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On the relevant domain of the Hilbert function of a finite multiprojective scheme

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Abstract. **Let X be a zero-dimensional reduced subscheme of a multiprojective space $\mathbb{V}$. Let s_i be the length of the projection of X onto the i-th component of $\mathbb{V}$. A result of Van Tuyl states that the Hilbert function of X is completely determined by its restriction to the product of the intervals $[0, s_i - 1]$. We extend this result to arbitrary zero-dimensional subschemes of $\mathbb{V}$.**

1 Introduction

Let q and $n_1, \ldots, n_q$ be positive integers. Let $\mathbb{K}$ be a field. Consider the multiprojective space $\mathbb{V} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ over $\mathbb{K}$. The coordinate ring of $\mathbb{V}$ is the $\mathbb{Z}^q$-graded algebra

$$\mathbb{S} = \mathbb{K}[x_{ij} \mid 1 \leq i \leq q,\ 0 \leq j \leq n_i].$$

We have $\deg(x_{ij}) = e_i$, where $e_i \in \mathbb{Z}^q$ is the i-th basis element. Let M be a finitely generated $\mathbb{Z}^q$-graded $\mathbb{S}$-module. The *Hilbert function* $\mathcal{H}_M : \mathbb{Z}^q \to \mathbb{Z}$ of M is defined by $\mathcal{H}_M(a) = \dim_{\mathbb{K}}(M_a)$. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme and let $I(X) \subset \mathbb{S}$ be the ideal generated by the $\mathbb{Z}^q$-homogeneous forms in $\mathbb{S}$ that vanish on X. The Hilbert function $\mathcal{H}_X$ of X is defined to be the Hilbert function of $\mathbb{S}/I(X)$.

The exploration of the Hilbert functions in the multiprojective setting is a natural extension of the rich theory of Hilbert functions of zero-dimensional subschemes of $\mathbb{P}^n$. The simplest case, when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$, was first investigated by Giuffrida et al. in [11]. This exploration was then continued by many authors. The case of $\mathbb{P}^1 \times \mathbb{P}^1$ remains the most assiduously studied case, see [3], [4], [7], [12], [13], [14], [15], [16], [17], [21], [22], [26]. For other ambient spaces $\mathbb{V}$ we refer to [1], [4], [8], [9], [25], [26]. The theory now follows three broad directions of development: Hilbert functions of sets of points, as in [8], [16], [17], [21], [22]; Hilbert functions of sets of fat points, as in [3], [4], [7], [13], [14], [15], [24]; and Hilbert functions of ACM schemes, as in [8], [9], [13], [14], [16], [17], [21], [22], [26].

In connection with the first and third directions of development, we mention two fundamental results belonging to Van Tuyl. According to [25], if X is reduced and $\mathbb{K}$ is algebraically closed, then $\mathcal{H}_X$ is uniquely determined by its restriction to a rectangular region of the form $R = [0, r_1] \times \cdots \times [0, r_q] \subset \mathbb{Z}^q$. More precisely, R is a relevant domain for $\mathcal{H}_X$ in the sense of Definition 2.1 and Lemma 2.2. Our first achievement is the generalization of this result to the case of an arbitrary zero-dimensional subscheme $X \subset \mathbb{V}$ over an arbitrary ground field $\mathbb{K}$. See Theorem 4.4. If $\mathbb{K}$ is algebraically closed, then, according to [26], the functions $\mathcal{H}_X$, where X runs through the zero-dimensional

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reduced ACM subschemes of $\mathbb{V}$, are precisely the functions $\mathcal{H}$ whose difference $\Delta\mathcal{H}$, defined at Equation (1), is the Hilbert function of an artinian $\mathbb{Z}^q$-graded quotient of $\mathbb{S}/(x_{10}, \dots, x_{q0})$. Our second achievement is the generalization of this result to the case of an arbitrary zero-dimensional subscheme $X \subset \mathbb{V}$ over an arbitrary infinite ground field $\mathbb{K}$. See Theorem 6.6.

We give three proofs to Theorem 4.4. The first proof consists of comparing the cohomology of the twists of $\mathcal{I}_X$ with the cohomology of the twists of the ideal sheaf of X in a smaller ambient space $W_i \subset \mathbb{V}$. Here W_i is obtained from $\mathbb{V}$ by replacing $\mathbb{P}^{n_i}$ with the projection of X onto $\mathbb{P}^{n_i}$. See Lemma 4.1. The second proof, located in Section 7, applies only in the case when $\mathbb{K}$ is algebraically closed, $\mathbb{V} = (\mathbb{P}^1)^q$ and X is ACM or sub-ACM (meaning depth$(\mathbb{S}/I(X)) = q - 1$). The technique we use draws on the technique of Giuffrida et al., who dealt with the case when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$. The key ingredients here are the constraints satisfied by the Hilbert function of an ACM or sub-ACM scheme. See Propositions 7.1 and 7.2. The third proof of Theorem 4.4, located in Section 9, applies only in the case when $\mathbb{K}$ is infinite and $\mathbb{V} = (\mathbb{P}^1)^q$. It is based on Macaulay's theorem and on a vanishing criterion for the difference $\Delta\mathcal{H}_{\mathbb{S}/J}$ of the Hilbert function of the quotient of $\mathbb{S}$ by a monomial ideal. See Proposition 8.4. We think that approaching Theorem 4.4 from three different angles provides a clearer picture of the subtleties that arise in the study of multiprojective Hilbert functions.

An important consequence of Theorem 4.4 is an upper estimate on the regularity index of $\mathbb{S}/I(X)$, regarded as a $\mathbb{Z}$-graded $\mathbb{S}$-module, in terms of the regularity indices of the projections of X onto the components $\mathbb{P}^{n_i}$ of $\mathbb{V}$. See Corollary 4.7.

Van Tuyl's method for proving his version of Theorem 6.6 consists of finding a regular sequence $\{u_1, \dots, u_q\}$ for $\mathbb{S}/I(X)$, as in Proposition 5.6. We adapt Van Tuyl's argument to the case when $X \subset \mathbb{V}$ is an arbitrary zero-dimensional subscheme and $\mathbb{K}$ is an arbitrary infinite field.

In this paper we also consider quasi-rectangular domains, that is, finite unions of rectangular domains, which are relevant to $\mathcal{H}_X$ in the sense of Definition 2.1. The third achievement of this paper is Proposition 9.5, which gives sufficient conditions for the existence of quasi-rectangular relevant domanins that are strictly contained in R. The problem of describing all quasi-rectangular domains $Q \subset R$ that are relevant for $\mathcal{H}_X$ remains open. An important class of schemes X for which this problem has been settled is the class of ACM subschemes of $(\mathbb{P}^1)^q$. See Corollary 6.7.

We now present the outline of the paper. In Section 2 we gather a few elementary facts about relevant domains. In Section 3 we examine complete intersections and we collect a few well-known facts about Hilbert functions of finite subschemes of $\mathbb{P}^n$. These facts will be needed in the proof of our first main theorem, concerning the rectangular relevant domain, to which Section 4 is devoted. In Section 5, whose role is to prepare the ground for the next two sections, we construct regular sequences for $\mathbb{S}/I(X)$ and for $I(X)$ in the case when X is ACM or sub-ACM. Section 6 contains our second main theorem, concerning ACM schemes. In Section 7 we combine the results of Section 5 with Lemma 3.5 in order to obtain inequalities involving the partial difference functions of $\mathcal{H}_X$ and $\mathcal{H}_{I(X)}$. As an application, we obtain our second proof of Theorem 4.4. In Section 8 we find a formula for $\Delta\mathcal{H}_{\mathbb{S}/J}$, where J is a monomial ideal. This leads us to our vanishing criterion for $\Delta\mathcal{H}_{\mathbb{S}/J}$. Section 9 contains our third proof of Theorem 4.4 and our procedure for detecting quasi-rectangular relevant domains.

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2 Relevant domains

Let q be a positive integer. Given $a = (a_1, \dots, a_q)$ and $b = (b_1, \dots, b_q)$ in $\mathbb{Z}^q$, we write $a \le b$ if $a_i \le b_i$ for all indices $i \in \{1, \dots, q\}$. Let $e_i = (0, \dots, 1, \dots, 0)$ be the element of $\mathbb{Z}^q$ that has entry 1 on position i and entries 0 elsewhere. Let $\mathcal{F} : \mathbb{Z}^q \to \mathbb{Z}$ be a function. We introduce the *difference* function $\Delta\mathcal{F} : \mathbb{Z}^q \to \mathbb{Z}$ by the formula

$$\Delta\mathcal{F}(a) = \mathcal{F}(a) + \sum_{1 \le p \le q} (-1)^p \sum_{1 \le i_1 < \cdots < i_p \le q} \mathcal{F}(a - e_{i_1} - \cdots - e_{i_p}). \tag{1}$$

In this paper we only consider functions $\mathcal{F}$ that vanish on the complement of the positive quadrant $\mathbb{Z}^q_+ = \{a \in \mathbb{Z}^q \mid a \ge 0\}$ because we are chiefly interested in Hilbert functions of ideals in $\mathbb{S}$. For such functions we can recover $\mathcal{F}$ from $\Delta\mathcal{F}$ by means of the formula

$$\mathcal{F}(a) = \sum_{0 \le b \le a} \Delta\mathcal{F}(b) \quad \text{for all} \quad a \in \mathbb{Z}^q_+. \tag{2}$$

Given $r, s \in \mathbb{Z}^q$ such that $r \le s$, we write $[r, s] = \{a \in \mathbb{Z}^q \mid r \le a \le s\}$. A *rectangular domain* in $\mathbb{Z}^q$ has the form $R = [0, r]$, for some $r \in \mathbb{Z}^q_+$. A *quasi-rectangular domain* $Q \subset \mathbb{Z}^q$ is a finite union of rectangular domains. The *boundary* B_Q of Q is defined to be the boundary of Q inside $\mathbb{Z}^q_+$:

$$B_Q = \{a \in Q \mid a + e_{i_1} + \cdots + e_{i_p} \notin Q \text{ for some indices } 1 \le i_1 < \cdots < i_p \le q\}.$$

In particular, for $R = [0, r]$, $B_R = \{a \in R \mid a_i = r_i \text{ for some index } 1 \le i \le q\}$.

Definition 2.1. Under the above notation, a quasi-rectangular domain Q is said to be *relevant* to $\mathcal{F}$ if $\Delta\mathcal{F}(a) = 0$ for all $a \in \mathbb{Z}^q \setminus Q$.

Lemma 2.2. *A rectangular domain $[0, r] \subset \mathbb{Z}^q$ is relevant to $\mathcal{F}$ if and only if for every i in $\{1, \dots, q\}$ and for every $a \in \mathbb{Z}^q$ such that $a_i \ge r_i$ we have $\mathcal{F}(a) = \mathcal{F}(a_1, \dots, r_i, \dots, a_q)$.*

Proof. Assume that $R = [0, r]$ is relevant to $\mathcal{F}$ and choose $a \in \mathbb{Z}^q_+$ such that $a_i \ge r_i$. Equation (2) can be rewritten as

$$\mathcal{F}(a) = \sum_{0 \le b_1 \le a_1} \cdots \sum_{0 \le b_i \le r_i} \cdots \sum_{0 \le b_q \le a_q} \Delta\mathcal{F}(b) + \sum_{0 \le b_1 \le a_1} \cdots \sum_{r_i < b_i \le a_i} \cdots \sum_{0 \le b_q \le a_q} \Delta\mathcal{F}(b).$$

The first summation equals $\mathcal{F}(a_1, \dots, r_i, \dots, a_q)$, again by virtue of Equation (2). The second summation vanishes because $\Delta\mathcal{F}(b) = 0$ if b lies outside R.

Conversely, assume that for every index $i \in \{1, \dots, q\}$ and for every $a \in \mathbb{Z}^q$ such that $a_i \ge r_i$ we have the equation $\mathcal{F}(a) = \mathcal{F}(a_1, \dots, r_i, \dots, a_q)$. This is equivalent to saying that for every index $i \in \{1, \dots, q\}$ and for every $a \in \mathbb{Z}^q$ such that $a_i > r_i$ we have the equation $\mathcal{F}(a) = \mathcal{F}(a - e_i)$. Choose $a \in \mathbb{Z}^q_+ \setminus R$. There is an index i such that $a_i > r_i$. Equation (1) can be rewritten in the form

$$\begin{aligned} \Delta\mathcal{F}(a) &= \big(\mathcal{F}(a) - \mathcal{F}(a - e_i)\big) \\ &+ \sum_{1 \le p \le q-1} (-1)^p \sum_{\substack{1 \le j_1 < \cdots < j_p \le q \\ j_1, \dots, j_p \ne i}} \big(\mathcal{F}(a - e_{j_1} - \cdots - e_{j_p}) - \mathcal{F}(a - e_i - e_{j_1} - \cdots - e_{j_p})\big). \end{aligned}$$

All terms in parentheses vanish, hence $\Delta\mathcal{F}(a) = 0$. Thus, R is relevant to $\mathcal{F}$. ■

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Lemma 2.3. *Assume that the rectangular domain* $[0, r]$ *is relevant to* $\mathcal{F}$*. Then* $\mathcal{F}(a) = \mathcal{F}(r)$ *for all* $a \in \mathbb{Z}^q$ *such that* $a \geq r$*.*

Proof. Applying Lemma 2.2 repeatedly, we obtain the equations

$$\mathcal{F}(a) = \mathcal{F}(r_1, a_2, \dots, a_q) = \mathcal{F}(r_1, r_2, a_3, \dots, a_q) = \cdots = \mathcal{F}(r). \quad \blacksquare$$

Remark 2.4. The above lemmas show that, if R is relevant to $\mathcal{F}$, then $\mathcal{F}|_{B_R}$ determines the function $\mathcal{F}$ on the complement of R. The same is true for a relevant quasi-rectangular domain Q. Take for instance $Q = [0, s] \setminus [r, s]$ in $\mathbb{Z}^2$, where $0 < r_1 < s_1$, $0 < r_2 < s_2$, and take $a \in [r, s]$. We have the equation

$$\mathcal{F}(a) = \mathcal{F}(a_1, r_2 - 1) + \mathcal{F}(r_1 - 1, a_2) - \mathcal{F}(r_1 - 1, r_2 - 1)$$

and $(a_1, r_2 - 1)$, $(r_1 - 1, a_2)$, respectively, $(r_1 - 1, r_2 - 1)$ lie on B_Q.

Given $a \in \mathbb{Z}^q_+$ we write $|a| = a_1 + \cdots + a_q$. Let $\mathcal{F} : \mathbb{Z}^q \to \mathbb{Z}$ be a function that vanishes on the complement of $\mathbb{Z}^q_+$. Let $\overline{\mathcal{F}} : \mathbb{Z}_+ \to \mathbb{Z}$ be given by the formula

$$\overline{\mathcal{F}}(d) = \sum_{a \in \mathbb{Z}^q_+, |a| = d} \mathcal{F}(a).$$

Lemma 2.5. *Let* $\mathcal{F} : \mathbb{Z}^q \to \mathbb{Z}$ *be a function that vanishes on the complement of* $\mathbb{Z}^q_+$*. Assume that* $[0, r]$ *is relevant to* $\mathcal{F}$*. Then the restriction of* $\overline{\mathcal{F}}$ *to* $[|r|, \infty)$ *is a polynomial function in the variable* d*, with rational coefficients and with dominant term* $\mathcal{F}(r)d^{q-1}/(q-1)!$

Proof. Let us write $R = [0, r]$ and $r = (r_1, \dots, r_q)$. Given $b \in B_R$ there is an integer $p = p_b \in \{1, \dots, q\}$ and there are indices $1 \leq i_1 < \cdots < i_p \leq q$ such that $b_i < r_i$ for $i \in \{1, \dots, q\} \setminus \{i_1, \dots, i_p\}$ and $b_{i_1} = r_{i_1}, \dots, b_{i_p} = r_{i_p}$. We consider the set

$$A(b) = \{a \in \mathbb{Z}^q_+ \mid a_{i_1} \geq r_{i_1}, \dots, a_{i_p} \geq r_{i_p},\ a_i = b_i \text{ for } i \in \{1, \dots, q\} \setminus \{i_1, \dots, i_p\}\}.$$

According to Lemma 2.2, $\mathcal{F}(a) = \mathcal{F}(b)$ for all $a \in A(b)$. For $d \geq |b|$ we consider the set

$$A_d(b) = \{a \in A(b) \mid |a| = d\}.$$

We now recall the fact that, for a fixed non-negative integer s, the number of integer solutions to the equation $c_1 + \cdots + c_p = s$, with unknowns $c_i \geq 0$, is $\binom{s+p-1}{p-1}$. For $a \in A_d(b)$ we have the equation $(a_{i_1} - r_{i_1}) + \cdots + (a_{i_p} - r_{i_p}) = d - |b|$, hence

$$|A_d(b)| = \binom{d - |b| + p - 1}{p - 1}.$$

Assume now that $d \geq |r|$. The decomposition $\{a \in \mathbb{Z}^q_+ \mid |a| = d\} = \bigsqcup_{b \in B_R} A_d(b)$ leads us to the following expression for $\overline{\mathcal{F}}(d)$:

$$\sum_{\substack{a \in \mathbb{Z}^q_+ \\ |a| = d}} \mathcal{F}(a) = \sum_{b \in B_R} \sum_{a \in A_d(b)} \mathcal{F}(a) = \sum_{b \in B_R} \mathcal{F}(b)|A_d(b)| = \sum_{b \in B_R} \mathcal{F}(b)\binom{d - |b| + p_b - 1}{p_b - 1}.$$

The r.h.s. is a polynomial function in d with rational coefficients. The assertion about the dominant term follows from the fact that $p_r = q$ while $p_b < q$ for $b \in B_R \setminus \{r\}$. $\blacksquare$

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Definition 2.6. The polynomial $\mathcal{P}(d) \in \mathbb{Q}[d]$ of Lemma 2.5, satisfying the relation $\mathcal{P}(d) = \overline{\mathcal{F}}(d)$ for $d \geq |r|$, will be called the *Poincarè polynomial* associated to $\mathcal{F}$.

Lemma 2.7. *Let $\mathcal{F} : \mathbb{Z}^q \to \mathbb{Z}$ be a function that vanishes on the complement of $\mathbb{Z}_+^q$. Assume that the rectangular domain $R = [0, r]$ is relevant to $\mathcal{F}$. Assume, in addition, that there is an integer $0 \leq \tau < |r|$ such that $\mathcal{F}$ is constant on the set $T = \{a \in R \mid |a| \geq \tau\}$. Then $\overline{\mathcal{F}}(d) = \mathcal{P}(d)$ for $d \geq \tau$, where $\mathcal{P}$ is the Poincarè polynomial associated to $\mathcal{F}$.*

Proof. We adopt the notation from the proof of Lemma 2.5. For $d \geq \tau$ we denote

$$C_d = \{a \in \mathbb{Z}_+^q \mid |a| = d\} \setminus \bigcup_{b \in B_R \setminus T} A_d(b).$$

By hypothesis, $\mathcal{F}(a) = \mathcal{F}(r)$ if $a \in T$. According to Lemma 2.2, $\mathcal{F}(a) = \mathcal{F}(r)$ if a lies in $A(b)$ for some $b \in B_R \cap T$. Thus, $\mathcal{F}(a) = \mathcal{F}(r)$ if $a \in C_d$. For $d \geq \tau$ we calculate:

$$\begin{aligned}
\overline{\mathcal{F}}(d) &= \sum_{a \in \mathbb{Z}_+^q, |a|=d} \mathcal{F}(a) \\
&= \sum_{a \in C_d} \mathcal{F}(a) + \sum_{b \in B_R \setminus T} \sum_{a \in A_d(b)} \mathcal{F}(a) \\
&= \mathcal{F}(r)|C_d| + \sum_{b \in B_R \setminus T} \mathcal{F}(b)|A_d(b)| \\
&= \mathcal{F}(r) \left[\binom{d+q-1}{q-1} - \sum_{b \in B_R \setminus T} \binom{d-|b|+p_b-1}{p_b-1} \right] \\
&\quad + \sum_{b \in B_R \setminus T} \mathcal{F}(b) \binom{d-|b|+p_b-1}{p_b-1}.
\end{aligned}$$

This is a polynomial expression in the variable d, which, in view of Lemma 2.5, must coincide with $\mathcal{P}(d)$. ■

3 Generalities concerning Hilbert functions

We let $\mathbb{V} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a multiprojective space over a field $\mathbb{K}$. We let $X \subset \mathbb{V}$ be a zero-dimensional subscheme with ideal sheaf $\mathcal{I}_X$ and structure sheaf $\mathcal{O}_X$. It is customary to denote $\text{length}(X) = \dim_{\mathbb{K}} \mathrm{H}^0(\mathcal{O}_X)$. We choose $a \in \mathbb{Z}^q$. From the short exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{V}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we obtain the exact sequence in cohomology

$$0 \longrightarrow I(X)_a \longrightarrow \mathbb{S}_a \longrightarrow \mathrm{H}^0(\mathcal{O}_X) \longrightarrow \mathrm{H}^1(\mathcal{I}_X(a)) \longrightarrow \mathrm{H}^1(\mathcal{O}_{\mathbb{V}}(a)).$$

The group on the right vanishes if a lies in the positive quadrant. Thus,

$$\mathcal{H}_X(a) = \text{length}(X) - \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a)) \quad \text{if} \quad a \in \mathbb{Z}_+^q. \tag{3}$$

Lemma 3.1. *Assume that $[0, r]$ is relevant for $\mathcal{H}_X$. Then $\mathcal{H}_X(a) = \text{length}(X)$ for all $a \geq r$.*

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Proof. It is well-known that $\mathrm{H}^1(\mathcal{I}_X(a))$ vanishes if $a_1, \ldots, a_q$ are sufficiently large. See [19, Theorem III.5.2]. Consequently, in view of Equation (3), $\mathcal{H}_X(a) = \mathrm{length}(X)$ if $a_1, \ldots, a_q$ are sufficiently large. We saw in Lemma 2.3 that $\mathcal{H}_X$ is constant on $[r, \infty)$. We conclude that $\mathcal{H}_X$ takes the value $\mathrm{length}(X)$ on $[r, \infty)$. ∎

It is well-known that $\mathcal{H}_X$ has a relevant domain when $\mathbb{V}$ is a projective space. The following proposition is a straightforward consequence of [10, Proposition 1.1].

Proposition 3.2. *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional subscheme. Then there is an integer $r \geq 0$ such that $\mathcal{H}_Z$ increases on the interval $[0, r]$ and is constant on the interval $[r, \infty)$. Thus, $[0, r]$ is the smallest relevant domain for $\mathcal{H}_X$.*

Notation 3.3. The integer $r = \mathrm{rin}(Z)$ is known as the *regularity index* of Z.

Notation 3.4. Let S be a $\mathbb{K}$-algebra and let M be an S-module. Consider elements $v_1, \ldots, v_p \in S$. We denote by $\{\epsilon_1, \ldots, \epsilon_p\}$ the standard basis of the $\mathbb{K}$-vector space $E = \mathbb{K}^p$. Consider the element $\mathbf{v} = \epsilon_1 \otimes v_1 + \cdots + \epsilon_p \otimes v_p \in E \otimes_{\mathbb{K}} S$. The sequence

$$0 \to M \xrightarrow{\cdot\mathbf{v}} E \otimes M \to \cdots \wedge^k E \otimes M \xrightarrow{\cdot\mathbf{v}} \wedge^{k+1} E \otimes M \to \cdots \wedge^{p-1} E \otimes M \xrightarrow{\cdot\mathbf{v}} \wedge^p E \otimes M$$

is the *Koszul complex* associated to $v_1, \ldots, v_p$ and M, denoted $K(v_1, \ldots, v_p) \otimes M$.

Lemma 3.5. *Let M be a $\mathbb{Z}^q$-graded $\mathbb{S}$-module. Assume that $\{u_1, \ldots, u_q\}$ is M-regular with $u_i \in \mathrm{span}\{x_{ij} \mid 0 \leq j \leq n_i\}$. Then $\Delta\mathcal{H}_M$ is the Hilbert function of $M/(u_1, \ldots, u_q)M$.*

Proof. We denote by $\{\epsilon_1, \ldots, \epsilon_q\}$ the standard basis of the $\mathbb{K}$-vector space $E = \mathbb{K}^q$. For each $k \in \{1, \ldots, q\}$ we endow $\wedge^k E \otimes_{\mathbb{K}} M$ with a $\mathbb{Z}^q$-grading as follows: if $h \in M$ is $\mathbb{Z}^q$-homogeneous and $1 \leq i_1 < \cdots < i_k \leq q$, then

$$\deg(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} \otimes h) = \deg(h) + \sum_{i \in \{1, \ldots, q\} \setminus \{i_1, \ldots, i_k\}} e_i.$$

The Koszul complex $K(u_1, \ldots, u_q) \otimes M$ introduced at Notation 3.4 becomes a complex of $\mathbb{Z}^q$-graded $\mathbb{S}$-modules. According to [5, Corollary 17.5], $K(u_1, \ldots, u_q) \otimes M$ is exact, by virtue of the fact that $\{u_1, \ldots, u_q\}$ is M-regular. Note the isomorphism $M \simeq \wedge^q E \otimes M$ of $\mathbb{Z}^q$-graded $\mathbb{S}$-modules given by $h \mapsto \epsilon_1 \wedge \cdots \wedge \epsilon_q \otimes h$. The cokernel of the last map in the Koszul complex is thus isomorphic to $M/(u_1, \ldots, u_q)M$. The lemma follows from the additivity of the Hilbert function on short exact sequences. ∎

We recall that the *projection* $\mathrm{pr}_i(X)$ *onto* $\mathbb{P}^{n_i}$ of a subscheme $X \subset \mathbb{V}$ is the subscheme of $\mathbb{P}^{n_i}$ defined by the ideal $I(X) \cap \mathbb{S}_i$. Here $\mathbb{S}_i = \mathbb{K}[x_{ij} \mid 0 \leq j \leq n_i]$. We recall that a zero-dimensional subscheme $X \subset \mathbb{V}$ is called a *complete intersection* if $I(X)$ is generated by a regular sequence of length $n_1 + \cdots + n_q$.

Proposition 3.6. *A zero-dimensional subscheme $X \subset \mathbb{V}$ is a* complete intersection *if and only if each $Z_i = \mathrm{pr}_i(X)$ is a complete intersection in $\mathbb{P}^{n_i}$ and $X = Z_1 \times \cdots \times Z_q$.*

Proof. Assume that X is a complete intersection, so $I(X)$ is generated by a regular sequence $\{f_1, \ldots, f_n\}$, where $n = n_1 + \cdots + n_q$. Write $d_k = (d_k^1, \ldots, d_k^q) = \deg(f_k)$. Arguing as in the proof of Lemma 3.5, we can show that $K(f_1, \ldots, f_n) \otimes \mathbb{S}$ is a resolution

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of $\mathbb{S}/I(X)$, hence, by the additivity of the Hilbert function on short exact sequences,

$$\mathcal{H}_X(a) = \mathcal{H}_{\mathbb{S}}(a) + \sum_{1\le p\le n} (-1)^p \sum_{1\le k_1<\cdots<k_p\le n} \mathcal{H}_{\mathbb{S}}(a - d_{k_1} - \cdots - d_{k_p}).$$

Assume now that $a = (a_1, 0, \ldots, 0)$. We have the equation

$$\mathcal{H}_{Z_1}(a_1) = \mathcal{H}_X(a) = \mathcal{H}_{\mathbb{S}_1}(a_1) + \sum_{1\le p\le n} (-1)^p \sum_{1\le k_1<\cdots<k_p\le n} \mathcal{H}_{\mathbb{S}}(a - d_{k_1} - \cdots - d_{k_p}). \quad (4)$$

Put $\{g_1, \ldots, g_m\} = \{f_1, \ldots, f_n\} \cap \mathbb{S}_1$ and write $e_k = \deg(g_k) \in \mathbb{Z}$ for $1 \le k \le m$. Notice that $\{g_1, \ldots, g_m\}$ is $\mathbb{S}_1$-regular, hence $m \le n_1$. Let $Z \subset \mathbb{P}^{n_1}$ be the subscheme defined by the ideal $(g_1, \ldots, g_m)$. If for some indices $1 \le \mu \le p$ and $2 \le i \le q$ we have $d^i_{k_\mu} > 0$, then $\mathcal{H}_{\mathbb{S}}(a - d_{k_1} - \cdots - d_{k_p}) = 0$. Discarding the superfluous terms, we can rewrite Equation (4) in the form

$$\mathcal{H}_{Z_1}(a_1) = \mathcal{H}_{\mathbb{S}_1}(a_1) + \sum_{1\le p\le m} (-1)^p \sum_{1\le k_1<\cdots<k_p\le m} \mathcal{H}_{\mathbb{S}_1}(a_1 - e_{k_1} - \cdots - e_{k_p}).$$

The r.h.s. equals $\mathcal{H}_Z(a_1)$ because $K(g_1, \ldots, g_m) \otimes \mathbb{S}_1$ is exact. Thus $\mathcal{H}_{Z_1} = \mathcal{H}_Z$. By construction, Z_1 is a subscheme of Z, hence $Z = Z_1$, and hence $m = n_1$. Moreover, Z_1 is a complete intersection. The same argument works for all Z_i, so each of them is a complete intersection. We have proved that $I(X) = I(Z_1) + \cdots + I(Z_q)$, so $X = Z_1 \times \cdots \times Z_q$. ∎

Notation 3.7. We assume that the zero-dimensional subscheme $Z \subset \mathbb{P}^n$ is a complete intersection. Say $I(Z) = (f_1, \ldots, f_n)$. We write

$$\delta(Z) = \begin{cases} \deg(f_1) - 1 & \text{if } n = 1, \\ \deg(f_1) + \cdots + \deg(f_n) - 2 & \text{if } n \ge 2. \end{cases}$$

We assume that the zero-dimensional subscheme $X \subset \mathbb{V}$ is a complete intersection. As per Proposition 3.6, each $Z_i = \mathrm{pr}_i(X)$ is a complete intersection, so we may write

$$\delta(X) = (\delta(Z_1), \ldots, \delta(Z_q)).$$

In the following proposition we collect several well-known properties of the regularity index. For the convenience of the reader we include their proofs. Let us recall that the *Castelnuovo-Mumford regularity* $\mathrm{reg}(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ on a projective space is the smallest integer ρ such that $\mathrm{H}^m(\mathcal{S}(\rho - m)) = \{0\}$ if $m \ge 1$. The Castelnuovo-Mumford regularity $\mathrm{reg}(Z)$ of a subscheme Z of a projective space is $\mathrm{reg}(\mathcal{I}_Z)$.

Proposition 3.8. *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional subscheme of length s and regularity index $\mathrm{rin}(Z)$. Then the following statements hold true:*

(i) $\mathcal{H}_Z(a) = s$ *for* $a \ge \mathrm{rin}(Z)$;
(ii) $\mathrm{H}^1(\mathcal{I}_Z(a)) = \{0\}$ *for* $a \ge \mathrm{rin}(Z)$;
(iii) $\mathrm{H}^m(\mathcal{I}_Z(a)) = \{0\}$ *for* $m \ge 2$ *and* $a \ge -n$;
(iv) $\mathrm{rin}(Z) \le s - 1$;
(v) $\mathrm{rin}(Z) = s - 1$ *if* $n = 1$;
(vi) $\mathrm{rin}(Z) + 1 = \mathrm{reg}(Z)$;
(vii) $\mathrm{rin}(Z) \le \delta(Z)$ *if Z is a complete intersection. See Notation 3.7.*

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Proof. Write $r = \operatorname{rin}(Z)$. To prove statement (i) we apply Proposition 3.2 and Lemma 3.1. In light of Equation (3), we have $\dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_Z(a)) = s - \mathcal{H}_Z(a) = 0$ for $a \geq r$. This proves (ii). To prove statement (iii) we employ the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Applying the long exact cohomology sequence we obtain the exact sequence

$$\mathrm{H}^{m-1}(\mathcal{O}_Z) \longrightarrow \mathrm{H}^m(\mathcal{I}_Z(a)) \longrightarrow \mathrm{H}^m(\mathcal{O}_{\mathbb{P}^n}(a)).$$

The group on the left vanishes because $\mathcal{O}_Z$ has support of dimension zero. The group on the right vanishes for $a \geq -n$. Thus, the group in the middle also vanishes.

(iv) According to Proposition 3.2, $\mathcal{H}_Z$ increases on the interval $[0, r]$. By definition, $\mathcal{H}_Z(0) = 1$, hence $\mathcal{H}_Z(r) \geq r + 1$, and hence, in view of statement (i), $s \geq r + 1$.

(v) For $0 \leq a \leq s - 1$ we have $\mathcal{H}_Z(a) = a + 1$ because there are no forms of degree a vanishing on a subscheme $Z \subset \mathbb{P}^1$ of length s. Thus, $\mathcal{H}_Z$ increases on the interval $[0, s - 1]$ forcing the inequality $s - 1 \leq r$. The reverse inequality was obtained at (iv).

(vi) From statements (ii) and (iii) we see that $\mathrm{H}^m(\mathcal{I}_Z(r + 1 - m)) = \{0\}$ if $m \geq 1$. From the definition of r and from Equation (3) it follows that $\mathrm{H}^1(\mathcal{I}_Z(r - 1)) \neq \{0\}$.

(vii) In the case when $n = 1$ we have $r = s - 1 = \delta(Z)$. Assume that $n \geq 2$. Write $d_i = \deg(f_i)$. The hypothesis that $\{f_1, \dots, f_n\}$ be a regular sequence implies that $K(f_1, \dots, f_n) \otimes \mathbb{S}$ is a resolution of $\mathbb{S}/I(Z)$. Consult the proof of Lemma 3.5. Thus,

$$\mathcal{H}_Z(a) = \mathcal{H}_{\mathbb{S}}(a) + \sum_{1 \leq p \leq n} (-1)^p \sum_{1 \leq k_1 < \cdots < k_p \leq n} \mathcal{H}_{\mathbb{S}}(a - d_{k_1} - \cdots - d_{k_p}).$$

Differentiating both sides, we obtain the equation

$$\Delta \mathcal{H}_Z(a) = \binom{a + n - 1}{n - 1} + \sum_{1 \leq p \leq n} (-1)^p \sum_{1 \leq k_1 < \cdots < k_p \leq n} \binom{a - d_{k_1} - \cdots - d_{k_p} + n - 1}{n - 1}.$$

We need to prove that $\Delta \mathcal{H}_Z(a) = 0$ for $a \geq \delta(Z) + 1$, that is, we need to prove the combinatorial equation

$$\binom{a + n - 1}{n - 1} = \sum_{1 \leq p \leq n} (-1)^{p+1} \sum_{1 \leq k_1 < \cdots < k_p \leq n} \binom{a - d_{k_1} - \cdots - d_{k_p} + n - 1}{n - 1} \qquad (5)$$

for $a \geq \delta(Z) + 1$. Put $A = \{1, 2, \dots, a + 1\}$. For $1 \leq k \leq n$, consider mutually disjoint subsets $D_k \subset A$ with d_k elements. Such subsets exist because, by hypothesis, $a + 1 \geq d_1 + \cdots + d_n$. Let C be the set of repeat combinations of $n - 1$ elements chosen in A. Let C_k be the set of repeat combinations of $n - 1$ elements chosen in $A \setminus D_k$. The l.h.s. of Equation (5) equals $|C|$ and

$$\binom{a - d_{k_1} - \cdots - d_{k_p} + n - 1}{n - 1} = \left| C_{k_1} \cap \cdots \cap C_{k_p} \right|.$$

Applying the inclusion-exclusion principle, we deduce that the r.h.s. of Equation (5) equals $\left| C_1 \cup \cdots \cup C_n \right|$. Equation (5) reduces to proving that $C = C_1 \cup \cdots \cup C_n$. Take $(c_1, \dots, c_{n-1}) \in C$. By construction, the sets $D_1, \dots, D_n$ are mutually disjoint, hence there is k such that $c_i \notin D_k$ for all indices $1 \leq i \leq n - 1$. Thus, $(c_1, \dots, c_{n-1}) \in C_k$. ■

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4 The rectangular relevant domain

Let $\mathbb{V} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a multiprojective space over a field $\mathbb{K}$. Let $\mathrm{pr}_i \colon \mathbb{V} \to \mathbb{P}^{n_i}$ be the projection onto the i-th component. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme and let $Z_i = \mathrm{pr}_i(X)$ be the zero-dimensional subscheme of $\mathbb{P}^{n_i}$ defined by the ideal $I(Z_i) = I(X) \cap \mathbb{K}[x_{ij} \mid 0 \le j \le n_i]$. Write $s_i = \mathrm{length}(Z_i)$. As per Notation 3.3, write $r_i = \mathrm{rin}(Z_i)$. Let $W_i = \mathrm{pr}_i^{-1}(Z_i)$ be the pull-back scheme, i.e. the subscheme of $\mathbb{V}$ defined by the ideal of $\mathbb{S}$ generated by $I(Z_i)$. Note that X is a subscheme of W_i. We denote by $\mathcal{I}_{X,W_i}$ the ideal sheaf of X in $\mathcal{O}_{W_i}$.

Our purpose in this section is to prove that the domain $[0, r_1] \times \cdots \times [0, r_q]$ is relevant for $\mathcal{H}_X$. As mentioned in the introduction, a similar result was proved by Van Tuyl. See the comments below Corollary 4.6. Van Tuyl's approach was based on his version of Proposition 4.2. Our approach is to replace the ambient space $\mathbb{V}$ with W_i. Our main technical tool is the following lemma.

Lemma 4.1. *Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Fix an arbitrary index i in $\{1, \ldots, q\}$ and let W_i and r_i be as defined above. Let $a \in \mathbb{Z}^q$ satisfy the conditions $a \ge 0$ and $a_i \ge r_i$. Then $\mathrm{H}^1(\mathcal{I}_X(a)) \simeq \mathrm{H}^1(\mathcal{I}_{X,W_i}(a - a_i e_i))$.*

Proof. By symmetry, we may assume that $i = 1$. By virtue of the Künneth formula,

$$\mathrm{H}^m(\mathcal{I}_{W_1}(a)) \simeq \bigoplus_{m_1+\cdots+m_q=m} \mathrm{H}^{m_1}(\mathcal{I}_{Z_1}(a_1)) \otimes \mathrm{H}^{m_2}(\mathcal{O}_{\mathbb{P}^{n_2}}(a_2)) \otimes \cdots \otimes \mathrm{H}^{m_q}(\mathcal{O}_{\mathbb{P}^{n_q}}(a_q)).$$

By hypothesis, $a_1 \ge r_1$, hence, in view of Proposition 3.8, $\mathrm{H}^{m_1}(\mathcal{I}_{Z_1}(a_1)) = \{0\}$ for $m_1 \ge 1$. By hypothesis, $a_2 \ge 0, \ldots, a_q \ge 0$, hence the higher cohomology groups of $\mathcal{O}_{\mathbb{P}^{n_2}}(a_2), \ldots, \mathcal{O}_{\mathbb{P}^{n_q}}(a_q)$ also vanish. We deduce that $\mathrm{H}^m(\mathcal{I}_{W_1}(a)) = \{0\}$ for $m \ge 1$. From the short exact sequence

$$0 \longrightarrow \mathcal{I}_{W_1} \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{I}_{X,W_1} \longrightarrow 0$$

of sheaves on $\mathbb{V}$ we obtain the long exact sequence

$$\{0\} = \mathrm{H}^1(\mathcal{I}_{W_1}(a)) \longrightarrow \mathrm{H}^1(\mathcal{I}_X(a)) \longrightarrow \mathrm{H}^1(\mathcal{I}_{X,W_1}(a)) \longrightarrow \mathrm{H}^2(\mathcal{I}_{W_1}(a)) = \{0\}.$$

The middle arrow becomes an isomorphism. The line bundle $\mathcal{O}_{\mathbb{P}^{n_1}}(a_1)$ is trivial on Z_1 because the latter is supported on finitely many points. We obtain the isomorphism $\mathcal{I}_{X,W_1}(a) \simeq \mathcal{I}_{X,W_1}(0, a_2, \ldots, a_q)$, which leads us to the desired isomorphism

$$\mathrm{H}^1(\mathcal{I}_X(a)) \simeq \mathrm{H}^1(\mathcal{I}_{X,W_1}(0, a_2, \ldots, a_q)). \quad \blacksquare$$

Proposition 4.2. *Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Assume that the ground field $\mathbb{K}$ is algebraically closed. Fix an arbitrary index $i \in \{1, \ldots, q\}$ and assume that Z_i is reduced, say $Z_i = \{P_1, \ldots, P_m\}$. For each index $k \in \{1, \ldots, m\}$ consider the scheme*

$$\mathbb{W}_k = \mathrm{pr}_i^{-1}(P_k) \simeq \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}}^{n_i} \times \cdots \times \mathbb{P}^{n_q}.$$

Set $Y_k = X \cap \mathbb{W}_k$. Let $\mathcal{H}_{Y_k}$ be the Hilbert function of Y_k as a subscheme of $\mathbb{W}_k$, the latter being regarded as a multiprojective space. Let $a \in \mathbb{Z}^q$ satisfy the conditions $a \ge 0$ and $a_i \ge r_i$. Then

$$\mathcal{H}_X(a) = \sum_{1 \le k \le m} \mathcal{H}_{Y_k}(a_1, \ldots, \widehat{a}_i, \ldots, a_q).$$

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Proof. From the decomposition $W_i = \mathbb{W}_1 \sqcup \cdots \sqcup \mathbb{W}_m$ we obtain the decomposition

$$\mathrm{H}^1(\mathcal{I}_{X,W_i}(a - a_i e_i)) \simeq \bigoplus_{1 \le k \le m} \mathrm{H}^1(\mathcal{I}_{Y_k, \mathbb{W}_k}(a_1, \ldots, \widehat{a}_i, \ldots, a_q)).$$

Applying Equation (3) and Lemma 4.1, we calculate:

$$\begin{aligned}
\mathcal{H}_X(a) &= \mathrm{length}(X) - \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a)) \\
&= \mathrm{length}(X) - \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_{X,W_i}(a - a_i e_i)) \\
&= \sum_{1 \le k \le m} \mathrm{length}(Y_k) - \sum_{1 \le k \le m} \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_{Y_k, \mathbb{W}_k}(a_1, \ldots, \widehat{a}_i, \ldots, a_q)) \\
&= \sum_{1 \le k \le m} \mathcal{H}_{Y_k}(a_1, \ldots, \widehat{a}_i, \ldots, a_q). \quad \blacksquare
\end{aligned}$$

The above result, in the particular case when X is reduced and $a_i \ge s_i - 1$, was obtained by Van Tuyl using different methods. Consult [25, Proposition 4.2].

Definition 4.3. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. For each $i \in \{1, \ldots, q\}$, let Z_i be the projection of X onto $\mathbb{P}^{n_i}$. Recall Notation 3.3. The tuple

$$\mathrm{rem}(X) = (\mathrm{rin}(Z_1), \ldots, \mathrm{rin}(Z_q))$$

will be called the *regularity multi-index* of X. We write $R(X) = [0,\ \mathrm{rem}(X)]$.

Theorem 4.4. *Let $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a zero-dimensional subscheme. We assert that $R(X) = [0,\ \mathrm{rem}(X)]$ is the smallest rectangular relevant domain for $\mathcal{H}_X$.*

Proof. Write $\mathrm{rem}(X) = (r_1, \ldots, r_q)$. Consider $a \in \mathbb{Z}_+^q$ satisfying the condition $a_i \ge r_i$ for some index $i \in \{1, \ldots, q\}$. According to Lemma 4.1, the expression $\dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(b))$ remains constant as b_i varies in the interval $[r_i, \infty)$ and b_j are nonnegative fixed integers for all indices $j \in \{1, \ldots, q\} \setminus \{i\}$. Thus,

$$\dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a)) = \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a_1, \ldots, r_i, \ldots, a_q)).$$

Applying Equation (3), we calculate:

$$\begin{aligned}
\mathcal{H}_X(a) &= \mathrm{length}(X) - \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a)) \\
&= \mathrm{length}(X) - \dim_{\mathbb{K}} \mathrm{H}^1(\mathcal{I}_X(a_1, \ldots, r_i, \ldots, a_q)) \\
&= \mathcal{H}_X(a_1, \ldots, r_i, \ldots, a_q).
\end{aligned}$$

From Lemma 2.2 we deduce that $R(X)$ is relevant to $\mathcal{H}_X$. We cannot shrink $R(X)$ to a smaller rectangular relevant domain because, as seen at Proposition 3.2, the function $\mathcal{H}_X(a_i e_i) = \mathcal{H}_{Z_i}(a_i)$ increases on the interval $[0, r_i]$. $\blacksquare$

The above result, in the particular case when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{K}$ is algebraically closed, was obtained by Giuffrida et al. We refer to [11, Remark 2.8 and Theorem 2.11]. Guardo and Van Tuyl gave a different proof to Giuffrida's result in the particular case when X is a union of fat points. Consult [14, Corollary 3.4].

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Proposition 4.5. *Let $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a zero-dimensional subscheme of length s and regularity multi-index* $\mathrm{rem}(X)$*. For each $i \in \{1, \ldots, q\}$, let s_i be the length of the projection of X onto $\mathbb{P}^{n_i}$. Then the following statements hold true:*

(i) $\mathcal{H}_X(a) = s$ *for* $a \geq \mathrm{rem}(X)$*;*
(ii) $\mathrm{H}^1(\mathcal{I}_X(a)) = \{0\}$ *for* $a \geq \mathrm{rem}(X)$*;*
(iii) $\mathrm{H}^m(\mathcal{I}_X(a)) = \{0\}$ *for* $m \geq 2$ *and* $a \geq 0$*;*
(iv) $\mathrm{rem}(X) \leq (s_1 - 1, \ldots, s_q - 1)$*;*
(v) $\mathrm{rem}(X) = (s_1-1, \ldots, s_q-1)$ *if* $\mathbb{V} = (\mathbb{P}^1)^q$*. Thus,* $R(X) = [0, s_1-1]\times\cdots\times[0, s_q-1]$*;*
(vi) $\mathrm{rem}(X) \leq \delta(X)$ *if X is a complete intersection. See Notation 3.7.*

Proof. Statement (i) follows from Theorem 4.4 and Lemma 3.1. To prove (ii) and (iii) we argue precisely as in the proof of Proposition 3.8(ii and iii). Statements (iv) and (v) follow from their counterparts at Proposition 3.8. Statement (vi) follows from Proposition 3.6 and Proposition 3.8(vii). ■

Proposition 4.5(i), in the particular case when X is a union of fat points and $\mathbb{K}$ is algebraically closed, was obtained by Sidman and Van Tuyl. See [24, Proposition 4.4].

Complete intersection schemes contained in a simplicial toric variety were studied in [23]. Restricting [23, Theorem 3.16] to the case when the ambient space is a multiprojective space we obtain the following result: "Let X be a zero-dimensional complete intersection scheme contained in a multiprojective space of dimension n defined over an algebraically closed field. Assume that $I(X) = (f_1, \ldots, f_n)$. Then $\mathcal{H}_X(a) = \mathrm{length}(X)$ for $a \geq \deg(f_1) + \cdots + \deg(f_n)$." In Proposition 4.5 we proved that the equation $\mathcal{H}_X(a) = \mathrm{length}(X)$ holds for the improved bound $a \geq \delta(X)$,

Corollary 4.6. *Let $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a zero-dimensional subscheme. For each index $i \in \{1, \ldots, q\}$, let s_i be the length of the projection of X onto $\mathbb{P}^{n_i}$. Then the rectangular domain $[0, s_1 - 1] \times \cdots \times [0, s_q - 1]$ is relevant to $\mathcal{H}_X$.*

The corollary follows from Theorem 4.4 and Proposition 4.5(iv). The above result, in the particular case when X is reduced and $\mathbb{K}$ is algebraically closed, was obtained by Van Tuyl. Consult [25, Proposition 4.6(ii) and Corollary 4.7].

Let M be a finitely generated $\mathbb{Z}^q$-graded $\mathbb{S}$-module. The canonical $\mathbb{Z}$-grading on $\mathbb{S}$ is given by the degree of a polynomial. Given $a \in \mathbb{Z}^q$, write $|a| = a_1 + \cdots + a_q$. We note that M is also a $\mathbb{Z}$-graded $\mathbb{S}$-module by setting $M_d = \oplus_{|a|=d} M_a$. The Hilbert function of this module is the function $\overline{\mathcal{H}}_M : \mathbb{Z} \to \mathbb{Z}$ given by

$$\overline{\mathcal{H}}_M(d) = \sum_{|a|=d} \mathcal{H}_M(a).$$

There exists a polynomial $\mathcal{P}_M$ in one variable, with rational coefficients, called the *Hilbert-Poincarè polynomial* of M, such that $\overline{\mathcal{H}}_M(d) = \mathcal{P}_M(d)$ for d sufficiently large. See [5, Theorem 1.11]. The *regularity index* $\mathrm{rin}(M)$ of M is the smallest integer with the property that $\overline{\mathcal{H}}_M = \mathcal{P}_M$ on $[\mathrm{rin}(M), \infty)$. Given a subscheme $X \subset \mathbb{V}$, we put $\overline{\mathcal{H}}_X = \overline{\mathcal{H}}_{\mathbb{S}/I(X)}$, $\mathcal{P}_X = \mathcal{P}_{\mathbb{S}/I(X)}$, and $\mathrm{rin}(X) = \mathrm{rin}(\mathbb{S}/I(X))$. These concepts become more familiar once we slightly change the point of view. Write $n = n_1 + \cdots + n_q + q - 1$ and let $\mathbb{P}^n$ have the coordinate ring $\mathbb{S}$, equipped with its $\mathbb{Z}$-grading. Let $\overline{X} \subset \mathbb{P}^n$ be the

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subscheme defined by the ideal $I(X)$. As an aside, note that, if X is reduced, then $\overline{X}$ is an arrangement of $(q-1)$-dimensional planes in $\mathbb{P}^n$. Clearly, $\overline{\mathcal{H}}_X$ is the usual Hilbert function of $\overline{X}$, $\mathcal{P}_X$ is the usual Hilbert-Poincarè polynomial of $\overline{X}$, and $\operatorname{rin}(X) = \operatorname{rin}(\overline{X})$.

Corollary 4.7. *Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Recall from Definition 4.3 the regularity multi-index* $\operatorname{rem}(X)$. *We assert that* $\operatorname{rin}(X) \leq |\operatorname{rem}(X)|$. *We further assert that $\mathcal{P}_X(d)$ has dominant term* $\operatorname{length}(X)d^{q-1}/(q-1)!$

Proof. Clearly, the Poincarè polynomial associated to $\mathcal{H}_X$, introduced at Definition 2.6, coincides with $\mathcal{P}_X$. We apply Lemma 2.5 to $\mathcal{H}_X$ and to its relevant domain $[0, \operatorname{rem}(X)]$. We deduce that $\mathcal{P}_X = \overline{\mathcal{H}}_X$ on $[|\operatorname{rem}(X)|, \infty)$, and that $\mathcal{P}_X$ has dominant term $\mathcal{H}_X(\operatorname{rem}(X))d^{q-1}/(q-1)!$ It follows that $\operatorname{rin}(X) \leq |\operatorname{rem}(X)|$. In accordance with Proposition 4.5(i), $\mathcal{H}_X(\operatorname{rem}(X)) = \operatorname{length}(X)$. This proves the second assertion. ∎

Lemma 4.8. *Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme of length s. Assume that there is an integer $0 \leq \tau < |\operatorname{rem}(X)|$ such that $\mathcal{H}_X(a) = s$ if $a \in R(X)$ and $|a| \geq \tau$. Then* $\operatorname{rin}(X) \leq \tau$.

Proof. We apply Lemma 2.7 to the function $\mathcal{F} = \mathcal{H}_X$. The hypothesis of Lemma 2.7 is satisfied because $\mathcal{H}_X$ takes the value s on the region $\{a \in R(X) \mid |a| \geq \tau\}$. We deduce that $\overline{\mathcal{H}}_X(d) = \mathcal{P}_X(d)$ for $d \geq \tau$, forcing the inequality $\operatorname{rin}(X) \leq \tau$. ∎

Proposition 4.9. *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional subscheme of length s. Let $\overline{X} \subset \mathbb{P}^3$ be the associated one-dimensional subscheme. Then* $\operatorname{rin}(X) \leq s-1$, *i.e.* $\operatorname{rin}(\overline{X}) \leq \deg(\overline{X}) - 1$.

Proof. According to Corollary 4.7, $\deg(\overline{X}) = s$, hence the two inequalities above are equivalent. Assume that $|\operatorname{rem}(X)| \leq s-1$. Applying Corollary 4.7 we get the inequalities $\operatorname{rin}(X) \leq |\operatorname{rem}(X)| \leq s-1$. Assume now that $s-1 < |\operatorname{rem}(X)|$. According to [20, Corollary 4.5], $\mathcal{H}_X(a) = s$ if $a \in R(X)$ and $|a| \geq s-1$. Applying Lemma 4.8 with $\tau = s-1$ we obtain the inequality $\operatorname{rin}(X) \leq s-1$. ∎

Let Y be a projective scheme. The inequality $\operatorname{rin}(Y) \leq \deg(Y) - 1$ is satisfied in the case when Y is zero-dimensional, as per Proposition 3.8(iv). This inequality is also satisfied in the case when $Y = \overline{X}$, as in Proposition 4.9. The question whether the inequality is satisfied for arbitrary Y remains open.

5 Regular sequences in the case of ACM and sub-ACM schemes

In this section we assume that the ground field $\mathbb{K}$ is infinite. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{S}$ generated by all the variables. The *depth* of a $\mathbb{Z}^q$-graded $\mathbb{S}$-module M is the maximal length of an M-regular sequence contained in $\mathfrak{m}$. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Since $\mathbb{K}$ is infinite, there exists a non-constant $\mathbb{Z}^q$-homogeneous form that does not vanish at any point of $\operatorname{red}(X)$. This form is a non-zerodivisor of $\mathbb{S}/I(X)$. Thus, we have the inequalities $1 \leq \operatorname{depth}(\mathbb{S}/I(X)) \leq \dim(\mathbb{S}/I(X)) = q$. We recall that X is said to be *arithmetically Cohen-Macaulay (ACM)* if $\operatorname{depth}(\mathbb{S}/I(X)) = q$. If $q = 1$, then X must be ACM. We say that X is *sub-ACM* if $q \geq 2$ and $\operatorname{depth}(\mathbb{S}/I(X)) = q-1$.

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Notation 5.1. Throughout this section we shall employ the following notation:

$$U = \operatorname{span}\{x_{ij} \mid 1 \le i \le q,\ 0 \le j \le n_i\},$$
$$U_i = \operatorname{span}\{x_{ij} \mid 0 \le j \le n_i\},$$
$$U_i^0 = \{u_i \in U_i \mid u_i \text{ does not vanish at any point of } \operatorname{red}(X)\}.$$

Remark 5.2. The spaces U_i^0 are non-empty and each $u_i \in U_i^0$ is a non-zerodivisor for $\mathbb{S}/I(X)$. Indeed, for each closed point $P \in \mathbb{V}$, $I(P) \cap U_i$ is a proper vector subspace of U_i. A vector space over an infinite field cannot be a finite union of proper subspaces, hence $U_i^0 \neq \emptyset$. The ideals $I(P)$ with $P \in \operatorname{red}(X)$ are the associated primes of X, hence u_i is a non-zerodivisor of $\mathbb{S}/I(X)$.

Remark 5.3. If $\mathbb{K}$ is algebraically closed and $\operatorname{red}(X) = \{P_1, \dots, P_m\}$, then, for each index $k \in \{1, \dots, m\}$, there are vector subspaces $U_{ki} \subset U_i$ of codimension one such that $I(P_k) = (U_{ki} \mid 1 \le i \le q)$. We have $U_i^0 = U_i \setminus \bigcup_{1 \le k \le m} U_{ki}$.

In the case when $\mathbb{K}$ is algebraically closed and X is reduced and ACM, Van Tuyl proved that we can choose a regular sequence $\{u_1, \dots, u_q\}$ for $\mathbb{S}/I(X)$ with $u_i \in U_i$. Consult [26, Proposition 3.2]. The aims of this section are, first, to generalize this result to the case when X is an arbitrary zero-dimensional ACM subscheme and $\mathbb{K}$ is an arbitrary infinite field (Proposition 5.6), second, to obtain a version of this result for sub-ACM schemes (Proposition 5.9), and, third, to show that the u_i above can be chosen generically (Proposition 5.10). These results and their corollaries will be used in Sections 6 and 7.

Lemma 5.4. *We assume that $\mathbb{K}$ is infinite. We write $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_n)$. We let $\mathfrak{p} \subset \mathbb{K}[\mathbf{x}, \mathbf{y}]$ be a prime ideal. We assert that $\operatorname{ht}(\mathfrak{p}) \ge \operatorname{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{x}]) + \operatorname{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{y}])$.*

Proof. Put $k = \dim \mathbb{K}[\mathbf{x}]/\mathfrak{p} \cap \mathbb{K}[\mathbf{x}]$ and $l = \dim \mathbb{K}[\mathbf{y}]/\mathfrak{p} \cap \mathbb{K}[\mathbf{y}]$. The assertion is equivalent to the inequality $\dim \mathbb{K}[\mathbf{x}, \mathbf{y}]/\mathfrak{p} \le k + l$, see [5, Corollary 13.4]. Applying the Noether normalization theorem [5, Theorem 13.3] we deduce that there are linearly independent one-forms $u_1, \dots, u_k \in \mathbb{K}[\mathbf{x}]$, respectively, $v_1, \dots, v_l \in \mathbb{K}[\mathbf{y}]$ such that the algebra $\mathbb{K}[\mathbf{x}]/\mathfrak{p} \cap \mathbb{K}[\mathbf{x}]$ is integral over $\mathbb{K}[\mathbf{u}]$ and the algebra $\mathbb{K}[\mathbf{y}]/\mathfrak{p} \cap \mathbb{K}[\mathbf{y}]$ is integral over $\mathbb{K}[\mathbf{v}]$. We wrote $\mathbf{u} = (u_1, \dots, u_k)$ and $\mathbf{v} = (v_1, \dots, v_l)$. The extension of algebras $\mathbb{K}[\mathbf{u}, \mathbf{v}]/\mathfrak{p} \cap \mathbb{K}[\mathbf{u}, \mathbf{v}] \subset \mathbb{K}[\mathbf{x}, \mathbf{y}]/\mathfrak{p}$ is integral, hence the two algebras have the same dimension, see [5, Theorem A, p. 286]. Thus, $\dim \mathbb{K}[\mathbf{x}, \mathbf{y}]/\mathfrak{p} \le \dim \mathbb{K}[\mathbf{u}, \mathbf{v}] = k + l$. ∎

Lemma 5.5. *Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Choose $u_i \in U_i^0$. Then (U_i) is contained in $\operatorname{rad}((u_i) + I(X))$.*

Proof. Recall that $I(X) \cap \mathbb{K}[x_{ij} \mid 0 \le j \le n_i]$ defines a zero-dimensional subscheme Z_i in $\mathbb{P}^{n_i}$. By hypothesis, u_i does not vanish at any point of $\operatorname{red}(Z_i)$, hence $(u_i) + I(Z_i)$ defines the empty subscheme in $\mathbb{P}^{n_i}$. Thus $(U_i) \subset \operatorname{rad}((u_i) + I(Z_i)) \subset \operatorname{rad}((u_i) + I(X))$. ∎

Proposition 5.6. *We assume that $\mathbb{K}$ is infinite. We let $X \subset \mathbb{V}$ be a zero-dimensional ACM subscheme. Then there are $u_i \in U_i$ such that $\{u_1, \dots, u_q\}$ is regular for $\mathbb{S}/I(X)$.*

Proof. Set $\mathbf{x}_i = (x_{i0}, \dots, x_{in_i})$. Performing induction on k, we will construct a regular sequence $\{u_1, \dots, u_k\}$ for $\mathbb{S}/I(X)$ with $u_i \in U_i^0$. To start the induction, choose u_1 in U_1^0 and recall Remark 5.2. For the induction step, assume that $k \in \{1, \dots, q-1\}$ and that $\{u_1, \dots, u_k\}$ has already been constructed. Write $J = (u_1) + \cdots + (u_k) + I(X)$.

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Choose a prime ideal $\mathfrak{p}$ that is associated to J. By hypothesis, $\mathbb{S}/I(X)$ is Cohen-Macaulay of dimension q, hence $\mathbb{S}/J$ is Cohen-Macaulay of dimension $q-k$. According to [5, Corollary 18.14], J is unmixed, hence $\dim(\mathfrak{p}) = q-k$ and $\mathrm{ht}(\mathfrak{p}) = n_1 + \cdots + n_q + k$. According to Lemma 5.5, $(U_1, \ldots, U_k)$ lies in $\mathrm{rad}(J)$, so it is contained in $\mathfrak{p}$. We claim that $\mathrm{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{x}_i]) = n_i$ for each index $i \in \{k+1, \ldots, q\}$. Indeed, the inequalities $\mathrm{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{x}_i]) \geq n_i$ follow from the fact that $\mathfrak{p} \cap \mathbb{K}[\mathbf{x}_i]$ contains the ideal of the projection of X onto $\mathbb{P}^{n_i}$. According to Lemma 5.4, we have the inequality

$$\mathrm{ht}(\mathfrak{p}) \geq \sum_{1 \leq i \leq q} \mathrm{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{x}_i]) \quad \text{or, equivalently,} \quad \sum_{k+1 \leq i \leq q} n_i \geq \sum_{k+1 \leq i \leq q} \mathrm{ht}(\mathfrak{p} \cap \mathbb{K}[\mathbf{x}_i]).$$

This proves the claim. The claim implies that U_{k+1} is not contained in $\mathfrak{p}$. The same is true for all associated primes of J. Since $\mathbb{K}$ is infinite, we can choose $u_{k+1} \in U^0_{k+1}$ such that u_{k+1} does not lie in any associated prime of J. Thus, u_{k+1} is a non-zerodivisor for $\mathbb{S}/J$, hence $\{u_1, \ldots, u_{k+1}\}$ is regular relative to $\mathbb{S}/I(X)$. ∎

Lemma 5.7. *We assume that $\mathbb{K}$ is algebraically closed. We let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. We write $\mathrm{red}(X) = \{P_1, \ldots, P_m\}$. We choose $p \in \{1, \ldots, q\}$ and $u_i \in U^0_i$ for $1 \leq i \leq p$. Then the ideals $\mathfrak{p}_k = (U_1, \ldots, U_p) + I(P_k)$ for $1 \leq k \leq m$ are the minimal prime ideals containing $(u_1, \ldots, u_p) + I(X)$.*

Proof. In view of Remark 5.3, we have $\mathfrak{p}_k = (U_1, \ldots, U_p) + (U_{k,p+1}, \ldots, U_{k,q})$. This is clearly a prime ideal. Some of these ideals may coincide, however, if $\mathfrak{p}_k \neq \mathfrak{p}_l$, then $\mathfrak{p}_k \not\subseteq \mathfrak{p}_l$ and $\mathfrak{p}_l \not\subseteq \mathfrak{p}_k$. The lemma reduces to proving that

$$\bigcap_{1 \leq k \leq m} \mathfrak{p}_k = \mathrm{rad}((u_1, \ldots, u_p) + I(X)).$$

The inclusion "$\supset$" is obvious, so we focus on proving the reverse inclusion. We denote by $\mathfrak{r}$ the ideal on the r.h.s. Take $f \in \bigcap_{1 \leq k \leq m} \mathfrak{p}_k$ and write $f = g + h$, where g lies in the ideal $(U_1, \ldots, U_p)$ and h is a polynomial in the variables x_{ij}, $p+1 \leq i \leq q$, $0 \leq j \leq n_i$. According to Lemma 5.5, $(U_1, \ldots, U_p) \subset \mathfrak{r}$, hence $g \in \mathfrak{r}$. By construction,

$$h \in \bigcap_{1 \leq k \leq m} (U_{ki} \mid p+1 \leq i \leq q) \subset \bigcap_{1 \leq k \leq m} I(P_k) = \mathrm{rad}(I(X)) \subset \mathfrak{r}.$$

We conclude that f lies in $\mathfrak{r}$. ∎

Lemma 5.8. *Let S be a $\mathbb{Z}$-graded $\mathbb{K}$-algebra and let M be a $\mathbb{Z}$-graded S-module. Let $\{v_1, \ldots, v_p\} \subset S$ be an M-regular sequence such that all v_i are homogeneous of the same degree. Consider a non-singular matrix $G = (\kappa_{ij})_{1 \leq i,j \leq p}$ with entries in $\mathbb{K}$. Then $\{\kappa_{i1} v_1 + \cdots + \kappa_{ip} v_p \mid 1 \leq i \leq p\}$ constitutes an M-regular sequence.*

Proof. If G is lower-triangular, then the lemma follows from the definition of an M-regular sequence. According to [5, Corollary 17.5 and Theorem 17.6], a sequence $\{w_1, \ldots, w_p\} \subset S$ of homogeneous elements is M-regular if and only if the Koszul complex $K(w_1, \ldots, w_p) \otimes M$, introduced at Notation 3.4, is exact. Permutations of $\{w_1, \ldots, w_p\}$ yield isomorphic Koszul complexes. Thus, every permutation of $\{v_1, \ldots, v_p\}$ remains an M-regular sequence. The lemma follows from the fact that the lower triangular matrices and the row permutations generate $\mathrm{GL}_p(\mathbb{K})$. ∎

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Proposition 5.9. *Assume that $\mathbb{K}$ is algebraically closed. Assume that $q \geq 3$. Let $X \subset \mathbb{V}$ be a zero-dimensional sub-ACM subscheme. For $1 \leq i \leq q$ choose $u_i \in U_i^0$. Then there is an index $p \in \{2, \ldots, q\}$ and there are scalars $\kappa_i \in \mathbb{K}$ for $i \in \{2, \ldots, q\} \setminus \{p\}$ such that $\{u_1\} \cup \{u_i + \kappa_i u_p \mid i \in \{2, \ldots, q\} \setminus \{p\}\}$ is a regular sequence for $\mathbb{S}/I(X)$.*

Proof. Write $\operatorname{red}(X) = \{P_1, \ldots, P_m\}$. According to Lemma 5.7, $\mathfrak{p}_k = (U_1) + I(P_k)$ for $1 \leq k \leq m$ are the minimal prime ideals containing $(u_1) + I(X)$. We denote $U' = \operatorname{span}\{u_2, \ldots, u_q\}$. Performing induction on $l \in \{2, \ldots, q-1\}$, we will construct a regular sequence $\{u_1, v_2, \ldots, v_l\}$ for $\mathbb{S}/I(X)$ with $v_i \in U'$. To start the induction, we consider the set $\mathfrak{A}_1$ of associated primes to $(u_1)+I(X)$. We claim that U' is not contained in any $\mathfrak{p} \in \mathfrak{A}_1$. To prove this, we argue by contradiction. Assume that $U' \subset \mathfrak{p}$ and that $\mathfrak{p} \in \mathfrak{A}_1$. This ideal must contain one of the minimal associated primes to $(u_1) + I(X)$, say $\mathfrak{p}_k \subset \mathfrak{p}$. Thus, $U_1 \subset \mathfrak{p}$ and $U_{ki} \subset \mathfrak{p}$ for $2 \leq i \leq q$. It follows that

$$\begin{aligned} U &= U_1 + U_2 + \cdots + U_q \\ &= U_1 + \operatorname{span}\{u_2, U_{k2}\} + \cdots + \operatorname{span}\{u_q, U_{kq}\} \\ &= U_1 + U_{k2} + \cdots + U_{kq} + U' \subset \mathfrak{p}, \end{aligned}$$

hence $\mathfrak{m} = (U) \subset \mathfrak{p}$, so every element of $\mathfrak{m}$ is a zerodivisor for $\mathbb{S}/((u_1) + I(X))$. On the other hand, by Remark 5.2, u_1 is a non-zerodivisor for $\mathbb{S}/I(X)$, hence

$$\operatorname{depth}(\mathbb{S}/((u_1) + I(X))) = \operatorname{depth}(\mathbb{S}/I(X)) - 1 = q - 2 \geq 1.$$

We have reached a contradiction, which proves the claim. We obtain a regular sequence $\{u_1, v_2\}$ relative to $\mathbb{S}/I(X)$ by choosing $v_2 \in U' \setminus \bigcup_{\mathfrak{p}\in\mathfrak{A}_1}(\mathfrak{p} \cap U')$.

We now perform the induction step. Assume that $l \in \{2, \ldots, q-2\}$ and that $\{u_1, v_2, \ldots, v_l\}$ has already been constructed. We denote by $\mathfrak{A}_l$ the set of associated primes to the ideal $(u_1, v_2, \ldots, v_l) + I(X)$. Arguing as above, we can prove that U' is not contained in any $\mathfrak{p}$ from $\mathfrak{A}_l$. Indeed, $\mathfrak{p}_k \subset \mathfrak{p}$ for some k, so, if $U' \subset \mathfrak{p}$, then $U \subset \mathfrak{p}$. It would follow that every element of $\mathfrak{m}$ is a zerodivisor for $\mathbb{S}/((u_1, v_2, \ldots, v_l) + I(X))$. On the other hand, this ring has depth $q - 1 - l \geq 1$. This would yield a contradiction. Choosing $v_{l+1} \in U' \setminus \bigcup_{\mathfrak{p}\in\mathfrak{A}_l}(\mathfrak{p} \cap U')$ we obtain a regular sequence $\{u_1, v_2, \ldots, v_{l+1}\}$ relative to $\mathbb{S}/I(X)$. This completes the induction step.

Thus far, we have constructed an $\mathbb{S}/I(X)$-regular sequence $\{u_1, v_2, \ldots, v_{q-1}\}$ such that $v_2, \ldots, v_{q-1}$ are linearly independent vectors in U'. Write $v_l = \sum_{2\leq i\leq q} \lambda_{li} u_i$. The matrix $\Lambda = (\lambda_{li})_{2\leq l\leq q-1,\, 2\leq i\leq q}$ has maximal rank. To simplify notation, we assume that the minor obtained by deleting the last column of Λ is non-zero. We now apply Lemma 5.8 to the $\mathbb{Z}$-graded ring $\mathbb{S}$, to the $\mathbb{Z}$-graded module $M = \mathbb{S}/((u_1) + I(X))$ and to the M-regular sequence $\{v_2, \ldots, v_{q-1}\}$. We take G to be the inverse of $(\lambda_{li})_{2\leq l,\, i\leq q-1}$. We obtain an M-regular sequence of the form $\{u_i + \kappa_i u_q \mid 2 \leq i \leq q-1\}$. In general, if the minor obtained by deleting column p of Λ is non-zero, then we obtain a regular sequence as in the proposition. ∎

Proposition 5.10. *Assume that $\mathbb{K}$ is algebraically closed. Let $X \subset \mathbb{V}$ be a zero-dimensional ACM subscheme. For $1 \leq i \leq q$ choose $u_i \in U_i^0$. Then $\{u_1, \ldots, u_q\}$ is $\mathbb{S}/I(X)$-regular.*

Proof. Performing induction on i, we will show that $\{u_1, \ldots, u_i\}$ is $\mathbb{S}/I(X)$-regular. By Remark 5.2, u_1 is a non-zerodivisor for $\mathbb{S}/I(X)$. Assume that $i \in \{1, \ldots, q-1\}$ and

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that $\{u_1, \dots, u_i\}$ is $\mathbb{S}/I(X)$-regular. By hypothesis, $\mathbb{S}/I(X)$ is Cohen-Macaulay, hence $\mathbb{S}/((u_1, \dots, u_i) + I(X))$ is also Cohen-Macaulay. Per [5, Corollary 18.14], this ring is unmixed. Thus, the associated primes of $(u_1, \dots, u_i) + I(X)$ are precisely the minimal primes. According to Lemma 5.7, they are of the form $\mathfrak{p}_k = (U_1, \dots, U_i) + I(P_k)$. By construction, u_{i+1} lies outside all ideals $\mathfrak{p}_k$, hence u_{i+1} is a non-zerodivisor of $\mathbb{S}/((u_1, \dots, u_i) + I(X))$, and hence $\{u_1, \dots, u_{i+1}\}$ is regular relative to $\mathbb{S}/I(X)$. ■

Lemma 5.11. *Let S be a commutative ring and let $I \subset S$ be an ideal. Assume that the sequence $\{u_1, \dots, u_p\} \subset S$ is S/I-regular. Then $(u_1, \dots, u_p)I = (u_1, \dots, u_p) \cap I$.*

Proof. The inclusion "$\subset$" is obvious, so we concentrate on proving the reverse inclusion. We perform induction on p. Assume that $p = 1$. Take $f \in (u_1) \cap I$ and write $f = u_1 g$. In S/I we have the relations $u_1 \hat{g} = \hat{u}_1 \hat{g} = \hat{f} = 0$. By hypothesis, u_1 is a non-zerodivisor for S/I, hence $\hat{g} = 0$, that is, $g \in I$, and hence $f \in (u_1)I$. Assume that $p > 1$ and that the lemma is true for the S/I-regular sequence $\{u_1, \dots, u_{p-1}\}$. Take $f \in (u_1, \dots, u_p) \cap I$ and write $f = u_1 g_1 + \cdots + u_p g_p$. In $S/((u_1 + \cdots + u_{p-1}) + I)$ we have the relations $u_p \hat{g}_p = \hat{u}_p \hat{g}_p = \hat{f} - \hat{u}_1 \hat{g}_1 - \cdots - \hat{u}_{p-1} \hat{g}_{p-1} = 0$. By hypothesis, u_p is a non-zerodivisor for $S/((u_1 + \cdots + u_{p-1}) + I)$, hence $\hat{g}_p = 0$. Write $g_p = u_1 h_1 + \cdots + u_{p-1} h_{p-1} + h_p$, where $h_p \in I$. From the relation

$$f - u_p h_p = u_1(g_1 + u_p h_1) + \cdots + u_{p-1}(g_{p-1} + u_p h_{p-1})$$

we see that $f - u_p h_p \in (u_1, \dots, u_{p-1}) \cap I$. By the induction hypothesis, this ideal coincides with $(u_1, \dots, u_{p-1})I$. We conclude that $f \in (u_1, \dots, u_p)I$. ■

Lemma 5.12. *Let S be a commutative ring and let $I \subset S$ be an ideal. Assume that the sequence $\{u_1, \dots, u_{p+1}\} \subset S$ is regular and that $\{u_1, \dots, u_p\}$ is S/I-regular. Then $\{u_1, \dots, u_{p+1}\}$ is also I-regular.*

Proof. By hypothesis, u_1 is a non-zerodivisor in S, hence u_1 is a non-zerodivisor for I. Take $i \in \{1, \dots, p\}$. We apply Lemma 5.11 to the S/I-regular sequence $\{u_1, \dots, u_i\}$. We deduce that $I/(u_1, \dots, u_i)I$ is isomorphic, as an S-module, to an ideal of $S/(u_1, \dots, u_i)$. By hypothesis, u_{i+1} is a non-zerodivisor for $S/(u_1, \dots, u_i)$, hence u_{i+1} is a non-zerodivisor for $I/(u_1, \dots, u_i)I$. ■

Proposition 5.13. *Assume that $\mathbb{K}$ is algebraically closed. Let $X \subset \mathbb{V}$ be a zero-dimensional sub-ACM subscheme. For $1 \le i \le q$, choose $u_i \in U_i^0$. Then $\{u_1, \dots, u_q\}$ is $I(X)$-regular.*

Proof. We assume that $q = 2$. According to Remark 5.2, $\{u_1\}$ is regular for $\mathbb{S}/I(X)$. Clearly, $\{u_1, u_2\}$ is $\mathbb{S}$-regular. From Lemma 5.12 we deduce that $\{u_1, u_2\}$ is $I(X)$-regular. We assume that $q \ge 3$. As in Proposition 5.9, we consider a $\mathbb{S}/I(X)$-regular sequence

$$\{w_1, \dots, w_{q-1}\} = \{u_i + \kappa_i u_p \mid i \in \{1, \dots, q\} \setminus \{p\}\}.$$

Here $\kappa_1 = 0$. We set $w_q = u_p$. Clearly, $\{w_1, \dots, w_q\}$ is $\mathbb{S}$-regular. From Lemma 5.12 we deduce that $\{w_1, \dots, w_q\}$ is also $I(X)$-regular. Let us now consider the column vectors $\mathbf{u} = (u_1, \dots, u_q)^{\mathrm{T}}$ and $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_q)^{\mathrm{T}}$. By construction, we have $\mathbf{w} = \Lambda \mathbf{u}$ for some $\Lambda \in \mathrm{GL}_q(\mathbb{K})$. We apply Lemma 5.8 to the $I(X)$-regular sequence $\{w_1, \dots, w_q\}$. We take $G = \Lambda^{-1}$. We conclude that $\{u_1, \dots, u_q\}$ is regular relative to $I(X)$. ■

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Proposition 5.14. *Assume that $\mathbb{K}$ is algebraically closed. Let $X \subset \mathbb{V}$ be a zero-dimensional ACM subscheme. For $1 \leq i \leq q$ choose $u_i \in U_i^0$. For $1 \leq i \leq q$ choose $v_i \in U_i \setminus \mathbb{K}u_i$. Then $\{u_1, \ldots, u_q, v_i\}$ is $I(X)$-regular for every index i.*

Proof. According to Proposition 5.10, $\{u_1, \ldots, u_q\}$ is regular relative to $\mathbb{S}/I(X)$. Clearly, $\{u_1, \ldots, u_q, v_i\}$ is $\mathbb{S}$-regular. The proposition follows from Lemma 5.12. ■

Proposition 5.15. *Assume that $\mathbb{K}$ is infinite. Assume that $X \subset \mathbb{V}$ is a zero-dimensional ACM subscheme. Then there are $u_i \in U_i$ and $v_i \in U_i \setminus \mathbb{K}u_i$ such that $\{u_1, \ldots, u_q, v_i\}$ is $I(X)$-regular for every index i.*

Proof. Proposition 5.6 provides an $\mathbb{S}/I(X)$-regular sequence $\{u_1, \ldots, u_q\}$. Clearly, $\{u_1, \ldots, u_q, v_i\}$ is $\mathbb{S}$-regular. The proposition follows from Lemma 5.12. ■

Proposition 5.16. *Assume that $\mathbb{K}$ is infinite. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. For $1 \leq i \leq q$, choose $u_i \in U_i^0$ and $v_i \in U_i \setminus \mathbb{K}u_i$. Then, for all indices i and j, $\{u_i, v_j\}$, is regular relative to $I(X)$.*

Proof. According to Remark 5.2, u_i is regular for $\mathbb{S}/I(X)$. Clearly, $\{u_i, v_j\}$ is $\mathbb{S}$-regular. The proposition follows from Lemma 5.12. ■

6 Finite ACM schemes

In this section we assume that the ground field $\mathbb{K}$ is infinite. We recall from Section 5 the notion of an ACM zero-dimensional scheme. Lemma 6.3 provides a class of examples of such schemes. We recall that all zero-dimensional subschemes $X \subset \mathbb{P}^n$ are ACM. The investigation of the next simplest case, when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$, was begun by Giuffrida et al. in [11]. For a summary of results in this case we refer to the monograph [18]. For later use, we cite at Theorem 6.2 some of the results in [11, Section 4].

Notation 6.1. The characteristic function $\mathcal{X}_T : \mathbb{Z}^q \longrightarrow \{0, 1\}$ of a subset $T \subset \mathbb{Z}^q$ is given by $\mathcal{X}_T(a) = 1$ if $a \in T$ and $\mathcal{X}_T(a) = 0$ if $a \in \mathbb{Z}^q \setminus T$.

Theorem 6.2 (Giuffrida et al.). *Consider the biprojective space $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{C}$, with $\mathbb{Z}^2$-graded coordinate ring $\mathbb{C}[x_0, x_1, y_0, y_1]$. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. Recall the relevant domain $R(X) = [0, s_1 - 1] \times [0, s_2 - 1]$ from Proposition 4.5(v). Then the following statements are equivalent:*

(i) *X is ACM;*

(ii) *there is an integer $m \geq 0$ and there are mutually incomparable elements $c_1, \ldots, c_m$ in $R(X)$, such that the quasi-rectangular domain*

$$Q(X) = R(X) \setminus \bigcup_{1 \leq k \leq m} [c_k, (s_1 - 1, s_2 - 1)] \subset \mathbb{Z}^2$$

satisfies the condition $\Delta\mathcal{H}_X = \mathcal{X}_{Q(X)}$. See Notation 6.1;

(iii) *there are homogeneous forms $\xi_1, \ldots, \xi_{m+1} \in \mathbb{C}[x_0, x_1]$ and $\upsilon_1, \ldots, \upsilon_{m+1} \in \mathbb{C}[y_0, y_1]$ such that $I(X) = (\upsilon_1 \cdots \upsilon_k \xi_{k+1} \cdots \xi_{m+1} \mid 0 \leq k \leq m + 1)$.*

Moreover, $\deg(\upsilon_1 \cdots \upsilon_k \xi_{k+1} \cdots \xi_{m+1}) = c_k$*, where $c_0 = (s_1, 0)$ and $c_{m+1} = (0, s_2)$.*

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As per Definition 2.1, $Q(X)$ is a relevant quasi-rectangular domain for $\mathcal{H}_X$, in fact, the smallest possible quasi-rectangular relevant domain.

Other characterizations of the ACM property for zero-dimensional subschemes X in $\mathbb{P}^1 \times \mathbb{P}^1$ are known in the case when X is reduced, see [26, Theorem 4.8], [21, Corollary 7.5], [22, Theorem 6.7], [16, Theorem 4.3] and [17, Theorem 8], and in the case when X is a union of fat points, see [13, Theorem 2.1] and [14, Theorem 4.8]. For other ambient spaces the focus has been entirely on reduced ACM schemes, see [16, Theorem 4.5 and Theorem 5.7], [8, Theorem 3.16], [9, Proposition 3.2 and Theorem 3.7].

We consider the algebra $\mathbb{C}[x_0, x_1, y_0, y_1]/(x_0, y_0) = \mathbb{C}[x_1, y_1]$ and we equip it with the inherited $\mathbb{Z}^2$-grading: $\deg(x_1) = e_1$ and $\deg(y_1) = e_2$. Condition (ii) from Theorem 6.2 is equivalent to saying that $\Delta\mathcal{H}_X$ is the Hilbert function of an artinian quotient of $\mathbb{C}[x_1, y_1]$ by a monomial ideal, i.e. an artinian $\mathbb{Z}^2$-graded quotient of $\mathbb{C}[x_1, y_1]$. This statement was partially generalized by Van Tuyl in [26]. We consider the algebra

$$\mathbb{S}_0 = \mathbb{S}/(x_{10}, \dots, x_{q0}) = \mathbb{K}[x_{ij} \mid 1 \le i \le q,\ 1 \le j \le n_i]$$

equipped with the induced $\mathbb{Z}^q$-grading: $\deg(x_{ij}) = e_i$. According to [26, Theorem 3.11], if $\mathbb{K}$ is algebraically closed and if $X \subset \mathbb{V}$ is zero-dimensional reduced and ACM, then $\Delta\mathcal{H}_X$ is the Hilbert function of an artinian $\mathbb{Z}^q$-graded quotient of $\mathbb{S}_0$. Conversely, for any artinian $\mathbb{Z}^q$-graded quotient A of $\mathbb{S}_0$, there exists a zero-dimensional reduced ACM subscheme $X \subset \mathbb{V}$ such that $\Delta\mathcal{H}_X = \mathcal{H}_A$. The aim of this section is to provide a version of this result that does not require X to be reduced or $\mathbb{K}$ to be algebraically closed.

Lemma 6.3. *Assume that the subscheme $X \subset \mathbb{V}$ is concentrated at a point and that $I(X)$ is a monomial ideal. Then X is ACM.*

Proof. By hypothesis, $\mathrm{red}(X) = \{P\}$ for a closed point $P \in \mathbb{V}$. By the multiprojective version of Hilbert's Nullstellensatz, $I(P) = \mathrm{rad}(I(X))$. As the radical of a monomial ideal, $I(P)$ itself is monomial. But $I(P)$ is also a prime ideal, hence $I(P)$ is generated by a subset of the set of variables. We may assume that $I(P) = (x_{ij} \mid 1 \le i \le q,\ 1 \le j \le n_i)$. If $x_{i0}\zeta \in I(X)$ for a monomial ζ, then, since x_{i0} does not vanish at P, ζ must lie in $I(X)$. This shows that the minimal generators of $I(X)$ are monomials in the same variables that generate $I(P)$. It has now become clear that $\{x_{i0} \mid 1 \le i \le q\}$ constitutes a regular sequence for $\mathbb{S}/I(X)$, hence $\mathrm{depth}(\mathbb{S}/I(X)) \ge q$, and hence $\mathrm{depth}(\mathbb{S}/I(X)) = q$. ∎

In the sequel we will need Macaulay's theorem, see [5, Theorem 15.3]. This theorem is usually stated for homogeneous ideals of polynomial rings, but it can easily be extended to the $\mathbb{Z}^q$-graded setting.

Notation 6.4. Let us fix a monomial well-ordering on $\mathbb{S}$. This is a well-ordering "$\le$" on the set $\mathbb{M}$ of monic monomials of $\mathbb{S}$ which is compatible with multiplication: if ζ_1, ζ_2 and ζ lie in $\mathbb{M}$ and $\zeta_1 < \zeta_2$, then $\zeta_1\zeta < \zeta_2\zeta$. We say that a monomial $\zeta \in \mathbb{M}$ *occurs* in a polynomial $f \in \mathbb{S}$ if $\kappa\zeta$ is one of the monomials of f for some $\kappa \in \mathbb{K} \setminus \{0\}$. For $f \in \mathbb{S} \setminus \{0\}$ we denote by $\mathrm{lead}(f)$ the largest monomial that occurs in f. For an ideal $I \subset \mathbb{S}$ we introduce the *leading ideal* $\mathrm{lead}(I) = (\mathrm{lead}(f) \mid f \in I)$.

Theorem 6.5 (Macaulay)**.** *Let $I \subset \mathbb{S}$ be a $\mathbb{Z}^q$-homogeneous ideal. We choose a monomial well-ordering on $\mathbb{S}$. Then* $\mathrm{lead}(I)$ *is $\mathbb{Z}^q$-homogeneous and has the same Hilbert function as I.*

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Theorem 6.6. *Assume that $\mathbb{K}$ is infinite. Let $X \subset \mathbb{V}$ be a zero-dimensional ACM subscheme. We assert that $\Delta\mathcal{H}_X = \mathcal{H}_A$ for an artinian $\mathbb{Z}^q$-graded quotient A of $\mathbb{S}_0$. Conversely, for any artinian $\mathbb{Z}^q$-graded quotient A of $\mathbb{S}_0$, we assert that there exists a zero-dimensional ACM subscheme $X \subset \mathbb{V}$ such that $\Delta\mathcal{H}_X = \mathcal{H}_A$.*

Proof. Assume that X is ACM. According to Proposition 5.6, there exists a regular sequence $\{u_1, \dots, u_q\}$ relative to $\mathbb{S}/I(X)$ with $u_i \in U_i$. In view of Lemma 3.5, $\Delta\mathcal{H}_X$ is the Hilbert function of $A = \mathbb{S}/((u_1, \dots, u_q) + I(X))$. Performing a linear change of coordinates on each $\mathbb{P}^{n_i}$, we may assume that $u_i = x_{i0}$, therefore A can be regarded as a $\mathbb{Z}^q$-graded quotient of $\mathbb{S}_0$. According to Theorem 4.4, $\mathcal{H}_A$ vanishes outside a rectangular domain, hence $\dim_{\mathbb{K}} A$ is finite, and hence A is an artinian $\mathbb{K}$-algebra.

Conversely, we assume we are given an artinian $\mathbb{Z}^q$-graded algebra $A = \mathbb{S}_0/I_0$. We recall Notation 6.4. We choose a monomial well-ordering on $\mathbb{S}_0$ and we apply Theorem 6.5 to the $\mathbb{Z}^q$-homogeneous ideal I_0. We find a monomial ideal $J_0 = \operatorname{lead}(I_0)$ such that $\mathcal{H}_A = \mathcal{H}_{\mathbb{S}_0/J_0}$. In particular, $\mathbb{S}_0/J_0$ is an artinian algebra, hence $\operatorname{rad}(J_0) = (x_{ij} \mid 1 \le i \le q, 1 \le j \le n_i)$. Let $J \subset \mathbb{S}$ be the ideal generated by J_0. Since J is generated by monomials that do not involve the variables x_{i0}, $1 \le i \le q$, it is obvious that J is saturated. Thus, $J = I(X)$ for a zero-dimensional subscheme $X \subset \mathbb{V}$ which is concentrated on the point given by the ideal $(x_{ij} \mid 1 \le i \le q, 1 \le j \le n_i)$. In view of Lemma 6.3, X is ACM. Since J is generated by monomials that do not involve the variables x_{i0}, $1 \le i \le q$, it is obvious that $\{x_{10}, \dots, x_{q0}\}$ is $\mathbb{S}/J$-regular. In view of Lemma 3.5, $\Delta\mathcal{H}_X$ must be the Hilbert function of $\mathbb{S}/((x_{10}, \dots, x_{q0}) + J) = \mathbb{S}_0/J_0$. ∎

The first assertion, in the particular case when $\mathbb{K} = \overline{\mathbb{K}}$ and X is reduced, was obtained by Van Tuyl in [26, Theorem 3.11]. The converse assertion, in the particular case when $\mathbb{K} = \overline{\mathbb{K}}$, already follows from op.cit. Indeed, Van Tuyl proved that for any A we can find a zero-dimensional reduced ACM subscheme $X \subset \mathbb{V}$ such that $\Delta\mathcal{H}_X = \mathcal{H}_A$.

Corollary 6.7. *Assume that $\mathbb{K}$ is infinite. Let $X \subset (\mathbb{P}^1)^q$ be a zero-dimensional ACM subscheme. We assert that there exists a quasi-rectangular domain $Q(X) \subset \mathbb{Z}^q$ such that $\Delta\mathcal{H}_X = \mathcal{X}_{Q(X)}$. See Notation 6.1. Conversely, for any quasi-rectangular domain $Q \subset \mathbb{Z}^q$, we assert that there exists a zero-dimensional ACM subscheme $X \subset (\mathbb{P}^1)^q$ such that $\Delta\mathcal{H}_X = \mathcal{X}_Q$.*

Proof. Note that $\mathbb{S}_0 = \mathbb{K}[x_{i1} \mid 1 \le i \le q]$, where $\deg(x_{i1}) = e_i$ for all i. An ideal of $\mathbb{S}_0$ is $\mathbb{Z}^q$-homogeneous if and only if it is monomial. If $A = \mathbb{S}_0/I_0$ is an artinian quotient by a monomial ideal, then I_0 must contain minimal generators of the form $x_{11}^{s_1}, \dots, x_{q1}^{s_q}$. Let $c_1, \dots, c_m$ be the degrees of the remaining minimal generators of I_0, if any. Write

$$Q = [0, s_1 - 1] \times \cdots \times [0, s_q - 1] \setminus \bigcup_{1 \le k \le m} [c_k, (s_1 - 1, \dots, s_q - 1)].$$

We have $\mathcal{H}_A = \mathcal{X}_Q$ Conversely, for any quasi-rectangular domain $Q \subset \mathbb{Z}^q$, we can find an artinian $\mathbb{Z}^q$-graded quotient $A = \mathbb{S}_0/I_0$ such that $\mathcal{H}_A = \mathcal{X}_Q$. ∎

The first assertion, in the particular case when $\mathbb{K} = \overline{\mathbb{K}}$ and X is reduced, was obtained by Van Tuyl in [26, Corollary 3.14]. The converse assertion, in the particular case when $\mathbb{K} = \overline{\mathbb{K}}$, already follows from op.cit. Indeed, Van Tuyl proved that for any Q we can find a zero-dimensional reduced ACM subscheme $X \subset (\mathbb{P}^1)^q$ such that $\Delta\mathcal{H}_X = \mathcal{X}_Q$.

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7 Further constraints on the Hilbert functions

In this section we assume that the ground field $\mathbb{K}$ is infinite. This section is devoted to a better understanding of the problem of classification of the functions $\mathbb{Z}^q \longrightarrow \mathbb{Z}$ that arise as Hilbert functions of zero-dimensional subschemes $X \subset \mathbb{V}$. The classical theorem [2, Theorem 4.2.10] of Macaulay provides a classification of the Hilbert functions of $\mathbb{Z}$-graded $\mathbb{K}$-algebras. The recent theorem [6, Theorem 4.8] of Favacchio provides a classification of the Hilbert functions of $\mathbb{Z}^2$-graded $\mathbb{K}$-algebras. Invoking Theorem 6.6, we obtain a characterization of the functions $\mathcal{H}_X$, for zero-dimensional subschemes $X \subset \mathbb{P}^n$, respectively, for zero-dimensional ACM subschemes $X \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. Yet the problem of describing the functions $\mathcal{H}_X$ in the case when $X \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ is sub-ACM or in the case when $q \geq 3$ remains open. In this section we make progress on this problem by exhibiting certain conditions that the functions $\mathcal{H}_X$ and $\mathcal{H}_{I(X)}$ must satisfy. These constraints are formulated in terms of the partial difference functions, defined below. The emphasis will be on ACM and sub-ACM schemes. All constraints will arise in the manner of Theorem 6.6, by exploiting the regular sequences from Section 5, and then by applying Lemma 3.5. At the end of the section we give a second proof to Theorem 4.4 in the particular case when $\mathbb{V} = (\mathbb{P}^1)^q$ and X is ACM or sub-ACM. Let $\mathcal{F} \colon \mathbb{Z}^q \longrightarrow \mathbb{Z}$ be a function. For $1 \leq i \leq q$ we consider the *partial difference* function

$$\Delta_i \mathcal{F} \colon \mathbb{Z}^q \longrightarrow \mathbb{Z} \quad \text{given by} \quad \Delta_i \mathcal{F}(a) = \mathcal{F}(a) - \mathcal{F}(a - e_i).$$

We will use the abbreviation $\Delta_{i_1 \ldots i_p} \mathcal{F} = \Delta_{i_1} \ldots \Delta_{i_p} \mathcal{F}$. Recalling Equation (1), we notice that $\Delta \mathcal{F} = \Delta_{1,\ldots,q} \mathcal{F}$ and that $\Delta \Delta_i \mathcal{F} = \Delta_{1,\ldots,q,i} \mathcal{F}$. Differentiating the equation

$$\mathcal{H}_{\mathbb{S}}(a) = \prod_{1 \leq i \leq q} \binom{a_i + n_i}{n_i} \text{ yields the equation } \Delta \mathcal{H}_{\mathbb{S}}(a) = \prod_{1 \leq i \leq q} \binom{a_i + n_i - 1}{n_i - 1}. \tag{6}$$

Assume that $\mathcal{F}$ vanishes on the complement of $\mathbb{Z}^q_+$. Fix indices $1 \leq i_1 < \cdots < i_p \leq q$. By analogy with Equation (2), we have the formula

$$\Delta_{i_1,\ldots,i_p} \mathcal{F}(a) = \sum_{\substack{0 \leq b \leq a \\ b_{i_1} = a_{i_1}, \ldots, b_{i_p} = a_{i_p}}} \Delta \mathcal{F}(b) \quad \text{for all} \quad a \in \mathbb{Z}^q_+. \tag{7}$$

Proposition 7.1. *Assume that $\mathbb{K}$ is algebraically closed. Assume that the zero-dimensional subscheme $X \subset \mathbb{V}$ is sub-ACM. Then the following statements hold true:*

(i) $\Delta \mathcal{H}_{I(X)} \geq 0$;
(ii) $\Delta \mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$;
(iii) $\Delta \mathcal{H}_X \leq \Delta \mathcal{H}_{\mathbb{S}}$;
(iv) *if* $\Delta \mathcal{H}_X(a) = \Delta \mathcal{H}_{\mathbb{S}}(a)$ *for some* $a \in \mathbb{Z}^q_+$, *then* $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ *on* $[0, a]$;
(v) $\Delta_{i_1 \ldots i_p} \mathcal{H}_{I(X)} \geq 0$ *for all indices* $1 \leq i_1 < \cdots < i_p \leq q$;
(vi) $\Delta_{i_1 \ldots i_p} \mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$ *and* $1 \leq i_1 < \cdots < i_p \leq q$.

Proof. Recall Notation 5.1. As per Proposition 5.13, we can construct $I(X)$-regular sequences $\{u_1, \ldots, u_q\}$ with generic $u_i \in U_i$. We write $N = I(X)/(u_1, \ldots, u_q)I(X)$. By applying Lemma 3.5, we deduce that $\Delta \mathcal{H}_{I(X)} = \mathcal{H}_N$. This function takes, of course,

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only non-negative values. We have proved statement (i). The same argument applies to statement (v), except that this time we consider the $I(X)$-regular sequence $\{u_{i_1}, \dots, u_{i_p}\}$.

If $I(X)_a \neq \{0\}$, then we can choose u_i such that the subvariety given by the ideal $(u_1, \dots, u_q)$ is not contained in the zero-set of $I(X)_a$. Thus, $N_a \neq \{0\}$, hence $\Delta\mathcal{H}_{I(X)}(a) > 0$. This proves statement (ii). The same argument applies to statement (vi), except that this time we consider the ideal $(u_{i_1}, \dots, u_{i_p})$. Statement (iii) follows from the equation $\Delta\mathcal{H}_X = \Delta\mathcal{H}_{\mathbb{S}} - \Delta\mathcal{H}_{I(X)}$ and from (i).

Assume that $\Delta\mathcal{H}_X(a) = \Delta\mathcal{H}_{\mathbb{S}}(a)$ for some $a \in \mathbb{Z}^q_+$. Thus, $\Delta\mathcal{H}_{I(X)}(a) = 0$. From (ii) we deduce that $I(X)_a = \{0\}$. A fortiori, $I(X)_b = \{0\}$ for $b \in [0, a]$, hence $\mathcal{H}_{I(X)} = 0$ on $[0, a]$, and hence $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ on $[0, a]$. This proves statement (iv). ■

Proposition 7.2. *Assume that $\mathbb{K}$ is algebraically closed.. Assume that the zero-dimensional subscheme $X \subset \mathbb{V}$ is ACM. As provided in Theorem 6.6, let A be an artinian algebra such that $\Delta\mathcal{H}_X = \mathcal{H}_A$. Then the following statements hold true:*

- (i) $\Delta\Delta_i\mathcal{H}_{I(X)} \geq 0$ *for all indices* $i \in \{1, \dots, q\}$;
- (ii) $\Delta\Delta_i\mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$ *and* $n_i \geq 2$;
- (iii) $\Delta_i\mathcal{H}_A \leq \Delta\Delta_i\mathcal{H}_{\mathbb{S}}$ *for all indices* $i \in \{1, \dots, q\}$;
- (iv) *if* $n_i \geq 2$ *and* $\Delta_i\mathcal{H}_A(a) = \Delta\Delta_i\mathcal{H}_{\mathbb{S}}(a)$ *for some* $a \in \mathbb{Z}^q_+$, *then* $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ *on* $[0, a]$;
- (v) $\Delta\mathcal{H}_{I(X)} \geq 0$;
- (vi) $\Delta\mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$;
- (vii) $\mathcal{H}_A \leq \Delta\mathcal{H}_{\mathbb{S}}$;
- (viii) *if* $\mathcal{H}_A(a) = \Delta\mathcal{H}_{\mathbb{S}}(a)$ *for some* $a \in \mathbb{Z}^q_+$, *then* $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ *on* $[0, a]$;
- (ix) $\Delta_{i_1 \dots i_p}\mathcal{H}_{I(X)} \geq 0$ *for all indices* $1 \leq i_1 < \dots < i_p \leq q$;
- (x) $\Delta_{i_1 \dots i_p}\mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$ *and* $1 \leq i_1 < \dots < i_p \leq q$.

Statements (i), (iii), (v), (vii) and (ix) also hold true under the weaker hypothesis that $\mathbb{K}$ be infinite.

Proof. Consider the $I(X)$-regular sequence $\{u_1, \dots, u_q, v_i\}$ from Proposition 5.15. Write $N = I(X)/(u_1, \dots, u_q, v_i)I(X)$. By analogy with Lemma 3.5, we can prove that $\Delta\Delta_i\mathcal{H}_{I(X)} = \mathcal{H}_N$. This function takes only non-negative values, proving statement (i). Assume that $I(X)_a \neq \{0\}$ and $n_i \geq 2$. According to Proposition 5.14, $u_1, \dots, u_q$ and v_i can be chosen generically. We choose them in such a way that the subvariety given by the ideal $(u_1, \dots, u_q, v_i)$ is not contained in the zero-set of $I(X)_a$. Thus, $N_a \neq \{0\}$, hence $\mathcal{H}_N(a) > 0$. This proves statement (ii). Statement (iii) follows from the equation $\Delta_i\mathcal{H}_A = \Delta\Delta_i\mathcal{H}_{\mathbb{S}} - \Delta\Delta_i\mathcal{H}_{I(X)}$ and from (i). To prove the remaining statements we can argue as in the proof of Proposition 7.1. Note also that (v) follows from the formula

$$\Delta\mathcal{H}_{I(X)} = \sum_{0 \leq k \leq a_i} \Delta\Delta_i\mathcal{H}_{I(X)}(a - ke_i)$$

and from (i). In the case when $n_i \geq 2$, statement (vi) follows from the above formula and from (i) and (ii). Statement (ix) follows from Equation (7) and from (v). Statement (x) follows from Equation (7) and from (v) and (vi). ■

Corollary 7.3. *Assume that $\mathbb{K}$ is algebraically closed. Assume that the zero-dimensional subscheme $X \subset (\mathbb{P}^1)^q$ is ACM or sub-ACM. Then the following statements hold true:*

- (i) $\Delta\mathcal{H}_X(a) \leq 1$ *for all* $a \in \mathbb{Z}^q_+$;

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(ii) *if* $\Delta\mathcal{H}_X(a) = 1$ *for some* $a \in \mathbb{Z}_+^q$, *then* $\Delta\mathcal{H}_X = 1$ *on* $[0, a]$.

Proof. Substituting $n_i = 1$ into Equation (6), we obtain $\Delta\mathcal{H}_{\mathbb{S}}(a) = 1$ for $a \in \mathbb{Z}_+^q$. Substituting this expression into Propositions 7.1(iii) and 7.2(vii) yields statement (i). Substituting this expression into Propositions 7.1(iv) and 7.2(viii) yields (ii). ■

In the case when X is ACM, the above corollary also follows from Corollary 6.7. The above result, in the particular case when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$, was obtained by Giuffrida et al. Consult [11, Proposition 2.7].

Proposition 7.4. *Assume that* $\mathbb{K}$ *is algebraically closed. Let* $X \subset \mathbb{V}$ *be a zero-dimensional subscheme. Then the following statements hold true:*

(i) $\Delta_{ij}\mathcal{H}_{I(X)} \geq 0$ *for all indices* $1 \leq i \leq j \leq q$;
(ii) $\Delta_{ij}\mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$ *and* $1 \leq i < j \leq q$;
(iii) $\Delta_{ii}\mathcal{H}_{I(X)}(a) > 0$ *if* $I(X)_a \neq \{0\}$ *and* $n_i \geq 2$;
(iv) $\Delta_{ij}\mathcal{H}_X \leq \Delta_{ij}\mathcal{H}_{\mathbb{S}}$ *for all indices* $1 \leq i < j \leq q$;
(v) *if* $\Delta_{ij}\mathcal{H}_X(a) = \Delta_{ij}\mathcal{H}_{\mathbb{S}}(a)$ *for some* $a \in \mathbb{Z}_+^q$ *and indices* $1 \leq i < j \leq q$, *then* $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ *on* $[0, a]$;
(vi) *if* $n_i \geq 2$ *and* $\Delta_{ii}\mathcal{H}_X(a) = \Delta_{ii}\mathcal{H}_{\mathbb{S}}(a)$ *for some* $a \in \mathbb{Z}_+^q$, *then* $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}}$ *on* $[0, a]$;
(vii) $\Delta_i\mathcal{H}_X \geq 0$ *for all indices* $i \in \{1, \ldots, q\}$.

Statements (i), (iv) and (vii) also hold true under the weaker hypothesis that $\mathbb{K}$ *be infinite.*

Proof. We use the $I(X)$-regular sequence $\{u_i, v_j\}$ provided by Proposition 5.16 and we repeat the arguments from the proof of Proposition 7.1. Statement (vii) follows from the fact that $\{u_i\}$ is regular for $\mathbb{S}/I(X)$, see Remark 5.2. ■

As an application of the above results, we will give a second proof to a particular case of Theorem 4.4. We formulate this as a separate proposition.

Proposition 7.5. *Assume that* $\mathbb{K}$ *is algebraically closed. Assume that the zero-dimensional subscheme* $X \subset (\mathbb{P}^1)^q$ *is ACM or sub-ACM. For each index* $i \in \{1, \ldots, q\}$, *let* s_i *be the length of the projection of* X *onto the* i*-th copy of* $\mathbb{P}^1$. *Then* $[0, s_1 - 1] \times \cdots \times [0, s_q - 1]$ *is the smallest rectangular relevant domain for* $\mathcal{H}_X$.

Proof. In view of Lemma 2.2, we must show that $\mathcal{H}_X(a) = \mathcal{H}_X(a - (a_i - s_i + 1)e_i)$ if $a_i \geq s_i - 1$. Equivalently, we must show that $\Delta_i\mathcal{H}_X(a) = 0$ if $a_i \geq s_i$. By symmetry, it is enough to prove that $\Delta_1\mathcal{H}_X(a) = 0$ if $a \in \mathbb{Z}_+^q$ and $a_1 \geq s_1$. Given $a \in \mathbb{Z}_+^q$, put $\sigma(a) = a_2 + \cdots + a_q$. We perform induction on $\sigma(a)$. To begin the induction, assume that $\sigma(a) = 0$. Write $Z_1 = \mathrm{pr}_1(X)$. According to Proposition 3.8(i and v), we have the equation $\mathcal{H}_X(b_1, 0, \ldots, 0) = \mathcal{H}_{Z_1}(b_1) = s_1$ for $b_1 \geq s_1 - 1$. This leads to the desired outcome $\Delta_1\mathcal{H}_X(a) = 0$. We now perform the induction step. We assume that $\sigma(a) > 0$. To simplify notation, we assume that $a_2, \ldots, a_p$ are positive and $a_{p+1}, \ldots, a_q$ are zero for some $p \in \{2, \ldots, q\}$. From the definition of the partial difference functions we obtain the equation

$$\Delta_1\mathcal{H}_X(a) = \Delta_{1,\ldots,p}\mathcal{H}_X(a) + \sum_{1 \leq k \leq p-1} \Delta_{1,\ldots,k}\mathcal{H}_X(a - e_{k+1}).$$

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Since $\sigma(a - e_{k+1}) < \sigma(a)$, $\Delta_{1,\ldots,k}\,\mathcal{H}_X(a - e_{k+1})$ is a finite sum of expressions of the form $\pm\Delta_1\,\mathcal{H}_X(b)$, with $\sigma(b) < \sigma(a)$ and with $b_1 = a_1$. By the induction hypothesis these expressions vanish. We obtain the equations

$$\begin{aligned}\Delta_1\,\mathcal{H}_X(a) &= \Delta_{1,\ldots,p}\,\mathcal{H}_X(a)\\ &= (a_{p+1}+1)\cdots(a_q+1) - \Delta_{1,\ldots,p}\,\mathcal{H}_{I(X)}(a)\\ &= 1 - \Delta_{1,\ldots,p}\,\mathcal{H}_{I(X)}(a).\end{aligned}$$

We know that $I(Z_1)_{s_1} \neq \{0\}$. It follows that $I(X)_a \neq \{0\}$. Since we are assuming that X is ACM or sub-ACM, we may apply Proposition 7.1(vi) and Proposition 7.2(x) in order to obtain the inequality $\Delta_{1,\ldots,p}\,\mathcal{H}_{I(X)}(a) > 0$. A fortiori, $\Delta_1\,\mathcal{H}_X(a) \leq 0$. According to Proposition 7.4(vii), the reverse inequality $\Delta_1\,\mathcal{H}_X(a) \geq 0$ is also satisfied. We obtain the desired outcome $\Delta_1\,\mathcal{H}_X(a) = 0$. This concludes the induction step. ∎

The above line of argument, in the particular case when $\mathbb{V} = \mathbb{P}^1 \times \mathbb{P}^1$, is due to Giuffrida et al. Consult [11, Remark 2.8 and Theorem 2.11]. We have adapted their proof to the case of arbitrary q. In the case when $q = 2$ there is no restriction on X because every zero-dimensional subscheme $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is ACM or sub-ACM.

8 A vanishing result for $\Delta\mathcal{H}$

In this section we assume that $\mathbb{V} = (\mathbb{P}^1)^q$ with coordinate ring $\mathbb{S} = \mathbb{K}[x_1, y_1, \ldots, x_q, y_q]$, where $\deg(x_i) = \deg(y_i) = e_i$. We saw in Theorem 6.2 and in Corollary 6.7 that zero-dimensional ACM subschemes $X \subset \mathbb{V}$ that are not complete intersections have a quasi-rectangular relevant domain $Q(X)$ which is strictly contained in the rectangular relevant domain $R(X)$ introduced in Theorem 4.4. This section and the next are devoted to finding a procedure (Proposition 9.5) for constructing a quasi-rectangular relevant domain $D(X) \subset R(X)$ that applies to schemes X which are not necessarily ACM. We restrict our attention only to schemes X for which $I(X)$ is a monomial ideal. The domain $D(X)$ may coincide with $R(X)$ or may be strictly contained in $R(X)$, depending on the scheme. At the end of Section 9 we shall give examples in which $D(X)$ is strictly contained in $R(X)$.

In this section we do some preparatory work. We obtain a vanishing criterion for $\Delta\mathcal{H}_{\mathbb{S}/J}$, where $J \subset \mathbb{S}$ is a monomial ideal. In order to achieve this, we need to take two preliminary steps. First, in Equation (9) we obtain a combinatorial formula for $\Delta\mathcal{H}_{\mathbb{S}/J}$. This formula actually holds for any $\mathbb{Z}^q$-graded polynomial ring, i.e. for arbitrary values of $n_1, \ldots, n_q$. The second step, Lemma 8.3, is also combinatorial and breaks down if there are more than two variables of degree e_i. This is the technical reason why our ambient space needs to be a product of projective lines.

We find it convenient to work with the function $\mathcal{H}_J$. Substituting $n_i = 1$ into Equation (6) we get $\Delta\mathcal{H}_{\mathbb{S}}(a) = 1$ for $a \in \mathbb{Z}^q_+$. A fortiori,

$$\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 1 - \Delta\mathcal{H}_J(a) \quad \text{for} \quad a \in \mathbb{Z}^q_+. \tag{8}$$

Notation 8.1. Let $J \subset \mathbb{S}$ be a monomial ideal. Let $\mathbb{M}$ be the set of monic monomials of $\mathbb{S}$. Let $\Gamma(J) = \{f_1, \ldots, f_m\} \subset \mathbb{M}$ be the set of minimal generators of J. Fix $a \in \mathbb{Z}^q$. For

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all integers $p \in \{1, \dots, m\}$ we write

$$\Gamma_a^p(J) = \{(f_{k_1}, \dots, f_{k_p}) \mid 1 \le k_1 < \dots < k_p \le m,\ \deg(\mathrm{lcm}(f_{k_1}, \dots, f_{k_p})) \le a\}.$$

Lemma 8.2. *We consider $a \in \mathbb{Z}_+^q$. We adopt the above notation. We assert that*

$$\Delta \mathcal{H}_{\mathbb{S}/J}(a) = 1 + \sum_{p \ge 1} (-1)^p \left|\Gamma_a^p(J)\right|. \tag{9}$$

Proof. Write $d_{k_1 \dots k_p} = (d^1_{k_1 \dots k_p}, \dots, d^q_{k_1 \dots k_p}) = \deg(\mathrm{lcm}(f_{k_1}, \dots, f_{k_p}))$. By definition,

$$\mathcal{H}_J(b) = \Big| \bigcup_{1 \le k \le m} \{\zeta \in \mathbb{M} \mid f_k \text{ divides } \zeta,\ \deg(\zeta) = b\} \Big|$$

for $b \in \mathbb{Z}^q$. Applying the inclusion-exclusion principle, we obtain the formula

$$\mathcal{H}_J(b) = \sum_{1 \le p \le m} (-1)^{p+1} \sum_{1 \le k_1 < \dots < k_p \le m} \left|\{\zeta \in \mathbb{M} \mid f_{k_1}, \dots, f_{k_p} \text{ divide } \zeta,\ \deg(\zeta) = b\}\right|.$$

Ignoring the empty sets on the r.h.s., we calculate:

$$\mathcal{H}_J(b) = \sum_{1 \le p \le m} (-1)^{p+1} \sum_{\substack{1 \le k_1 < \dots < k_p \le m \\ d_{k_1 \dots k_p} \le b}} (b_1 - d^1_{k_1 \dots k_p} + 1) \cdots (b_q - d^q_{k_1 \dots k_p} + 1).$$

Assume now that $b = a + c$, where $c_i \in \{-1, 0\}$ for all indices $i \in \{1, \dots, q\}$. If $d \in \mathbb{Z}^q$ satisfies the conditions $d \le a$ and $d \nleq b$, then there is an index i such that $d_i = a_i = b_i + 1$, forcing the equation $(b_1 - d_1 + 1) \cdots (b_q - d_q + 1) = 0$. Thus, on the r.h.s. of the above formula we can add all the terms for which $d_{k_1 \dots k_p} \le a$ but $d_{k_1 \dots k_p} \nleq b$, i.e. we can replace b by a under the summation sign:

$$\mathcal{H}_J(b) = \sum_{1 \le p \le m} (-1)^{p+1} \sum_{\substack{1 \le k_1 < \dots < k_p \le m \\ d_{k_1 \dots k_p} \le a}} (b_1 - d^1_{k_1 \dots k_p} + 1) \cdots (b_q - d^q_{k_1 \dots k_p} + 1).$$

This equation holds for $b = a + c$ for all possible $c \in \{-1, 0\}^q$, hence we may apply the Δ operator calculated at a to both sides:

$$\begin{aligned}\Delta \mathcal{H}_J(a) &= \sum_{1 \le p \le m} (-1)^{p+1} \sum_{\substack{1 \le k_1 < \dots < k_p \le m \\ d_{k_1 \dots k_p} \le a}} \Delta \left((b_1 - d^1_{k_1 \dots k_p} + 1) \cdots (b_q - d^q_{k_1 \dots k_p} + 1)\right)|_{b=a} \\ &= \sum_{1 \le p \le m} (-1)^{p+1} \sum_{\substack{1 \le k_1 < \dots < k_p \le m \\ d_{k_1 \dots k_p} \le a}} 1 \quad = \sum_{1 \le p \le m} (-1)^{p+1} \left|\Gamma_a^p(J)\right|.\end{aligned}$$

To conclude the proof of the lemma we employ Equation (8). ∎

Lemma 8.3. *We assume that $2 \le p \le m$. We recall Notation 8.1. We assert that*

$$\Gamma_a^p(J) = \big\{(f_{k_1}, \dots, f_{k_p}) \mid 1 \le k_1 < \dots < k_p \le m,\ (f_{k_\mu}, f_{k_\nu}) \in \Gamma_a^2(J) \text{ for all indices } 1 \le \mu < \nu \le p\big\}.$$

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Proof. Assume that $(f_{k_1}, \dots, f_{k_p})$ lies in $\Gamma_a^p(J)$. Then, for all indices $1 \le \mu < \nu \le p$,

$$\deg(\operatorname{lcm}(f_{k_\mu}, f_{k_\nu})) \le \deg(\operatorname{lcm}(f_{k_1}, \dots, f_{k_p})) \le a.$$

This proves the inclusion "$\subset$". Conversely, assume that $(f_{k_1}, \dots, f_{k_p})$ belongs to the set on the r.h.s. For each index $\mu \in \{1, \dots, p\}$ write $f_{k_\mu} = x_1^{\alpha_{\mu 1}} y_1^{\beta_{\mu 1}} \cdots x_q^{\alpha_{\mu q}} y_q^{\beta_{\mu q}}$. For all indices $i \in \{1, \dots, q\}$ put $\alpha_i = \max_{1 \le \mu \le p} \alpha_{\mu i}$ and $\beta_i = \max_{1 \le \mu \le p} \beta_{\mu i}$. Thus,

$$\operatorname{lcm}(f_{k_1}, \dots, f_{k_p}) = x_1^{\alpha_1} y_1^{\beta_1} \cdots x_q^{\alpha_q} y_q^{\beta_q}.$$

For a fixed index $i \in \{1, \dots, q\}$ choose indices $\mu, \nu \in \{1, \dots, p\}$ such that $\alpha_i = \alpha_{\mu i}$ and $\beta_i = \beta_{\nu i}$. If $\mu = \nu$, then $\alpha_i + \beta_i = \alpha_{\mu i} + \beta_{\mu i} = \deg(f_{k_\mu})_i \le a_i$. If $\mu \ne \nu$, then $\alpha_i + \beta_i = \alpha_{\mu i} + \beta_{\nu i} = \deg(\operatorname{lcm}(f_{k_\mu}, f_{k_\nu}))_i \le a_i$. Since i was chosen arbitrarily, we get

$$\deg(\operatorname{lcm}(f_{k_1}, \dots, f_{k_p})) = (\alpha_1 + \beta_1, \dots, \alpha_q + \beta_q) \le a.$$

Thus, $(f_{k_1}, \dots, f_{k_p})$ must lie in $\Gamma_a^p(J)$. This proves the reverse inclusion "$\supset$". ∎

Proposition 8.4. *Let $J \subset \mathbb{K}[x_1, y_1, \dots, x_q, y_q]$ be a monomial ideal. Consider $a \in \mathbb{Z}_+^q$ and let $\{g_1, \dots, g_n\}$ be the set of minimal generators of J whose degree is less or equal to a. If $n = 1$, then $\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 0$. If $n \ge 2$ and $\deg(\operatorname{lcm}(g_1, g_l)) \le a$ for all indices $l \in \{2, \dots, n\}$, then, again, $\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 0$.*

Proof. We recall Notation 8.1. By hypothesis, (g_1, g_l) lies in $\Gamma_a^2(J)$ for all indices l in $\{2, \dots, n\}$. In view of Lemma 8.3, for $2 \le p \le n$ we can write $\Gamma_a^p(J) = \Phi^p \sqcup \Psi^p$, where

$$\Phi^p = \{(g_1, g_{l_2}, \dots, g_{l_p}) \mid 2 \le l_2 < \cdots < l_p \le n,\ (g_{l_2}, \dots, g_{l_p}) \in \Gamma_a^{p-1}(J)\}$$

and

$$\Psi^p = \{(g_{l_1}, \dots, g_{l_p}) \in \Gamma_a^p(J) \mid 2 \le l_1 < \cdots < l_p \le n\}.$$

We notice that $|\Phi^p| = \left|\Psi^{p-1}\right|$, where, by convention, $\Psi^1 = \{g_2, \dots, g_n\}$. We also notice that $\Psi^n = \emptyset$. Applying Equation (9), we calculate:

$$\begin{aligned}
\Delta\mathcal{H}_{\mathbb{S}/J}(a) &= 1 + \sum_{1 \le p \le n} (-1)^p \left|\Gamma_a^p(J)\right| \\
&= 1 - \left|\Gamma_a^1(J)\right| + \sum_{2 \le p \le n} (-1)^p (|\Phi^p| + |\Psi^p|) \\
&= 1 - n + \sum_{2 \le p \le n} (-1)^p \left|\Psi^{p-1}\right| + \sum_{2 \le p \le n-1} (-1)^p \, |\Psi^p| \\
&= 1 - n + \left|\Psi^1\right| + \sum_{2 \le p \le n-1} (-1)^{p+1} \, |\Psi^p| + \sum_{2 \le p \le n-1} (-1)^p \, |\Psi^p| \\
&= 1 - n + (n-1) + \sum_{2 \le p \le n-1} ((-1)^{p+1} + (-1)^p) \, |\Psi^p| \\
&= 0.
\end{aligned}$$

In the case when $n = 1$, we can notice directly that $\left|\Gamma_a^1(J)\right| = 1$ and $\left|\Gamma_a^p(J)\right| = 0$ for $p \ge 2$. Substituting these values into Equation (9) yields the equation $\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 0$. ∎

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9 Finite subschemes of a product of projective lines

In this section we assume that $\mathbb{K}$ is infinite and that $\mathbb{V} = (\mathbb{P}^1)^q$, where $q \geq 2$. We write $\mathbb{S} = \mathbb{K}[x_1, y_1, \ldots, x_q, y_q]$, where $\deg(x_i) = e_i$ and $\deg(y_i) = e_i$. Let $X \subset \mathbb{V}$ be a zero-dimensional subscheme. We recall from Section 4 that s_i denotes the length of the projection Z_i of X onto the i-th component of $\mathbb{V}$. Our first goal is to give a different proof for the fact that the domain $[0, s_1 - 1] \times \cdots \times [0, s_q - 1]$ is relevant to $\mathcal{H}_X$. Our second goal is to give a procedure for detecting quasi-rectangular domains that are relevant to $\mathcal{H}_X$. All results of this section are applications of Theorem 6.5 and of Proposition 8.4.

Lemma 9.1. *We adopt the above notation. We consider the lexicographic monomial ordering on $\mathbb{S}$ such that $x_1 > y_1 > \cdots > x_q > y_q$. We assume that y_1 does not vanish at any point of* $\operatorname{red}(X)$*. We assert that $x_1^{s_1}$ is a minimal generator of* $\operatorname{lead}(I(X))$*. See Notation 6.4. We further assert that every other minimal generator of* $\operatorname{lead}(I(X))$ *has the form $x_1^{\alpha_1} x_2^{\alpha_2} y_2^{\beta_2} \cdots x_q^{\alpha_q} y_q^{\beta_q}$ with $0 \leq \alpha_1 \leq s_1 - 1$.*

Proof. We set $J = \operatorname{lead}(I(X))$. We claim that y_1 does not divide any minimal generator of J. To prove this claim we argue by contradiction. Assume that there existed a minimal generator g of J of the form $g = x_1^{\alpha_1} y_1^{\beta_1} \cdots x_q^{\alpha_q} y_q^{\beta_q}$ with $\beta_1 > 0$. Write $g = \operatorname{lead}(f)$ for some $\mathbb{Z}^q$-homogeneous polynomial $f \in I(X)$. For any other monomial $\zeta = x_1^{\gamma_1} y_1^{\delta_1} \cdots x_q^{\gamma_q} y_q^{\delta_q}$ occurring in f we have the inequality $\alpha_1 \geq \gamma_1$ because $g > \zeta$ and we have the equation $\alpha_1 + \beta_1 = \gamma_1 + \delta_1$ because $\deg(g) = \deg(\zeta)$. It follows that $\beta_1 \leq \delta_1$. Since ζ was chosen arbitrarily, it follows that f is divisible by $y_1^{\beta_1}$. Since y_1 does not vanish at any point of $\operatorname{red}(X)$, it follows that $f/y_1^{\beta_1}$ lies in $I(X)$. Thus, $g/y_1^{\beta_1} = \operatorname{lead}(f/y_1^{\beta_1})$ belongs to J. This contradicts the fact that g is a minimal generator of J and concludes the proof of the claim. The ideal $J_1 = J \cap \mathbb{K}[x_1, y_1]$ is a $\mathbb{Z}$-homogeneous ideal of the $\mathbb{Z}$-graded ring $\mathbb{S}_1 = \mathbb{K}[x_1, y_1]$. According to Theorem 6.5, $\mathcal{H}_{\mathbb{S}/J} = \mathcal{H}_X$ hence

$$\begin{aligned} \mathcal{H}_{\mathbb{S}_1/J_1}(a_1) &= \mathcal{H}_{\mathbb{S}/J}(a_1, 0, \ldots, 0) = \mathcal{H}_X(a_1, 0, \ldots, 0) \\ &= \mathcal{H}_{Z_1}(a_1) = \begin{cases} a_1 + 1 & \text{if} \quad 0 \leq a_1 \leq s_1 - 1, \\ s_1 & \text{if} \quad a_1 \geq s_1. \end{cases} \end{aligned}$$

It follows that J_1 is generated by a single monomial of degree s_1, which, according to the above claim, is not divisible by y_1. We deduce that J_1 is generated by $x_1^{s_1}$. This monomial must be a minimal generator of J. Every other minimal generator of J is not divisible by y_1 or by $x_1^{s_1}$, so it has the form given in the lemma. ■

As an application of our methods we obtain a third proof for a particular case of Theorem 4.4. We formulate this as a separate proposition.

Proposition 9.2. *Assume that $\mathbb{K}$ is infinite. Let $X \subset (\mathbb{P}^1)^q$ be a zero-dimensional subscheme. For each $i \in \{1, \ldots, q\}$, let s_i be the length of the projection of X onto the i-th copy of $\mathbb{P}^1$. Then $R(X) = [0, s_1 - 1] \times \cdots \times [0, s_q - 1]$ is the smallest rectangular relevant domain for $\mathcal{H}_X$.*

Proof. We must show that $\Delta\mathcal{H}_X$ vanishes on the complement of $R(X)$, i.e., we must show that $\Delta\mathcal{H}_X(a) = 0$ if $a_i \geq s_i$ for some index $i \in \{1, \ldots, q\}$. By symmetry, it is enough to consider only the case when $i = 1$. Performing, if necessary, a linear change

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of coordinates on the first copy of $\mathbb{P}^1$, we may assume that y_1 does not vanish at any point of $\operatorname{red}(X)$. We choose a monomial ordering on $\mathbb{S}$ as in Lemma 9.1 and we consider the ideal $J = \operatorname{lead}(I(X))$. See Notation 6.4. According to Theorem 6.5, $\mathcal{H}_X = \mathcal{H}_{\mathbb{S}/J}$, so the theorem reduces to showing that $\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 0$ if $a_1 \geq s_1$ and $a \geq 0$. This will follow from Proposition 8.4.

We now verify the hypotheses of Proposition 8.4. Consider the set $\{g_1, \dots, g_n\}$ of minimal generators of J whose degree is less or equal to a. According to Lemma 9.1, $x_1^{s_1}$ is a minimal generator of J. By hypothesis $\deg(x_1^{s_1}) = (s_1, 0, \dots, 0) \leq a$, so we may take $g_1 = x_1^{s_1}$. Applying again Lemma 9.1, we see that for each index $l \in \{2, \dots, n\}$ we may write $g_l = x_1^{\alpha_1} x_2^{\alpha_2} y_2^{\beta_2} \cdots x_q^{\alpha_q} y_q^{\beta_q}$ with $0 \leq \alpha_1 \leq s_1 - 1$. Thus, $\operatorname{lcm}(g_1, g_l) = x_1^{s_1} x_2^{\alpha_2} y_2^{\beta_2} \cdots x_q^{\alpha_q} y_q^{\beta_q}$. We have $\deg(\operatorname{lcm}(g_1, g_l)) \leq a$ because $s_1 \leq a_1$ and $\alpha_j + \beta_j \leq a_j$ for all indices $j \in \{2, \dots, q\}$, by virtue of the fact that $\deg(g_l) \leq a$. Thus, the hypotheses of Proposition 8.4 are satisfied and we conclude that $\Delta\mathcal{H}_{\mathbb{S}/J}(a) = 0$. ∎

Remark 9.3. Let $I \subset \mathbb{S}$ be a $\mathbb{Z}^q$-homogeneous ideal. Let F and G be two finite sets of generators of I consisting of $\mathbb{Z}^q$-homogeneous non-zero polynomials. We assume that F is minimal, i.e. no proper subset of F can generate I. We assert that for every $f \in F$ there is $g \in G$ such that $\deg(f) = \deg(g)$. Indeed, write $f = \sum_{g \in G} \eta_g g$. For each $g \in G$ write $g = \sum_{h \in F} \theta_{gh} h$. We may assume that all η_g and θ_{gh} are $\mathbb{Z}^q$-homogeneous. We claim that there is $g \in G$ such that $\eta_g \neq 0$ and $\theta_{gf} \neq 0$. If this were not the case, then f would be a combination of elements in $F \setminus \{f\}$, which would contradict the minimality of F. We have the relations $\deg(f) = \deg(\eta_g) + \deg(g)$ and $\deg(g) = \deg(\theta_{gf}) + \deg(f)$, hence $\deg(f) \geq \deg(g) \geq \deg(f)$.

Lemma 9.4. *Let $X \subset (\mathbb{P}^1)^q$ be a zero-dimensional subscheme. We assume that $I(X)$ is a monomial ideal. We assert that the set of minimal generators of $I(X)$ is of the form $\{f_1, \dots, f_q, g_1, \dots, g_n\}$, where $n \geq 0$, $f_i = x_i^{\alpha_i} y_i^{s_i - \alpha_i}$ for all indices $i \in \{1, \dots, q\}$, and $\deg(g_l) \leq (s_1 - 1, \dots, s_q - 1)$ for all indices $l \in \{1, \dots, n\}$.*

Proof. We let f_i be the sole generator of $I(X) \cap \mathbb{K}[x_i, y_i]$ and we let $g_1, \dots, g_n$ be the minimal generators of $I(X)$ that do not lie in any $\mathbb{K}[x_i, y_i]$. We concentrate on proving the inequalities $\deg(g_l) \leq (s_1 - 1, \dots, s_q - 1)$, the rest of the lemma being obvious. Since $\mathbb{K}$ is infinite, we can find $\kappa_1 \in \mathbb{K} \setminus \{0\}$ such that $z_1 = \kappa_1 x_1 + y_1$ does not vanish at any point of $\operatorname{red}(X)$. Regarding $\mathbb{S}$ as a polynomial ring in $x_1, z_1, x_2, y_2, \dots, x_q, y_q$, we consider the lexicographic ordering on $\mathbb{S}$ such that $x_1 > z_1 > x_2 > y_2 > \cdots > x_q > y_q$. Put $J = \operatorname{lead}(I(X))$, as in Notation 6.4. According to Lemma 9.1, $x_1^{s_1} = \operatorname{lead}(f_1)$ is a minimal generator of J and every other minimal generator h of J satisfies the condition $\deg(h)_1 \leq s_1 - 1$. Let G be a Gröbner basis of $I(X)$ containing f_1 and consisting of $\mathbb{Z}^q$-homogeneous polynomials, such that $\operatorname{lead}(G)$ is the set of minimal generators of J. For every $g \in G \setminus \{f_1\}$ we have the relations $\deg(g)_1 = \deg(\operatorname{lead}(g))_1 \leq s_1 - 1$. We apply Remark 9.3 to the set F of minimal generators of $I(X)$ and to G. For each g_l there is $g \in G$ such that $\deg(g_l) = \deg(g)$. Since $g_l \notin \mathbb{K}[x_1, y_1]$, it follows that $g \neq f_1$, hence $\deg(g_l)_1 = \deg(g)_1 \leq s_1 - 1$.

In the same manner, for all indices $l \in \{1, \dots, n\}$ and $i \in \{1, \dots, q\}$, by replacing the variable y_i with a suitable variable $z_i = \kappa_i x_i + y_i$, chosen so as not to vanish at any point of $\operatorname{red}(X)$, we can prove the inequality $\deg(g_l)_i \leq s_i - 1$. ∎

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Let X be as above. We recall, from Proposition 4.5(v), the rectangular relevant domain

$$R(X) = [0, s_1 - 1] \times \cdots \times [0, s_q - 1] = [0, \mathrm{rem}(X)] \subset \mathbb{Z}^q.$$

If $n = 1$, we put $R_1 = [\deg(g_1), \mathrm{rem}(X)]$. If $n > 1$, for each index $l \in \{1, \ldots, n\}$, we put

$$R_l = \bigcap_{k \in \{1,\ldots,n\} \setminus \{l\}} [\deg(\mathrm{lcm}(g_k, g_l)),\ \mathrm{rem}(X)].$$

Proposition 9.5. *Assume that $\mathbb{K}$ is infinite. Let $X \subset (\mathbb{P}^1)^q$ be a zero-dimensional subscheme. Assume that $I(X)$ is a monomial ideal and that X is not a complete intersection. Then the quasi-rectangular domain*

$$D(X) = R(X) \setminus \bigcup_{1 \le l \le n} R_l \subset \mathbb{Z}^q$$

is a relevant domain for $\mathcal{H}_X$. If $I(X)$ has $q + 1$ minimal generators, then $D(X) \neq R(X)$.

Proof. We adopt the notation of Lemma 9.4. We have the inequality $n \ge 1$ because X is not a complete intersection. We must show that $\Delta\mathcal{H}_X(a) = 0$ for all $a \in \mathbb{Z}^q \setminus D(X)$. We already know from Theorem 4.4 that $R(X)$ is a relevant domain for $\mathcal{H}_X$, hence we may assume that $a \in R(X)$, that is, $a \in R_l$ for some index $l \in \{1, \ldots, n\}$. Relabeling $\{g_1, \ldots, g_n\}$, if necessary, we may take $l = 1$. We desire to apply Proposition 8.4 to $J = I(X)$. We now verify the hypotheses of Proposition 8.4. By the construction of R_1, $a \ge \deg(\mathrm{lcm}(g_1, g_k)) \ge \deg(g_k)$ for all indices $k \in \{2, \ldots, n\}$ and $a \ge \deg(g_1)$. Since a belongs to $R(X)$, $a \ngeq \deg(f_i)$ for all indices $i \in \{1, \ldots, q\}$. From Lemma 9.4 we deduce that $\{g_1, \ldots, g_n\}$ is the set of minimal generators of $I(X)$ whose degree is less or equal to a. The inequality from Proposition 8.4 is satisfied by the definition of R_1. Thus, the hypotheses of Proposition 8.4 are satisfied and we conclude that $\Delta\mathcal{H}_X(a) = 0$.

If $I(X)$ has $q + 1$ minimal generators, that is, if $n = 1$, then, in view of Lemma 9.4, $R_1 \neq \emptyset$, forcing $D(X)$ to be strictly contained in $R(X)$. ■

If $n > 1$, then R_l may be empty for all indices l, i.e. $D(X)$ may coincide with $R(X)$. We finish this section with two examples in which $n > 1$, X is non-ACM and $D(X) \neq R(X)$.

Example 9.6. Take $X \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ to be the union of two multiple points with ideals $(x_1^{\alpha_1}, x_1x_2, x_2^{\alpha_2})$, respectively, $(y_1^{\beta_1}, y_1y_2, y_2^{\beta_2})$. Here $\alpha_1, \alpha_2, \beta_1, \beta_2 \ge 2$. We have

$$I(X) = (x_1^{\alpha_1} y_1^{\beta_1},\ x_2^{\alpha_2} y_2^{\beta_2},\ g_1, \ldots, g_7),$$

where

$$\begin{aligned} &g_1 = x_1^{\alpha_1} y_1 y_2, && g_2 = x_1^{\alpha_1} y_2^{\beta_2}, && g_3 = x_1 y_1^{\beta_1} x_2, && g_4 = x_1 y_1 x_2 y_2, \\ &g_5 = x_1 x_2 y_2^{\beta_2}, && g_6 = y_1^{\beta_1} x_2^{\alpha_2}, && g_7 = y_1 x_2^{\alpha_2} y_2. \end{aligned}$$

We have the equations $s_1 = \alpha_1 + \beta_1$ and $s_2 = \alpha_2 + \beta_2$. We have the equations

$$\begin{aligned} &\mathrm{lcm}(g_1, g_3) = x_1^{\alpha_1} y_1^{\beta_1} x_2 y_2, && \deg(\mathrm{lcm}(g_1, g_3)) = (\alpha_1 + \beta_1, 2), \\ &\mathrm{lcm}(g_2, g_6) = x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}, && \deg(\mathrm{lcm}(g_2, g_6)) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2), \\ &\mathrm{lcm}(g_5, g_7) = x_1 y_1 x_2^{\alpha_2} y_2^{\beta_2}, && \deg(\mathrm{lcm}(g_5, g_7)) = (2, \alpha_2 + \beta_2). \end{aligned}$$

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None of the expressions on the r.h.s. is less or equal to $(s_1 - 1, s_2 - 1)$. We deduce that R_1, R_3, R_2, R_6, R_5 and R_7 are empty. We have the equations

$$\mathrm{lcm}(g_4, g_1) = x_1^{\alpha_1} y_1 x_2 y_2, \quad \mathrm{lcm}(g_4, g_2) = x_1^{\alpha_1} y_1 x_2 y_2^{\beta_2}, \quad \mathrm{lcm}(g_4, g_3) = x_1 y_1^{\beta_1} x_2 y_2,$$
$$\mathrm{lcm}(g_4, g_5) = x_1 y_1 x_2 y_2^{\beta_2}, \quad \mathrm{lcm}(g_4, g_6) = x_1 y_1^{\beta_1} x_2^{\alpha_2} y_2, \quad \mathrm{lcm}(g_4, g_7) = x_1 y_1 x_2^{\alpha_2} y_2.$$

From these we obtain the relations

$$\max_{k=1,2,3,5,6,7} \deg(\mathrm{lcm}(g_4, g_k))_1 = 1 + \max\{\alpha_1, \beta_1\} \leq s_1 - 1,$$
$$\max_{k=1,2,3,5,6,7} \deg(\mathrm{lcm}(g_4, g_k))_2 = 1 + \max\{\alpha_2, \beta_2\} \leq s_2 - 1.$$

We deduce that R_4 is not empty. In fact,

$$R_4 = [1 + \max\{\alpha_1, \beta_1\},\ \alpha_1 + \beta_1 - 1] \times [1 + \max\{\alpha_2, \beta_2\},\ \alpha_2 + \beta_2 - 1].$$

We conclude that

$$D(X) = [0,\ \alpha_1 + \beta_1 - 1] \times [0,\ \alpha_2 + \beta_2 - 1] \setminus R_4.$$

If X were ACM, then, in view of Theorem 6.2(iii), the degrees of $g_1, \ldots, g_7$ would be incomparable. However, $\deg(g_4) = (2, 2) \leq \deg(g_2) = (\alpha_1, \beta_2)$. Thus, X is not ACM.

Example 9.7. Take $X \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ to be the union of three multiple points with ideals $(x_1^{\alpha_1}, x_2^{\alpha_2})$, $(x_1^{\alpha}, y_2^{\beta})$, respectively, $(y_1^{\beta_1}, y_2^{\beta_2})$. We assume that $\alpha_1 < \alpha$ and $\beta < \beta_2$. We have

$$I(X) = (x_1^{\alpha} y_1^{\beta_1},\ x_2^{\alpha_2} y_2^{\beta_2},\ g_1,\ g_2,\ g_3),$$

where $g_1 = x_1^{\alpha_1} y_1^{\beta_1} y_2^{\beta}$, $g_2 = x_1^{\alpha_1} y_2^{\beta_2}$ and $g_3 = y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta}$. Note that $s_1 = \alpha + \beta_1$ and $s_2 = \alpha_2 + \beta_2$. We have the relations

$$\mathrm{lcm}(g_1, g_2) = x_1^{\alpha_1} y_1^{\beta_1} y_2^{\beta_2}, \qquad \deg(\mathrm{lcm}(g_1, g_2)) = (\alpha_1 + \beta_1, \beta_2) \leq (s_1 - 1, s_2 - 1),$$
$$\mathrm{lcm}(g_1, g_3) = x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta}, \quad \deg(\mathrm{lcm}(g_1, g_3)) = (\alpha_1 + \beta_1, \alpha_2 + \beta) \leq (s_1 - 1, s_2 - 1),$$
$$\mathrm{lcm}(g_2, g_3) = x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}, \quad \deg(\mathrm{lcm}(g_2, g_3)) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \nleq (s_1 - 1, s_2 - 1).$$

We deduce that R_2 and R_3 are empty and that

$$R_1 = [\alpha_1 + \beta_1,\ \alpha + \beta_1 - 1] \times [\max\{\beta_2, \alpha_2 + \beta\},\ \alpha_2 + \beta_2 - 1].$$

We conclude that

$$D(X) = [0,\ \alpha + \beta_1 - 1] \times [0,\ \alpha_2 + \beta_2 - 1] \setminus R_1.$$

Assume that X were ACM. Then, as mentioned at Theorem 6.2(iii), there would exist homogeneous polynomials $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{C}[x_1, y_1]$ and $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in \mathbb{C}[x_2, y_2]$ such that

$$I(X) = (\xi_1\xi_2\xi_3\xi_4,\ \upsilon_1\xi_2\xi_3\xi_4,\ \upsilon_1\upsilon_2\xi_3\xi_4,\ \upsilon_1\upsilon_2\upsilon_3\xi_4,\ \upsilon_1\upsilon_2\upsilon_3\upsilon_4).$$

We choose κ and λ in $\mathbb{C}^*$ such that $\upsilon_i(\kappa, \lambda) \neq 0$ for $i = 1, 2, 3, 4$. We reduce the above equality of ideals modulo $(x_2 - \kappa,\ y_2 - \lambda)$. It follows that $\mathbb{C}[x_1, y_1]$ is generated by the polynomials $\xi_1\xi_2\xi_3\xi_4,\ \xi_2\xi_3\xi_4,\ \xi_3\xi_4$ and ξ_4. This is absurd. Thus, X is not ACM.

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