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ON ELEMENTARY ABELIAN CARTESIAN GROUPS

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ABSTRACT. J. Hayden [2] proved that, if a finite abelian group is a Cartesian group satisfying a certain "homogeneity condition", then it must be an elementary abelian group. His proof required the character theory of finite abelian groups. In this note we present a shorter, elementary proof of his result.

Hayden [2] proved that, if a finite abelian group is a *cartesian group* satisfying a certain "homogeneity condition", then it must be an elementary abelian group.

His proof required the character theory of finite abelian groups. In this note we present a shorter, elementary proof of his result.

Let G be an abelian group of order n. G is called a cartesian group if there exist bijections $\theta_1, \ldots, \theta_{n-2}: G \longrightarrow G$ such that $\theta_i(0) = 0$ for $i = 1, \ldots, n-2$, the mappings $\eta_i: x \longrightarrow \theta_i(x) - x$ are bijections for $i = 1, \ldots, n-2$, and the mappings $\delta_{ij}: x \longrightarrow \theta_i(x) - \theta_j(x)$ are bijections for $i, j = 1, \ldots, n-2$, $i \neq j$. From these mappings we can construct an affine plane of order n as follows. The points of the plane are ordered pairs of elements of G, lines being given by the equations $x = c$, $y = c$, $y = x + b$, and $y = \theta_i(x) + b$ for $i = 1, \ldots, n-2$. This plane is (∞, l_∞)-transistive and it is well-known that any (∞, l_∞)-transistive plane can be constructed from some cartesian group (see Dembowski [1, p. 129]).

If G is an abelian cartesian group and $\theta_1, \ldots, \theta_{n-2}$ are corresponding bijections, then we say that condition (H) is satisfied if the following is satisfied.

(H) $\theta_i(rx) = r\theta_i(x)$ for $i = 1, \ldots, n-2, x \in G, r \in N, (r, n) = 1$.

If A is an (∞, l_∞)-transistive plane, then we say that A satisfies condition (H') if it satisfies the following.

(H') $H_r: (a, b) \longrightarrow (ra, rb)$, $r \in N$, $(r, n) = 1$, are all homologies of the plane with axis l_∞ and center $(0, 0)$.

Hayden [2] showed these two conditions to be equivalent and so we can paraphrase his theorem as follows.

THEOREM. *Let G be an abelian cartesian group of order n and let A be a corresponding* (∞, l_∞)*-transistive plane. If A satisfies condition* (H')*, then G is elementary abelian.*

PROOF. As H_r is a homology of the plane, if the mapping $x \longrightarrow rx$, $x \in G$, $(r, n) = 1$, fixes any nonidentity element of G, then it must fix all elements of G. There are three

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cases to consider.

CASE 1. n odd, not a power of a prime. Let $p_1 < p_2$ be the two smallest prime divisors of n. Then $r = p_1 + 1$ is relatively prime to n and $x \longrightarrow rx$ fixes all elements of order p_1 and no element of order p_2. A contradiction.

CASE 2. n even, not a power of 2. Let $2 < p_1 < \cdots < p_n$ be the prime divisors of n. Then $r = 2p_1 \ldots p_n - 1$ is relatively prime to n and $x \longrightarrow rx$ fixes only the identity and elements of order 2. A contradiction.

CASE 3. n is a power of a prime p. As the case n a prime is trivial, we shall assume that n is not a prime. Then $r = p + 1$ is relatively prime to n and $x \longrightarrow rx$ fixes only the identity and elements of order p. Hence all elements of order G are of order p and so G is elementary abelian.

NOTE. In the proof we needed only one value of r, relatively prime to n, for which the mapping $x \longrightarrow rx$ fixed some but not all of the nonidentity elements of G, to obtain the desired conclusion. This leads naturally to many similar results involving smaller homology groups.

REFERENCES

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