ON GENERALIZED AVERAGING OPERATORS

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1. Introduction. Let

$$\nabla_h f(z) = \tfrac{1}{2}[f(z+h) + f(z)].$$

Sumner **(4)** has discussed $\nabla_h^\lambda f(z)$ for arbitrary real λ and h, where $f(z)$ is an entire function of exponential type $\tau < \pi/|h|$. I shall show that in this case an alternative definition of ∇_h^λ, which leads to Sumner's results more quickly, is equivalent to Sumner's. (However, Sumner's definition is, in principle, applicable to a wider class of functions.) I also make some remarks on Sumner's question about the existence of nontrivial solutions of $\nabla_h^\lambda f = 0$. In particular, I shall show that in a certain sense there are such solutions for every positive λ.

2. Definitions. Let $f(z)$ be an entire function of exponential type τ, let S be its conjugate indicator diagram, and let $F(w)$ be the Borel-Laplace transform of f. (For terminology, see, for example, **(1)**.) Then we have the Pólya representation

$$f(z) = (2\pi i)^{-1} \int_C F(w) e^{zw} dw, \tag{1}$$

where C is a contour surrounding S; since S is a subset of the disk $|w| \leqslant \tau$, C can in particular be the circumference $|w| = \tau + \epsilon$, $\epsilon > 0$.

Let $\phi(w)$ be regular on S, hence on contours C which are sufficiently close to the boundary of S. If D stands for d/dz, we can define the operator $\phi(D)$ by

$$\phi(D)f(z) = (2\pi i)^{-1} \int_C F(w)\phi(w) e^{zw} dw, \tag{2}$$

where the definition is independent of the particular contour C that is employed provided that we admit only contours lying within the domain of regularity of ϕ. If $K[S]$ denotes the class of entire functions of exponential type whose conjugate indicator diagrams are subsets of S, the operator $\phi(D)$ applies to all elements of $K[S]$ and transforms them into elements of $K[S]$. (If ϕ has poles in S we can still define $\phi(D)$ by (2) but we get, in general, different values for $\phi(D)f$ according to which C we take, if there are poles of ϕ not in the conjugate indicator diagram of f.)

If ψ is regular over the range of $\phi(w)$ for w in S, we have

$$\psi[\phi(D)]f = (2\pi i)^{-1} \int_C F(w)\psi[\phi(w)] e^{zw} dw,$$

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so in particular

(3) $$[\phi(D)]^\lambda f = (2\pi i)^{-1}\int_C F(w)[\phi(w)]^\lambda e^{zw}dw$$

for any positive integral λ. If $\phi(w)$ is zero-free on S, (3) holds for arbitrary (real or imaginary) λ; we must of course specify which regular branch of $[\phi(w)]^\lambda$ is to be taken. If, for example, $[\phi(w)]^\lambda = e^{\lambda \log \phi(w)}$ with the same branch of $\log \phi(w)$ for all λ, it is then immediate that

(4) $$[\phi(D)]^\lambda\{[\phi(D)]^\mu f(z)\} = [\phi(D)]^{\lambda+\mu} f(z).$$

As a special case, we see that $[\phi(D)]^{-\lambda}$ inverts $[\phi(D)]^\lambda$.

For example, if ϕ is regular and zero-free in a region containing $w = a$, we have

(5) $$[\phi(D)]^\lambda e^{az} = [\phi(a)]^\lambda e^{az}.$$

An alternative representation is obtained by expanding $\phi(w)e^{w(z-t)}$ in powers of w:

$$\phi(w)e^{w(z-t)} = \sum_{n=0}^\infty w^n p_n(z-t).$$

The $p_n(z)$ are the Appell polynomials **(3)** generated by $\phi(w)$, and (2) can be written

(6) $$\phi(D)f(z) = \sum_{n=0}^\infty f^{(n)}(t)p_n(z-t),$$

where the series converges if S is inside the largest circle of regularity of $\phi(w)$ with center at 0, and is Mittag-Leffler summable **(2**, p. 79**)** if S is inside the Mittag-Leffler star of ϕ with respect to 0. In particular, if $t = z$ and

$$\phi(w) = \sum_{n=0}^\infty c_n w^n,$$

(6) becomes

$$\phi(D)f(z) = \sum_{n=0}^\infty c_n f^{(n)}(z) = \sum_{n=0}^\infty c_n D^n f(z),$$

which is a very natural interpretation of $\phi(D)f$.

In Sumner's case $\phi(D) = \nabla_h = \frac{1}{2}(e^{hD}+1)$, so that $\phi(w) = \frac{1}{2}(e^{hw}+1)$. If h is any non-zero complex number, the zeros of ϕ closest to 0 are at $w = \pm i\pi/h$, so that (3) is valid (in the sense of transforming an element of $K[S]$ into an element of $K[S]$) if f is an entire function of exponential type less than $\pi/|h|$, and more generally if S avoids the points $(2k+1)i\pi/h$. For example, (3) defines ∇_h^λ for all λ if h is real and $f(z)$ is an exponential sum

$$\sum_{j=0}^n a_j e^{b_j z}$$

with real b_j.

If we choose $\log\{\frac{1}{2}(e^{hw}+1)\}$ so that it reduces to 0 at $w = 0$, we then have

$$\lim_{h\to 0} \nabla_h^\lambda f(z) = f(z),$$

uniformly on compact sets, if f is an entire function of exponential type (the left-hand side being defined when $|h|$ is sufficiently small). We also have

$$\lim_{\lambda\to 0} \nabla_h^\lambda f(z) = f(z)$$

when $f(z)$ is of exponential type less than $\pi/|h|$.

3. Equivalence of the definitions. We now compare our definition of ∇_h^λ with Sumner's. Since both versions satisfy (4), and coincide for integral λ, and Sumner's was given only for real λ, it is enough to consider the range $0 < \lambda < 1$; for convenience we take $h > 0$.

For this range, Sumner's definition is

$$(S)\nabla_h^\lambda f(z) = \frac{1}{2^\lambda\Gamma(1-\lambda)\cdot 2\pi i}\int_{b-i\infty}^{b+i\infty} \{f(z-hw) + f(z+h-hw)\} \times\Gamma(w)\Gamma(1-\lambda-w)dw,$$

where $0 < b < 1 - \lambda$ and $f(z)$ is of exponential type less than π/h. Representing $f(z)$ by (1), we have

$$(7)\qquad (S)\nabla_h^\lambda f(z) = \frac{1}{2^\lambda\Gamma(1-\lambda)(2\pi i)^2}\int_{b-i\infty}^{b+i\infty}\Gamma(w)\Gamma(1-\lambda-w)dw \times\int_C (e^{zt} + e^{(z+h)t})e^{-hwt}F(t)dt.$$

If we change the order of integration, this becomes

$$(S)\nabla_h^\lambda f(z) = \frac{1}{2^\lambda\Gamma(1-\lambda)\ (2\pi i)^2}\int_C F(t)\ (e^{zt} + e^{(z+h)t})dt \times\int_{b-i\infty}^{b+i\infty} e^{-hwt}\Gamma(w)\Gamma(1-\lambda-w)dw$$

$$= \frac{1}{2^\lambda\cdot 2\pi i}\int_C e^{zt}(1+e^{ht})F(t)\ (1+e^{ht})^{\lambda-1}dt,$$

which coincides with (3) when $\phi(w) = \frac{1}{2}(e^{hw+1})$. Here we have used a definite integral quoted by Sumner (**4**, p. 438).

It remains to justify the change of order of integration. It is sufficient to verify that the iterated integral is absolutely convergent. Now, for fixed z and h, $F(t)(e^{zt} + e^{(z+h)t})$ is bounded on C, so it is enough to show that

$$(8)\qquad \int_{b-i\infty}^{b+i\infty} |e^{-hwt}\Gamma(w)\Gamma(1-\lambda-w)dw|$$

converges uniformly for t on C, where $0 < b < 1 - \lambda$. We can take C to be a circumference $|t| = (\pi/h) - \epsilon$, $\epsilon > 0$, so that

$$|e^{-hwt}| < \exp\{h|y|[(\pi/h) - \epsilon]\} = O(e^{\sigma|y|})$$

with $\sigma < \pi$ and independent of t. Now we have

$$\Gamma(1-\lambda-w) = \frac{\pi\csc\pi(1-\lambda-w)}{\Gamma(\lambda+w)},$$

so that

$$|\Gamma(w)\Gamma(1-\lambda-w)| \leqslant \pi|\csc \pi(1-\lambda-w)|\,|\Gamma(w)/\Gamma(\lambda+w)|.$$

Since $\csc \pi(1-\lambda-w) = O(e^{-\pi|y|})$ and

$$\Gamma(w)/\Gamma(\lambda+w) = O(|w^{-\lambda}|) = O(|y|^{-\lambda})$$

as $w = b + iy \to \infty$, the integrand in (8) is $O(e^{-\delta|y|})$ with $\delta > 0$ and independent of t. Hence (8) does converge uniformly for t on C and the change of order of integration in (7) is legitimate.

4. The existence of eigenvalues. Sumner noted that there are nontrivial solutions of $\nabla_h^\lambda f(z) = 0$ which are entire functions of exponential type, although not of type less than $\pi/|h|$, when λ is a positive integer. It is easy to show that there are no nontrivial solutions which are of exponential type less than $\pi/|h|$. More generally, if $\phi(D)f(z)$ is defined by (2), where $\phi(w)$ is regular on S and $F(w)$ is regular outside S, we have, if $\phi(D)f(z) \equiv 0$,

$$\int_C F(w)\,\phi(w)e^{zw}dw \equiv 0.$$

This implies, by a lemma of Pólya's (**1**, p. 110), that $F(w)\phi(w)$ is regular inside C. If $\phi(w)$ has no zeros inside C, $F(w)$ must be regular inside C, and so everywhere; hence, since F vanishes at ∞, $F(w) \equiv 0$ and so $f(z) \equiv 0$.

On the other hand, if $\phi(w)$ has a zero at $w = a \in S$, (2) shows that $\phi(D)$ (and all its positive integral powers) annul the function f whose Borel-Laplace transform F is $F(w) = 1/(w-a)$, namely $f(z) = e^{az}$. Moreover, $[\phi(D)]^2$ also annuls the inverse Laplace transform of $1/(w-a)^2$, namely $f(z) = ze^{az}$; and so on.

In particular, $\nabla_h^\lambda f(z) = 0$ for positive integral λ if $f(z) = e^{\pm i\pi z/h}$ (cf. (5)), or $\sin(\pi z/h)$ or $\cos(\pi z/h)$; $\nabla_h^\lambda f(z) = 0$ for $\lambda = 2,3,\ldots$ if $f(z) = z\sin(\pi z/h)$ or $z\cos(\pi z/h)$; and so on.

Sumner raised the question of whether there is a set of values of λ which are eigenvalues for ∇_h^λ in the sense that for these, and only for these, there are nontrivial solutions (eigenfunctions) of $\nabla_h^\lambda f(z) = 0$. We have, of course, defined $\nabla_h^\lambda f(z)$, when λ is not a positive integer, only when f is of exponential type less than $\pi/|h|$, but it would be natural to extend the definition as follows. If $|k| < |h|$, and f is of exponential type $\pi/|h|$, $\nabla_k^\lambda f(z)$ is defined and we take

$$\nabla_h^\lambda f(z) = \lim_{k\to h} \nabla_k^\lambda f(z) \qquad (|k| < |h|), \tag{9}$$

if the limit exists. We may, of course, lose property (4) with the extended definition.

Now by (5) we have, if $|k| < |h|$,

$$\nabla_k^\lambda e^{\pm i\pi z/h} = \{\tfrac{1}{2}(e^{\pm i\pi k/h} + 1)\}^\lambda e^{\pm i\pi z/h},$$

and if $\Re(\lambda) > 0$, this approaches 0 as $k \to h$. Thus if ∇_h^λ is defined by (9), every λ of positive real part is an eigenvalue and $e^{\pm i\pi z/h}$ are eigenfunctions, just as they were for positive integral λ. Furthermore, $ze^{\pm i\pi z/h}$ are also eigenfunctions when $\Re(\lambda) > 1$, and so on.

It is interesting to note that there are still other eigenfunctions. To see this, suppose that the Borel-Laplace transform $F(w)$ of $f(z)$ is such that $F(w)(w \pm i\pi/k)^{\lambda}$ is uniformly dominated by an integrable function in an annulus $\pi/|h| < |w| < a$. We can then shrink the contour C in (2) to $|w| = \pi/|h|$ and obtain

$$\nabla_k^{\lambda} f(x) = (2\pi i)^{-1} \int_{|w|=\pi/|h|} F(w)\{\tfrac{1}{2}(e^{kw}+1)\}^{\lambda} e^{zw}\, dw;$$

then we let $k \to h$, and obtain finally

$$\nabla_h^{\lambda} f(x) = (2\pi i)^{-1} \int_{|w|=\pi/|h|} F(w)\{\tfrac{1}{2}(e^{hw}+1)\}^{\lambda} e^{zw}\, dw, \tag{10}$$

where the branch of the power is one that is regular in the plane cut from $\pm i\pi/h$ to ∞.

Now consider (for the sake of simplicity) positive values of h, and apply (10) to $f(z) = J_0(\pi z/h)$. Then $F(w) = (w^2 + \pi^2/h^2)^{-\frac{1}{2}}$ for $|w| > h$, and satisfies the hypotheses required for (10) if $\lambda = \frac{1}{2}$. Thus

$$\nabla_h^{\frac{1}{2}} f(z) = (2\pi i)^{-1} \int_{|w|=\pi/h} \{w^2 + \pi^2/h^2\}^{-\frac{1}{2}}\{\tfrac{1}{2}(e^{hw}+1)\}^{\frac{1}{2}} e^{zw} dw. \tag{11}$$

Here the first square root is defined in the plane cut from $i\pi/h$ to $-i\pi/h$. However, continuing either square root around $\pm i\pi/h$ replaces it by its negative, so the integrand in (11) is regular in the closed disk $|w| \leqslant \pi/h$. Hence the integral in (11) is zero. Thus $J_0(\pi z/h)$ is still another eigenfunction corresponding to $\lambda = \frac{1}{2}$.

References

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