a^2x-a \in Z$ are commutative, that is, then $ax=xa$.

**Proof.** Since $a^2x-a \in Z$, $(a^2x-a)a = a(a^2x-a)$, and so (1) $a^3(ax-xa)=0$. Using (1), it follows that $[a(ax-xa)a]^2=0$, so that, since $A$ is $CN$, (2) $a(ax-xa)a \in Z$. Commuting this with $a$ and using (1), there results (3) $a(ax-xa)a^3=0$. Since $a^2x-a \in Z$, one easily verifies that

\[(4) \quad ax-xa=[a(ax-xa)+(ax-xa)a]x.\]

If (4) is multiplied on the left by $a$, using (1), the result can be simplified to $a(ax-xa)=a(ax-xa)a\sqrt{x}$, so that, by (2), one has

\[(5) \quad a(ax-xa)=a(ax-xa)a=xa(ax-xa)a.\]

Multiplying (5) on the right by $a$ produces (6) $a(ax-xa)a=xa(ax-xa)a^3$, which is $=0$ by (3). Reapplying this latter fact to (5) yields (7) $a(ax-xa)=0$. Thus, (4) can be simplified to (8) $ax-xa=(ax-xa)a\sqrt{x}$. From (7) $[(ax-xa)a]^2=0$, so that $(ax-xa)a \in Z$. Commuting this with $a$, and using (7), one obtains (9) $(ax-xa)a^3=0$. Since $(ax-xa)a \in Z$, (8) becomes $(ax-xa)=x(ax-xa)a$, and so, by (9), one has (10) $(ax-xa)a=xa(ax-xa)a^3x=0$. Then (8) reduces to $ax=xa$, which is the desired result.

**Corollary 1.2.** If $a$ and $x$ are elements of a ring $A$, and if the element $a^2x-a$ and all nilpotent elements of $A$ commute with both $a$ and $x$, then $a$ and $x$ commute.

**Proof.** Let $Q$ denote the subring of $A$ generated by $a$ and $x$, and let $\mathfrak{z}$ denote the center of $Q$. Then the condition of the corollary implies that $Q$ is a $CN$-ring, and that $a^2x-a \in \mathfrak{z}$, so that the corollary follows from the proposition.

An element $a$ of a ring $A$ is (von Neumann) **regular** if $axa=a$, and **strongly**

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$^3$ A $CN$-ring is a ring in which every nilpotent element belongs to the center. It seems that in each case where I assume that a ring is $CN$, I actually require only that the center contains all nilpotent elements of index two. I do not know whether this latter condition is equivalent to the $CN$ hypothesis.
regular\ (Arens and Kaplansky) if $a^2x = a$, for suitable $x \in A$. The next corollary in the case $n = 1$ shows that in CN-rings strong regularity of an element $a$ implies its regularity. The corollary follows from the proposition by observing that $a^{n+1}x = a^n$ implies $(a^n)^2x^n = a^n$.

**Corollary 1.3.** If $A$ is a CN-ring, then the equation $a^{n+1}x = a^n$ for two elements $a$ and $x$ in $A$, and a natural number $n$, implies the equation $a^n x^n = x^n a^n$.

**Lemma 1.4.** A directly irreducible CN-ring $A$ has an identity element $1 \neq 0$ if and only if there exist $a, x \in A$ such that $a^2x = a \neq 0$. Then $ax = xa = 1$.

**Proof.** By the corollary $a^2x = a$ implies $ax = xa$, so that $e = ax$ is a nonzero idempotent when $a \neq 0$. Since the sets $eA(1-e), (1-e)Ae$ are central, they commute with $e$, whereby they are $= 0$. By the direct irreducibility of $A, e = 1$. The converse is trivial.

In a s.r. extension of a division ring, to each element $a$ there corresponds an element $x$ such that $a^2x - a$ has certain regularity properties. This situation for directly irreducible rings is slightly generalized directly below, and following this a similar generalization of s.r. extensions of integral domains is considered.

**Theorem 1.5.** Let $A$ be a directly irreducible CN-ring containing a left identity $1 \neq 0$, and such that to each $a \in A$ there correspond $b \in A$ and a natural number $n = n_a$ such that either $a^{n+1}b - a^n = 0$, or else $a^{n+1}b - a^n$ has a right inverse in $A$. Then the totality $N$ of nilpotent elements of $A$ is an ideal, and $A - N$ is a division ring.

**Proof.** By Lemma 1.4, 1 is a two-sided identity. If $a^{n+1}b - a^n$ has the right inverse $x$, then $a$ has the right inverse $a^n bx - a^{n-1}x$. If $a^{n+1}b = a^n$, then, by Corollary 1.3, $a^n b^n = b^n a^n$, so that $e = a^n b^n$ satisfies $e^2 = e$. Then, by Lemma 1.4, either $e = 1$, whence $a$ has right inverse $a^{-1}b^n$, or else $0 = e = ea^n = a^n$. Thus, every nonnilpotent element has a right inverse. It is easy to see that this means that every nonnilpotent element has a two-sided inverse. Then (e.g., [9, p. 21]), since $N$ is a (central) ideal of $A, A - N$ is division.

**Remark.** One can show in general that in a ring $A$ with identity such that every nonnilpotent element has an inverse, that $N$ is a two-sided ideal such
that $A - N$ is division (cf. the proof of Lemma 1.9 below.)

**Corollary 1.6.** If $A$ is directly irreducible with a left identity $1 \neq 0$ such that to each $a \in A$ there corresponds $b \in A$ such that either $a^2b = a = 0$, or else $ab - a$ has a right inverse, then $A$ is division.

**Proof.** It is trivial to show that $N = 0$, so that $A$ is $CN$, and the theorem applies.

The corollary shows that a nonzero directly irreducible s.r. ring is division if only if there exists a left identity. In such a ring a two-sided identity exists, according to Lemma 1.4.

**Corollary 1.7.** A nonzero directly irreducible ring is s.r. if and only if it is division.

**Theorem 1.8.** If $A$ is a directly irreducible $CN$-ring, and if to each $a \in A$ there corresponds $b \in A$ and a natural number $n = n_a$ such that $a^{n+1}b - a^n$ is not a proper right divisor of zero in $A$, then the set $N$ of nilpotent elements of $A$ is a nil ideal, and $A - N$ is an integral domain.

**Proof.** In a way completely analogous to the proof of the last theorem, one sees that $N$ coincides with the set $D$ of all right divisors of $0$ in $A$. Thus, the theorem is a consequence of the following lemma. The lemma no doubt is known, but I have not been able to find a published proof. For this reason, I include one here.

**Lemma 1.9.** If $N = D$ in a ring $A$, then $N$ is an ideal of $A$, and $A - N$ is an integral domain.

**Proof.** If $N = 0$, there is nothing to prove. Now let $0 \neq x \in N$ have index of nilpotency $= m$. Then, since $(ax)x^{m-1} = 0$, $ax \in D = N$, for all $a \in A$, that is, $Ax \subseteq N$, for all $x \in N$. Since $(ax)^m = 0$ implies that $(xa)^m = 0$, this shows that $Ax \subseteq N$ implies that $xA \subseteq N$, so that $AxA \subseteq N$, for all $x \in N$. In order to show that $N$ is an ideal, it remains to show that $N$ is closed under addition. If $x, y \in N$, then, since $(x + y)^m \subseteq AxA + AyA$, it follows that $(x + y) \in N$. It remains to show that $A - N$ is integral. It suffices to show that $a \in N$, $b \in N$, $ab - q \in N$ leads to a contradiction. Clearly, $q \neq 0$, so $q$ is nilpotent of index $m \geq 2$. Since $(ab)^m = [(ab)^{m-1}a]b = 0$, $b \in D$ implies that $(ab)^{m-1}a = 0$. But $a \notin D$ implies that $(ab)^{m-1} = 0$, which is the desired contradiction.
A consequence of Corollary 1.6 is that a directly irreducible s.r. extension of a division ring is a division ring. In analogy with this fact one has

**Corollary 1.10.** A directly irreducible s.r. extension of an integral domain is an integral domain.

**Proof.** If $A$ is the extension, and $B$ the integral domain, then $A$ contains no nilpotent elements $\neq 0$. Thus, if $a, y \in A$, then $ay = 0$ if and only if $ya = 0$. Since $A$ is a $CN$-ring, by the theorem it suffices to show that $a - a^2b \in B$ is not a proper right divisor of zero in $A$. Hence assume that $0 \neq y \in A$ is such that $y(a - a^2b) = 0$, and $a - a^2b \neq 0$. Then $(a - a^2b)y = 0$, so that $(a - a^2b)(y - y^3c) = 0$, where $c$ can be chosen such that $y - y^3c \in B$. Since $B$ is integral, $y = y^3c$, so that, by Lemma 1.4, $yc = cy$ is the ring identity, which contradicts the choice of $y$ as a proper left divisor of 0.

Since a s.r. ring is a s.r. extension of every subring, it would seem that the hypothesis “$A$ is a s.r. extension of $B$” would have more force if one assumes at the outset that $A$ is not an s.r. ring. (Then $B$ is not s.r.!) For these rings the structure theory can be reduced in some cases to that of directly irreducible s.r. extensions.

**Proposition 1.11.** Let $B$ be a simple ring with identity $e$, and let $A$ be a s.r. extension of $B$, $A$ not a s.r. ring. Then $A = Q \oplus P$, where $Q$ is a directly irreducible s.r. extension of $B$ having the identity $e$, and $P$ is a s.r. ring. Conversely, $Q \oplus R$ is a s.r. extension of $B$, if $Q$ is any s.r. extension of $B$, and $P$ is any s.r. ring.

**Proof.** The sufficiency is clear. The necessity requires the following lemma which is also of interest in more general situations.

**Lemma.** If $B$ is a ring with a central idempotent $e$, and if $A/B$ is a s.r. extension, then $e$ is a central element of $A$.

**Proof of the Lemma.** $B$ contains all nilpotent elements of index two, so that $B$ contains the sets $eA(1 - e), (1 - e)Ae$. Since $e$ is central in $B$, these sets $= 0$, so $A = eAe \oplus (1 - e)A(1 - e)$, and $e$ is central.

Going back to the proof of the proposition, since $(1 - e)A(1 - e) \cap B = 0$, $P = (1 - e)A(1 - e)$ is s.r. as required. It remains to show that $Q = eAe$ is directly irreducible. To this end assume that $Q = M \oplus N$, where $M$ and $N$ are
ideals. \( M \cap B = 0 \) implies that \( M \) is s.r. If both \( M \) and \( N \) were s.r., then so would \( Q \), whence \( A \), be s.r., contrary to assumption. On the other hand, if \( M \cap B \neq 0 \), then \( M \cong B \), so that \( M = Q, N = 0 \), and \( Q \) is directly irreducible.

The existence of directly irreducible s.r. proper extensions, not division rings, of simple rings is guaranteed by the example in §4 of [4].

§2. Semisimple strongly regular extensions. The next theorem shows that in order that a ring \( B \) possess a primitive s.r. extension, it is necessary that \( B \) be a primitive ring, or an integral domain.

2.1. Structure Theorem. Let \( A \) be a primitive ring, not a division ring, which is represented as a dense ring of l.t.'s in a vector space \( V \) over a division ring \( D \). Then: if \( B \) is any subring of \( A \) such that \( A/B \) is s.r., then \( B \) is isomorphic to a dense ring of l.t.'s in \( V \).

Proof. Let \( V_n \) be a vector subspace of \( V \) of finite dimension \( n \), let \( U = \{ a \in A | V_n a \subseteq V_n \} \), and let \( K = \{ a \in A | V_n a = 0 \} \). Then, as is well known [9], the difference ring \( \overline{U} = U - K \) is isomorphic to \( D_n \), the complete ring of \( n \times n \) matrices over \( D \). First assume that \( n > 1 \), and let \( u \in U \) be such that \( u^2 \in K \). Then, if \( c \in A \) is such that \( u - u^2 c \in B \), then, since \( u^2 c \in K \), it follows that \( u - u^2 c \in Q = B \cap U \). Thus, the subring \( \overline{Q} \) determined by \( Q \) under the canonical homomorphism \( U \twoheadrightarrow \overline{U} \) contains every \( \overline{u} \in \overline{U} \) satisfying \( \overline{u}^2 = 0 \). By [7, p. 602, Proposition 1], \( \overline{Q} = \overline{U} \), that is, \( U = Q + K \), and, consequently, every l.t. of \( V_n \) is induced by an element of \( B \), in case \( n > 1 \). Now \( V_1 \) is contained in a subspace \( V_2 \), and if \( \overline{a} \) is any l.t. in \( V_1 \), there exists a l.t. \( \overline{a}_2 \) in \( V_2 \) such that \( a_2 \) induces \( \overline{a}_2 \). Then, if \( b \in B \) induces \( \overline{a}_2 \), then \( b \) also induces \( \overline{a}_2 \). Thus, in all cases, the l.t.'s in \( V_n \) can be induced by elements of \( B \). This establishes that \( B \) is isomorphic to a dense ring of l.t.'s in \( V \).

(1) of the next corollary is immediate.

Corollary 2.2. Let \( A \) be a s.r. extension of a ring \( B \). (1) If \( A \) is a primitive ring, not a division ring, then \( B \) is a primitive ring, and so is any intermediate ring of \( A/B \). (2) If \( A \) is semisimple, then \( B \) is a subdirect sum of primitive rings, and integral domains.

Proof. (2) Let \( \langle P \rangle \) denote the collection of primitive ideals in \( A \). Since
∩ \mathcal{P} = 0$, necessarily \( \cap (P \cap B) = 0 \), so that \( B \) is a subdirect sum of the rings \( \{B - (P \cap B)\} \). Now \( A - P \) is a s.r. extension of \((P + B) - P\), so that by the theorem: if \( A - P \) is not a division ring, then \((P + B) - P\) is primitive; \((P + B) - P\) is an integral domain, otherwise. (2) is completed by observing that \((P + B) - P\) is isomorphic to \(B - (P \cap B)\), for each \( P \in \{P\} \).

**Corollary 2.3.** Let \( A \) be a semisimple ring which is a s.r. extension of a commutative subring \( B \). Then \( A \) is a subdirect sum of division rings, and \( B \) is a subdirect sum of (commutative) integral domains. (If in addition \( A \) is subdirectly irreducible, then \( A \) is a division ring).

**Proof.** Let \( A' \) be any primitive homomorph of \( A \), and let \( B' \) denote the corresponding map of \( B \). Since \( A'/B' \) is s.r., and \( B' \) is commutative, density of \( B' \) in \( A' \) would imply commutativity of \( A' \), which in turn would imply that \( A' \) is a field. Thus, by the theorem, \( A' \) is a division ring, so that \( A \) is a subdirect sum of division rings. By the corollary, \( B \) must be a subdirect sum of (commutative) primitive rings and integral domains. Since a commutative primitive ring is a field, \( B \) has the desired structure. (The parenthetical remark is obvious).

3. **Commutativity theorems.** If \( S \) is a nonempty subset of a ring \( A \), then \([S]\) denotes the subring generated by \( S \). If \( R \) is a subring, then \( R[S] \) denotes the subring generated by \( R \) and \( S \). If \( A \) is a division ring, and if \( R \) is a division subring, \( R(S) \) is the division subring generated by \( R \) and \( S \).

Let \( \Phi \) be a commutative ring with identity. A ring \( A \) is a \( \Phi \)-ring (in the sense of Jacobson [8, p. 55]) if \( A \) is a unitary left \( \Phi \)-module satisfying \( c(xy) = (cx)y = x(cy) \) for all \( c \in \Phi \), and all \( x, y \in A \).

**Definition.** Let \( \Phi \) be a commutative ring with identity which contains a (possibly 0) subring \( K \) with the property that (1) a nonzero homomorph \( K' \) of \( K \) is an integral domain if and only if \( K' \) is an algebraically closed field, and (2) there exist finitely many \( c_1, \ldots, c_r \in \Phi \) such that \( \Phi = K[c_1, \ldots, c_r] \). Let \( A \) be a \( \Phi \)-ring, and \( B \) a \( \Phi \)-subring of \( A \) such that to each \( a \in A \) there corresponds a polynomial \( P_a(x) \) in the polynomial ring \( \Phi[x] \) such that

\[
a^n - a^{n+1} p_a(a) \in B
\]

for some natural number \( n \) depending on \( a \). Then \( A/B \) is an \( N \)-extension. If
A/B is an $N$-extension, then it is an $N_1$-extension if $n = 1$ for all $a \in A$, and it is an $N_2$-extension if $B$ contains all idempotents of $A$.

$N$-extensions have been studied extensively by Nakayama [12] (and others, see [12, References]) where the main result states that any ring $A$ which is an $N_1$-extension of its center $Z$ is commutative (or, more generally, any $CN$-ring which is an $N$-extension of its center is commutative.) This result had been obtained earlier by Nakayama in the case $K = 0$. In this case it is also true that a division ring $A$ is commutative if it is an $N$-extension of a division $\Phi$-subring $\cong A$ (Faith [4, Theorem 1]), a result which is extended to the $K \neq 0$ case below.

**Theorem 3.1.** Let $A$ be a division $\Phi$-ring, $\Phi$ as in the definition, and let $B$ be a $\Phi$-subring such that $A/B$ is an $N$-extension. Then: if $B$ is commutative, or if $B$ is a division subring $\cong A$, then $A$ is a field.

**Proof.** If $B$ is commutative, so is the division subring $(B)$ generated by $B$. If $(B) = A$, then $A$ is a field as required. Hence, it suffices to consider only the case where $A (\neq Z)$ is an $N$-extension of the division ring $B \cong A$.

Let $\mathbf{1}$ be the identity of $A$, and set $\psi = \Phi \mathbf{1}$. Since $\psi \neq 0$, $A$ and $B$ are algebras over the field $\overline{\psi}$ of quotients of $\psi$. In this case the results of [6] are applicable. The hypotheses imply that to each $a \in A$ there corresponds $p_a(x)$ with coefficients in $\psi(\cong \overline{\psi})$ such that $a^n - a^{n+1}p_a(a) \in B$. Under these conditions [6, Theorem 1.5] asserts that to each $b \in A$ there corresponds a polynomial $F_b(x)$ over $\overline{\psi}$ such that (i) $F_b(b) \in Z$, and (ii) $F_b(x)$ is the composition of finitely many of the polynomials in the set

$$\{x^n - x^{n+1}p_a(x) | a \in A, \ n = 1, 2, \ldots \}.$$ 

Clearly, then, the polynomial $F_b(x)$ has the form

$$F_b(x) = x^m - x^{m+1}g_b(x),$$

with $m = m(b) > 1$, and $g_b(x) \in \overline{\psi}[x]$. (It is important to note that the $F_b(x)$ are polynomials over $\overline{\psi}$.) The effect of all of this is to show that $A/Z$ is an $N$-extension, as defined above, so that $A = Z$ by the result of Nakayama.

**Theorem 3.2.** Let $A$ be a $\Phi$-ring, $\Phi$ as in the definition, and let $B$ be a commutative $\Phi$-subring such that $A/B$ is an $N_1$-extension. If either $A$ is semi-
simple, or $B \cap J(A) = 0$, where $J(A)$ denotes the Jacobson radical of $A$, then $A$ is commutative.

**Proof.** Since $A$ is semisimple (if $B \cap J(A) = 0$, then $J$ is s.r., so $J = 0$), $A$ is a subdirect sum of division rings $A'$ by Corollary 2.3. Each $A'$ can be regarded as a $\Phi$-ring, and it follows that each $A'$ is an $N_1$-extension of a commutative subring, so that each $A'$ is commutative by Theorem 3.1. Then $A$ is commutative.

Below, if a ring is an $N_1$-extension of $0$, then it is an $N_1$-ring. By Nakayama's result, every $N_1$-ring is commutative. If $A$ is an $N_1$-extension of a simple subring $B$, and if $B$ has an identity $e$, it follows from the lemma to Proposition 1.11 that $A = eAe \oplus (1 - e)A(1 - e)$. Since $(1 - e)A(1 - e)$ is an $N_1$-ring, it is commutative. Now suppose that $eAe = M \oplus N$, where $M$ and $N$ are ideals. If both $M$ and $N$ are disjoint from $B$, then both $M$ and $N$ are $N_1$-rings, whence they are commutative. Thus, if $eAe$ is noncommutative, it can be assumed that, say, $B \cap M \neq 0$. Then, by the simplicity of $B$, $B \cong M$, and, since $M$ now contains the identity $e$ of $eAe$, $M = eAe$, $N = 0$, so that $eAe$ is directly irreducible. This establishes the lemma.

**Lemma 3.3.** If $A$ is an $N_1$-extension of a simple $\Phi$-subring $B$, and if $B$ contains an identity element $e$, then

$$A = Q \oplus P,$$

where $Q = eAe$, and $P = (1 - e)A(1 - e)$ is a (commutative) $N_1$-ring. Furthermore, either $A$ is commutative, or else $eAe = Q$ is directly irreducible.

Now suppose that $B$ in the lemma is a division $\Phi$-subring. Then, if $A$ is noncommutative, $Q$ is a directly irreducible $N_1$-extension of $B$. Since Corollary 1.7 shows that $Q$ is a division ring, it follows from Theorem 3.1 that either $B = Q$, or else $Q$ is a field. This completes the proof of the next theorem.

**Theorem 3.4.** Let $A$ be a $\Phi$-ring, and $B$ a division $\Phi$-subring such that $A/B$ is an $N_1$-extension. Then, either $A$ is commutative, or else $A = B \oplus P$, where $P$ is a (commutative) $N_1$-ring. Furthermore, if $A$ is directly irreducible, and $B \nleq A$, then $A$ is a field.

The theorem and the discussion preceding have the corollary.
Corollary 3.5. If $A$ is a $\Phi$-ring which is an $N_v$-extension of a $\Phi$-subfield $B$, then $A$ is commutative.

§ 4. $\xi_v$-extensions. The extension $A/B$ is a $\xi_v$-extension in case to each $a \in A$ there exist $x = x_v \in A$ and a natural number $n = n_v$ such that $a^n - a^{n+1}x \in B$. If $x$ can be chosen such that $x^n$ commutes with $a^n$, for every $a \in A$, then a $\xi_v$-extension is a $\xi_v'$-extension. A $\xi_v$-extension is $\xi_v$, if $B$ contains all idempotents of $A$, and $\xi_v'$ if it is both $\xi_v$ and $\xi_v'$.

If $A$ is a $\Phi$-ring, where $\Phi$ is a commutative ring with identity, and if $B$ is a $\Phi$-subring such that to each $a \in A$ there correspond $p_a(x) \in \Phi[x]$ and a natural number $n = n_v$ such that $a^n - a^{n+1}p_a(a) \in B$, then $A/B$ is a $\xi_v'$-extension; it is $\xi_v'$ if $p_a(e) = 0$ for each idempotent $e \in A$. Thus, the results of this section are applicable to these extensions; in particular, they are applicable to $N_v$-extensions.

A ring $A$ is a $\xi_v'$-ring if it is a $\xi_v'$-extension of 0. It is trivial to verify that any $\xi_v'$-ring is a nil ring, and conversely. If $A/B$ is $\xi_v'$, and if $L$ is any left ideal disjoint from $B$, then $L$ is nil. To see this, if $a \in L$, and if $a^n - a^{n+1}x \in B$, then $0 = a^n - a^{n+1}x \in B \cap L = 0$. Since $a^n x^n = x^n a^n$, this implies that $e = a^n x^n$ is idempotent. Since $e \in L \cap B = 0$, then $a^n = ea^n = e = 0$, so that $L$ is nil. This fact is used several times below.

Theorem 4.1. If $A$ is a $\xi_v'$-extension of a simple ring $B$, and if $J(A) \neq A$, then $J(A)$ is nil, and $A - J(A)$ is primitive.

Proof. Suppose for the moment that $J(A) \nsubseteq B$. Then $A - J(A)$ would be a $\xi_v'$-ring, whence it is a nil ring. This would imply that $A = J(A)$, which is excluded by hypothesis. Hence $J(A) \nsubseteq B$, so that $J(A) \cap B = 0$, whence $J(A)$ is nil. Now $B$ cannot be contained in every primitive ideal of $A$, since the intersection of these is $J(A)$. Hence there exists a primitive ideal $P$ which is disjoint from $B$. Then $P$ is nil, whence $P = J(A)$, and $A - J(A)$ is primitive.

Now suppose that $A$ is a ring with no nil ideals $\neq 0$ which is a $\xi_v'$-extension of a division subring $B$. By the theorem, $A$ is primitive, but, as a matter of fact, $A$ is division. The proof of this is similar to the proof of the theorem, except that one considers the modular maximal left ideals (m.m.l.-ideals) of $A$ instead of the primitive ideals. Since $A$ contains no nil left ideals, one concludes that 0 is a m.m.l.-ideal, that is, that $A$ is a division ring. This fact is stated
in the next theorem.

**Theorem 4.2.** If $A$ is a ring with no nil ideals $\neq 0$, and if $A$ is a $\xi'$-extension of a division subring, then $A$ is a division ring.

The corollary below is a consequence of the theorem, and of Theorem 3.1.

**Corollary 4.3.** If $A$ is a ring with no nil ideals $\neq 0$, and if $A$ is a $N\xi'$-extension of a division subring $B \cong A$, then $A$ is a field.

The last two results can be restated as follows: If $A$ is a ring containing no nonzero idempotents $\neq 1$, and containing no nonzero nil ideals, and if $A$ is a $\xi'$-extension (resp. $N\xi'$-extension) of a division subring $B \cong A$, then $A$ is a division ring (resp. field.)

The corollary generalizes results on radical extensions of [4] and [5].

If $A$ is a radical extension of an integral domain, then to each $a \in A$ there corresponds a natural number $n$ such that $a^n$ has certain regularity properties. The situation is generalized below.

**Theorem 4.4.** Let $A$ be a ring with the property that to each $a \in A$ there corresponds a natural number $n = n_a$ such that $a^n$ is not a proper right divisor of zero in $A$. Then the set $N$ of nilpotent elements is an ideal, and $A - N$ is an integral domain.

**Proof.** Let $D$ denote the set of all right divisors of zero in $A$. The condition of the theorem implies that $N = D$, so that the theorem follows from Lemma 1.9.

**Remark.** If $A$ is a ring with a nil ideal $N$ such that $A - N$ is integral, then, of course, $D = N$ in $A$, and $A$ has the property of the theorem.

Now let $A$ be a radical extension of an integral domain $B$, that is, such that to each $a \in A$ there corresponds a natural number $n = n_a$ such that $a^n \in B$. Assume that $A$ contains no nil left ideals $\neq 0$, let $x \in A$ be nonnilpotent, and let $y \in L_x = \langle a \in A | ax = 0 \rangle$. Then, since $y^m x^n = 0$, $m = m_y$. $n = n_x$, since $B$ is integral, and since $x^n \neq 0$, then $y^m = 0$. $L_x$ is therefore nil, so $L_x = 0$. This shows that each $a \in A$ has the property stated in the theorem, and completes the proof of the corollary.

**Corollary 4.5.** If $A$ is a ring with no nil left ideals $\neq 0$, and if $A$ is a radical extension of an integral domain, then $A$ is an integral domain.
A commutative integral domain $A$ can be radical over a subring $B$ only under very special circumstances. For then, if $A^*$ and $B^*$ denote the respective quotient fields of $A$ and $B$, then $A^*$ is radical over $B^*$. It follows from the work of Kaplansky [Canad. J. Math. vol. 3 (1951) 290-292] that either $A^* = B^*$, or else, $A^*$ has characteristic $p > 0$, and either $A^*/B^*$ is purely inseparable, or else $A^*$ is algebraic over $GF(p)$. It would be interesting to know the corresponding situation for noncommutative integral domains (cf. [5] for some results with added hypotheses on $A$ and $B$).

5. Generation of primitive rings. If $A$ is a ring, let $T(A)$ denote the subring generated by all nilpotent elements of index two, and let $E(A)$ be the subring generated by all idempotents. If $A$ is primitive, and $A/B$ is s.r., then by Corollary 2.2, $B$ is dense in the finite topology on $A$, if $A$ is not division. In view of the fact that $B$ contains $T(A)$ when $A/B$ is s.r., it would be interesting to know if any subring of $A$ which contains $0 \neq T(A)$ is dense in $A$. Positive results abound in special cases, making a counterexample hard to find.

**Theorem 5.1.** If $A$ is a primitive ring with a minimal left ideal, and if $A$ is not a division ring, then $T(A)$ and $E(A)$ are dense in the finite topology on $A$. (Then $T(A)$ and $E(A)$ are primitive rings).

Let $S$ denote the socle of $A$. It suffices to show that $T(S) = E(S) = S$, since then density follows from the inclusions $T(A) \supseteq S$, $E(A) \supseteq S$. Thus, the theorem is a consequence of the lemma below. (In case $A$ does not satisfy the minimum condition, then the theorem follows immediately from Rosenberg's generalization [Proc. Amer. Math. Soc. vol. 7 (1956) p. 897, Corollary 5] of a theorem of Kasch [10]).

**Lemma 5.2.** (a) If $A$ is a simple ring containing a nontrivial idempotent, then $T(A) = A$. If, in addition, (b) $A$ is an algebra over a field $\Phi \cong GF(2)$, or (c) if $A$ contains a minimal left ideal, then $E(A) = A$.

**Proof.** (a) Let $T$ denote the additive subgroup generated by all nilpotent elements of index two, and let, for any subset $S$ of $A$, $[S, S]$ denote the additive subgroup generated by all $[a, b] = ab - ba$, $a, b \in S$. If $u, v \in T$ are nilpotent of index two, then so is

$$w = (1 + u) v (1 - u),$$
Then,

\[ [u, v] = w + uvu - v \in \mathcal{I} . \]

It easily follows from this that \( \mathcal{I} \) is a Lie ring with respect to \([a, b]\). Then Amitsur's [1, Lemma 2] shows that \( \mathcal{I} \cong [A, A] \), so that \( T(A) \cong [A, A] \). If \( e \) is any nontrivial idempotent in \( A \), then \( eAf, fAe \subseteq \mathcal{I} \subseteq T(A) \), where (formally) \( f = 1 - e \). But \( T(A) \) also contains the product

\[ eAe = e(AfA)e = (eAf)(fAe), \]

similarly, \( fAf \subseteq T(A) \). Then \( T(A) = A = eAe + eAf + fAe + fAf \), as needed.

(b) In this case Amitsur's [1, Theorem 1] states that \( A \) contains no non-invariant noncentral subalgebras \( \neq A \), unless \( A \) is 4-dimensional over a field \( F \) of characteristic two. Since \( E(A), T(A) \) are invariant noncentral subalgebras, equality \( E(A) = T(A) = A \) follows when \( \dim A/F \neq 4 \). In this exceptional case, \( A \) is a simple matrix algebra. A general property of arbitrary matrix algebras \( A = R_n, n > 1 \), implied by [7, Prop. 1] is that \( E(A) = T(A) = A \). This latter result also suffices for the case (c), since \( A \) is then locally a complete matrix ring \( R_n, n > 1 \), by Litoff's theorem [8, p. 90].

**Theorem 5.3.** Let \( A \) be an algebraic algebra over the field \( \Phi \). (a) If \( A \) is primitive, but not division, then \( E(A) \) and \( T(A) \) are dense in the finite topology on \( A \). (Then \( E(A) \) and \( T(A) \) are primitive algebras.) (b) If \( A \) is semisimple, so is \( E(A) \).

**Proof.** (a) The proof is analogous to that of Theorem 2.1. Adopting the terminology there, with \( B = E(A) \) (resp. \( B = T(A) \)), if \( \bar{e} \) is any element in a complete set of matrix units for \( \bar{U} \), by [9, p. 239 ff.,] there exists an element \( \bar{f} \) in a complete set of matrix units in \( U \) such that \( \bar{f} = \bar{e} \). If \( \bar{e}^2 = \bar{e} \) (resp. \( \bar{e}^2 = 0 \)), then, since \( f \in E(A) \) (resp. \( f \in T(A) \)), it follows that \( \bar{e} \in \bar{Q} \). Since any automorphism of \( \bar{U} \) maps a complete set of matrix units onto another complete set, this latter assertion shows that \( \bar{Q} \) contains all conjugates of \( \bar{e} \). Since \( \bar{U} = D_n, n > 1 \), by [7, Prop. 1], \( \bar{U} \) is generated by the conjugates of \( \bar{e} \), so that \( \bar{U} = \bar{Q} \). The rest of the proof is unchanged.

(b) It is not hard to show that a subring (subalgebra) \( B \) of a semisimple ring (algebra) \( A \) is itself semisimple, if each homomorphism of \( A \) which maps \( A \) onto a primitive ring (algebra) also maps \( B \) onto a primitive ring (algebra).
(The proof of this is related to that of Corollary 2.2). Thus, if $A$ is a semi-simple algebraic algebra, and if $P$ is any primitive ideal of $A$, then [9, p. 239 ff.] shows that the canonical map $A \rightarrow A - P$ maps $B = E(A)$ onto $E(A - P)$. If $A - P$ is not division, then $E(A - P)$ is primitive by (a), while if $A - P$ is division, since it is an algebraic division algebra, every nonzero subalgebra is a division algebra. Thus $E(A - P)$ is primitive in this case too, and the semisimplicity of $B$ follows from the remark above.

Relating to Lemma 5.2 is the question whether $T(A) = A$ in a simple ring (algebra) $A$ implies the equality $E(A) = A$.

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