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ON THE PREDICTION ERROR FOR TWO-PARAMETER STATIONARY RANDOM FIELDS

R. Cheng

A number of Szegö-type prediction error formulas are given for two-parameter stationary random fields. These give rise to an array of elementary inequalities and illustrate a general duality relation.

1. Introduction

Suppose that μ is a finite nonnegative Borel measure on the unit circle, I. Consider the problem of estimating the constant function 1 within the span of $\{e^{in\theta}\}_{n=1}^{\infty}$ in $L^2(\mu)$. A formula for the least-squares error ε_1 was discovered by Szegö [8] and was amplified by Kolmogorov, Kreĭn and Wiener:

Theorem 1.1. *Let $d\mu = w\,d\sigma + d\lambda$ be the Lebesgue decomposition of μ with respect to normalised Lebesgue measure σ. We have*

$$\varepsilon_1^2 = \exp \int \log w \, d\sigma,$$

where the right hand side is interpreted as zero if $\log w$ is not integrable.

Likewise, Kolmogorov [6] derived an expression for the least-squares error ε_2 in estimating 1 within the span of $\{e^{in\theta}\}_{n\neq 1}$.

Theorem 1.2. *Let $d\mu = w\,d\sigma + d\lambda$. We have*

$$\varepsilon_2^2 = \left(\int w^{-1} d\sigma\right)^{-1},$$

where the right hand side is interpreted as zero if the integral diverges.

Of course, these results have given rise to a number of variations and generalisations. In particular, let us now take μ to be a finite nonnegative Borel measure on the torus I^2. For $U \subseteq Z^2$, let $\varepsilon(U, \mu)$ be the least-squares error in estimating 1 within the span of $\{e^{ims+int} \colon (m, n) \in U\}$ in $L^2(\mu)$. This is the bivariate analogue of the above problems of Kolmogorov and Szegö.

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In this article, formulas for $\varepsilon(U,\mu)$ are presented for several natural choices of U. A few of these were previously known; others can be derived via Theorems 1.1 and 1.2; still others are extensions of the work of Kallianpur, Miamee and Niemi [5]. In the last case, a result of [1] is used to remove restrictions on μ. Together, these error formulas yield an array of elementary inequalities (see Corollary 3.9), not all of which are obvious. Lastly, we shall examine a duality relation discovered by Miamee and Pourahmadi [6] in the univariate picture. This principle extends to the multivariate case, and is illustrated by the error formulas treated below.

2. Preliminaries

In light of the formulas in Theorems 1.1 and 1.2, we might expect a study of $\varepsilon(U,\mu)$ to involve Lebesgue decompositions, logarithmic integrals and so on. This is indeed the case, and the following structures will be needed. First, with μ given, let μ_1 and μ_2 be the associated marginals:

$$\mu_1(E)=\mu(E\times I)$$
$$\mu_2(E)=\mu(I\times E)$$

for all Borel sets E of I. Next, perform the Lebesgue decompositions

$$\begin{aligned}d\mu_2&=g\,d\sigma+d\xi\\ d\mu&=w\,d\sigma^2+d\lambda\\ d\mu&=1_\Gamma w_R d(\sigma\times\mu_2)+1_{\Gamma^c}d\lambda_R\\ d\mu&=1_\Delta w_T d(\mu_1\times\sigma)+l_{\Delta^c}d\lambda_T.\end{aligned}$$

Let

$$A=\{e^{it}\in I\colon \int\log w(e^{is},e^{it})\,d\sigma(e^{is})>-\infty\}$$
$$B=\{e^{is}\in I\colon \int\log w(e^{is},e^{it})\,d\sigma(e^{it})>-\infty\}.$$

For $U\subseteq Z^2$, let $M(U,\mu)$ be the span of $\{e^{ims+int}\colon(m,n)\in U\}$ in $L^2(\mu)$. In particular, let $R_N(=R_N(\mu))$ be the subspace $M(\{(m,n)\colon m\geqslant N,\,n\in Z\},\mu)$, and $R_\infty=\cap R_N$, the "right remote space." Similarly, define $T_N=M(\{(m,n)\colon m\in Z,\,n\geqslant N\},\mu)$, and $T_\infty=\cap T_N$, the "top remote space." A useful reduction occurs whenever $M(U,\mu)$ contains R_∞ or T_∞.

Lemma 2.1. [1, 2.2] The following identifications hold:

$$R_\infty^\perp=L^2(1_{\Gamma\cap(I\times A)}\mu)$$
$$T_\infty^\perp=L^2(1_{\Delta\cap(B\times I)}\mu).$$

For then we have:

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Lemma 2.2. *Let $U \subseteq Z^2$. Suppose that $\{(m, n): m \geqslant N, n \in Z\} \subseteq U$ for some N. Then $\varepsilon(U, \mu) = \varepsilon(U, 1_{\Gamma\cap(I\times A)}\mu)$.*

Proof: Clearly, $\varepsilon(U, \mu) \geqslant \varepsilon(U, 1_{\Gamma\cap(I\times A)}\mu)$.

Conversely, note that $1_{\Gamma^c\cup(I\times A^c)} \in R_\infty \subseteq R_0$, and $1 \in R_0$, hence $1_{\Gamma\cap(I\times A)} \in R_0$. It follows that if p and q are finite trigonometric sums in R_N, then

$$1_{\Gamma\cap(I\times A)}p + 1_{\Gamma^c\cup(I\times A^c)}q$$

lies in R_N, and hence in $M(U, \mu)$. Now

$$\begin{aligned}\varepsilon^2(S, \mu) &= \inf\left\{\int |1+f|^2\,d\mu : f \in M(S, \mu)\right\} \\ &\leqslant \inf\left\{\int \left|1 + 1_{\Gamma\cap(I\times A)}p + 1_{\Gamma^c\cup(I\times A^c)}q\right|^2 d\mu : p \text{ and } q \text{ as above}\right\} \\ &= \inf\left\{\int |1+p|^2\, 1_{\Gamma\cap(I\times A)}d\mu : p\right\} + \inf\left\{\int |1+q|^2\, 1_{\Gamma^c\cup(I\times A^c)}d\mu : q\right\}.\end{aligned}$$

The second term is zero, since $L^2\left(1_{\Gamma^c\cup(1\times A^c)}\mu\right)$ is right-remote; that is, by Lemma 2.1, $1 \in L^2\left(1_{\Gamma^c\cup(I\times A^c)}\mu\right) = R_n\left(1_{\Gamma^c\cup(I\times A^c)}\mu\right)$ for all n. The remaining term is simply $\varepsilon^2\left(S, 1_{\Gamma\cap(I\times A)}\mu\right)$. ∎

An analogous statement holds for U containing $\{(m, n) : m \in Z, n \geqslant N\}$.

3. Error formulas

We now investigate $\varepsilon^2(U, \mu)$ for a number of interesting choices of U. Let us interpret divergent integrals in the obvious ways.

Let $U_1 = \{(0, n) \in Z^2 : n \geqslant 1\}$.

Theorem 3.1. *We have*

$$\varepsilon^2(U_1, \mu) = \exp\int \log g\, d\sigma,$$

$$\varepsilon^2\left(U_1, w\, d\sigma^2\right) = \exp\int \log\left(\int w\left(e^{is}, e^{it}\right)d\sigma\left(e^{is}\right)\right)d\sigma\left(e^{it}\right).$$

Proof:

$$\begin{aligned}\varepsilon^2(U_1, \mu) &= \inf\left\{\int |1+f|^2\,d\mu : f \in M(U_1, \mu)\right\} \\ &= \inf\left\{\int \left|1+f\left(e^{it}\right)\right|^2 d\mu_2\left(e^{it}\right) : f \in M(U_1, \mu)\right\}.\end{aligned}$$

By Theorem 1.1, the last expression is equal to $\exp\int \log g\, d\sigma$. In case $d\mu = w\, d\sigma^2$, the density $g(\cdot)$ becomes $\int w\left(e^{is}, \cdot\right)d\sigma\left(e^{is}\right)$. ∎

Let $U_2 = \{(m, n) \in Z^2 : m \in Z, n \geqslant 1\}$.

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Theorem 3.2. *We have*

$$\varepsilon^2(U_2, \mu) = \int \exp\left(\int \log w_T(e^{is}, e^{it})\, d\sigma(e^{it})\right) d\mu_1(e^{is}),$$

$$\varepsilon^2(U_2, w\, d\sigma^2) = \int \exp\left(\int \log w(e^{is}, e^{it})\, d\sigma(e^{it})\right) d\sigma(e^{is}).$$

Proof: This is [2, 4.5], an extension of [5, Theorem II.1]. ∎

Let $U_3 = U_2 \cup \{(m, 0)\colon m \geqslant 1\}$. This is an "augmented halfplane," first studied by Helson and Lowdenslager [3].

Theorem 3.3. *We have*

$$\varepsilon^2(U_3, \mu) = \exp \int \log w\, d\sigma^2.$$

Proof: This is [3, Theorem 1]. ∎

The formula remains valid if U_3 is replaced by any other augmented halfplane, such as $Z^2 \setminus (U_3 \cup \{(0, 0)\})$.

Let $U_4 = \{(m, n)\colon m \in Z, n \geqslant 0\} \setminus \{(0, 0)\}$.

Theorem 3.4. *We have*

$$\varepsilon^2(U_4, \mu) = \left[\int \exp\left(\int \log\left[1/w(e^{is}, e^{it})\right]\right) d\sigma(e^{it})\, d\sigma(e^{is})\right]^{-1}.$$

Proof: See [5, Theorem III.7]. ∎

Let $U_5 = \{(0, n)\colon n \neq 0\}$.

Theorem 3.5. *We have*

$$\varepsilon^2(U_5, \mu) = \left(\int g^{-1} d\sigma\right)^{-1},$$

$$\varepsilon^2(U_5, w d\sigma^2) = \left[\int \left(\int w(e^{is}, e^{it})\, d\sigma(e^{is})\right)^{-1} d\sigma(e^{it})\right]^{-1}.$$

Proof: This follows from Theorem 1.2. ∎

Let $U_6 = \{(m, n)\colon m \in Z, n \neq 0\}$.

Theorem 3.6. *We have*

$$\varepsilon^2(U_6, \mu) = \int \left(\int \left[1/w_T(e^{is}, e^{it})\right] d\sigma(e^{it})\right)^{-1} d\mu_1(e^{is}),$$

$$e^2(U_6, w\, d\sigma^2) = \int \left(\int \left[1/w(e^{is}, e^{it})\right] d\sigma(e^{it})\right)^{-1} d\sigma(e^{is}).$$

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PROOF: Note that $\{(m, n) \in Z^2 : m \in Z, n \geqslant 1\} \subseteq U_6$.

By Lemma 2.2, $\varepsilon^2(U_6, \mu) = \varepsilon^2\big(U_6, 1_{\Delta \cap (B \times I)}\mu\big)$. That is, we can assume that $d\mu$ is of the form $d\mu = w_T d(\mu_1 \times \sigma)$. In that case the assertion follows from [5, Theorem III.10]. ∎

Let $U_7 = Z^2 \setminus \{(m, 0) : m \leqslant 0\}$.

THEOREM 3.7. *We have*

$$\varepsilon^2(U_7, \mu) = \exp \int \log \left(\int \big[1/w(e^{is}, e^{it})\big]\, d\sigma(e^{it}) \right)^{-1} d\sigma(e^{is}).$$

PROOF: Let u_m be the projection of e^{ims} onto $M(U_6, \mu)$, $m \in Z$. Then $\varepsilon(U_7, \mu)$ is equal to the least squares error of estimating u_0 within the span of $\{u_m\}_{m=1}^{\infty}$. To see this, let P and Q be the projection operators of $L^2(\mu)$ onto $M(U_6, \mu)$ and $M(U_7, \mu)$, respectively. Then

$$\begin{aligned} 1 - Q1 &= 1 - P1 - (Q - P)1 \\ &= 1 - P1 - (Q - P)(1 - P1) \\ &= u_0 - (Q - P)u_0. \end{aligned}$$

But the range of $Q - P$ is exactly the span of $\{u_m\}_{m=1}^{\infty}$. By [5, Theorem III.11], the measure ν on I given by

$$\nu(\cdot) = \left(\int \big[1/w_T(\cdot, e^{it})\big]\, d\sigma(e^{it}) \right)^{-1} d\mu_1(\cdot)$$

has the property

$$\langle u_j, u_k \rangle = \int e^{i(j-k)\theta} d\nu(e^{i\theta}).$$

Hence $\varepsilon(U_7, \mu)$ is equal to the least-squares error in estimating 1 within the span of $\{e^{im\theta}\}_{m=1}^{\infty}$ in $L^2(\nu)$. Taking $d\mu_1 = h\,d\sigma + d\eta$, Theorem 1.1 now gives

$$\begin{aligned} \varepsilon^2(U_7, \mu) &= \exp \int \log \left[h(e^{is}) \left(\int \big[1/w_T(e^{is}, e^{it})\big]\, d\sigma(e^{it}) \right)^{-1} \right] d\sigma(e^{is}). \\ &= \exp \int \log \left(\int \big[1/h(e^{is}) w_T(e^{is}, e^{it})\big]\, d\sigma(e^{it}) \right)^{-1} d\sigma(e^{is}). \\ &= \exp \int \log \left(\int \big[1/w(e^{is}, e^{it})\big]\, d\sigma(e^{it}) \right)^{-1} d\sigma(e^{is}). \end{aligned}$$

∎

Let $U_8 = Z^2 \setminus \{(0, 0)\}$.

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THEOREM 3.8. *We have*

$$\varepsilon^2(U_8, \mu) = \left(\int w^{-1} d\sigma^2\right)^{-1}.$$

PROOF: Let $\{u_m\}_{m\in Z}$, ν and h be defined as in the proof of Theorem 3.7. Then $\varepsilon^2(U_8, \mu)$ is equal to the least squares error in estimating u_0 by $\{u_m\}_{m\neq 0}$. Now applying Theorem 1.2 to the measure μ gives

$$\begin{aligned}\varepsilon^2(U_8, \mu) &= \left(\int h(e^{is})^{-1}\left(\int [1/w_T(e^{is}, e^{it})]\, d\sigma(e^{it})\right)^{-1} d\sigma(e^{is})\right)^{-1}\\ &= \left(\iint [1/w(e^{is}, e^{it})]\, d\sigma(e^{is}) d\sigma(e^{it})\right)^{-1}.\end{aligned}$$

□

Figure 1, over, shows graphs of the parameter sets $U_1, U_2, \ldots, U_8$ on the Z^2 array. Note that the sets are decreasing by containment from top to bottom, and from left to right. Since $U \subseteq V$ implies that $\varepsilon(U, \mu) \geqslant \varepsilon(V, \mu)$, this yields:

COROLLARY 3.9. *Suppose that $W(x, y)$ is a nonnegative, $[dx\, dy]$-integrable function on $[0, 1] \times [0, 1]$. Then the following inequalities hold, provided that the reciprocals of divergent integrals are interpreted as zero:*

$$\begin{array}{ccc}
\iint W\,dx\,dy & \geqslant & \\
\exp\int\left(\log\int W\,dx\right)dy & \geqslant & \left(\left(\int W\,dx\right)^{-1}dy\right)^{-1}\\
\geqslant & & \geqslant\\
\int \exp\left(\int \log W\,dy\right)dx & \geqslant & \int\left(\int W^{-1}dy\right)^{-1}dx\\
\geqslant & & \geqslant\\
\exp\iint \log W\,dx\,dy & \geqslant & \exp\int\log\left(\left(\int W^{-1}dy\right)^{-1}\right)dx\\
\geqslant & & \geqslant\\
\left(\int\left(\exp\int\log W^{-1}dy\right)dx\right)^{-1} & \geqslant & \left(\iint W^{-1}dx\,dy\right)^{-1}.
\end{array}$$

4. A DUALITY RELATION

The following result is an adaptation of [7, Theorem 3.1] to the present context. For any subset U of $Z^2 \setminus \{(0, 0)\}$, let $U^{-1} = \left(Z^2 \setminus \{(0, 0)\}\right) \setminus U$.

THEOREM 4.1. *Suppose that $w \geqslant 0$, $\int w\, d\sigma^2 < \infty$ and $\int w^{-1} d\sigma^2 < \infty$. Then*

$$\varepsilon\left(U, w d\sigma^2\right) = \varepsilon\left(U^{-1}, w^{-1} d\sigma^2\right)^{-1}$$

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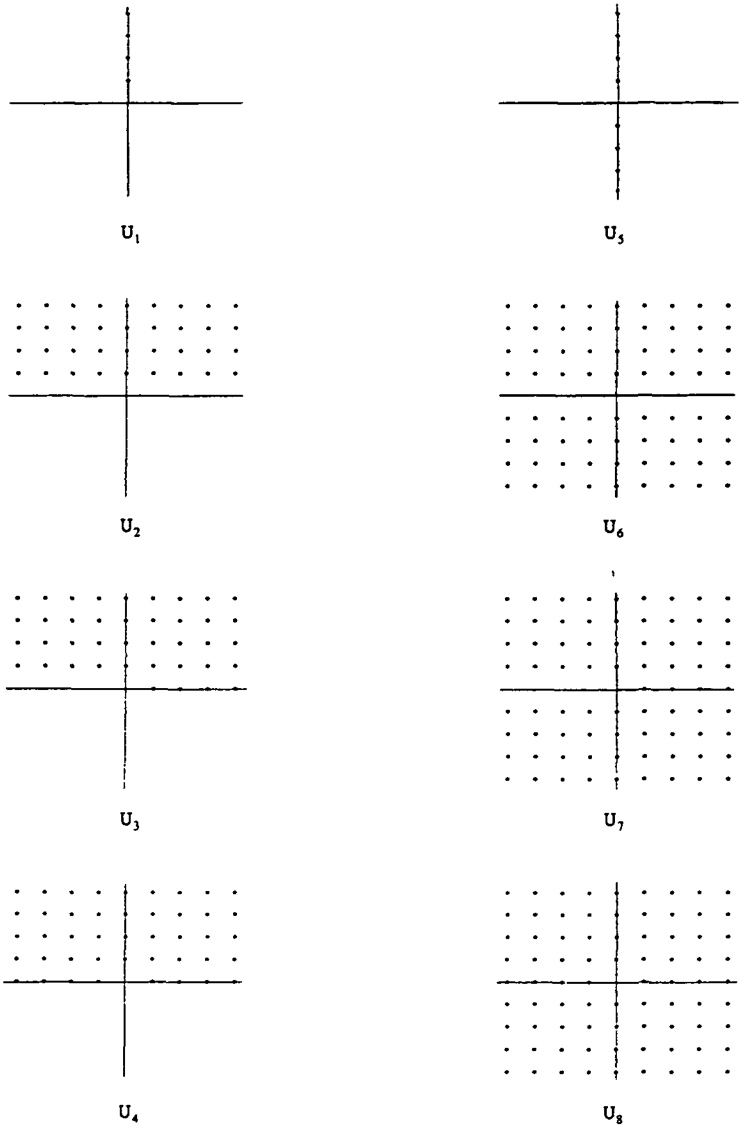

Figure 1

for any $U \subseteq Z^2 \setminus \{(0, 0)\}$.

PROOF: Let $M_0(U)$ be the collection of finite linear combinations of

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$\{e^{ims+int}: (m, n) \in U\}$.

$$\begin{aligned}\varepsilon^2\left(U, w\, d\sigma^2\right) &= \inf\left\{\int |1+p|^2\, w\, d\sigma^2 : p \in M_0(U)\right\}\\ &= \inf\left\{\frac{\int |f|^2\, w\, d\sigma^2}{\left(\int f\, d\sigma^2\right)^2} : f \in M_0(U \cup \{(0,0)\})\right\}\\ &= \inf\left\{\frac{\int |fw|^2\, w^{-1}\, d\sigma^2}{\left(\int (fw) w^{-1}\, d\sigma^2\right)^2} : f \in M_0(U \cup \{(0,0)\})\right\}\\ &= \left[\sup\left\{\frac{\left(\int (fw) w^{-1}\, d\sigma^2\right)^2}{\int |fw|^2\, w^{-1}\, d\sigma^2} : f \in M_0(U \cup \{(0,0)\})\right\}\right]^{-1}.\end{aligned}$$

The supremum is exactly the squared norm of 1 as a bounded linear functional on the span G of $\{fw: f \in M_0(U \cup \{(0,0)\})\}$ in $L^2\left(w^{-1}\, d\sigma^2\right)$. This, in turn, is equal to the squared distance in $L^2\left(w^{-1}\, d\sigma^2\right)$ from 1 to $G^\perp$. But $G^\perp$ is spanned by $M_0\left(U^{-1}\right)$. Thus the chain of equations continues

$$\begin{aligned}&= \left[\inf\left\{\int |1+q|^2\, w^{-1}\, d\sigma^2 : q \in M_0\left(U^{-1}\right)\right\}\right]^{-1}\\ &= \varepsilon^2\left(U^{-1}, w^{-1}\, d\sigma^2\right)^{-1}.\end{aligned}$$

□

It can be checked by inspection that the formulas of Theorems 3.1 through 3.9 are consistent with Theorem 4.1. First note that U_1^{-1} is a rotation of U_7; U_2^{-1} is a rotation of U_4; U_5^{-1} is a rotation of U_8; U_3^{-1} is a rotation of itself; U_8^{-1} is the empty set (this corresponds to prediction of 1 by 0). After the appropriate variable changes to account for the rotations, the associated pairs of formulas do indeed illustrate Theorem 4.1.

References

[1] R. Cheng, 'A natural decomposition of L^2 spaces on the torus', (preprint 1990).

[2] R. Cheng, 'A strong mixing condition for second-order stationary random fields', (preprint 1990), *Studia Math.* (to appear).

[3] H. Helson and D. Lowdenslager, 'Prediction theory and Fourier series in several variables', *Acta Math.* **99** (1959), 165–202.

[4] H. Helson and D. Lowdenslager, 'Prediction theory and Fourier series in several variables II', *Acta Math.* **106** (1962), 175–213.

[5] G. Kallianpur, A.G. Miamee and H. Niemi, *On the prediction theory of two-parameter stationary random fields: Center for Stochastic Processes*, Technical Report No. 178 (UNC, Chapel Hill, 1987).

https://doi.org/10.1017/S0004972700011771 Published online by Cambridge University Press

[6] A.N. Kolmogorov, 'Interpolation and Extrapolation von stationären zufälligen Folgen', *Izv. Akad. Nauk SSSR* **5** (1941), 3–14.

[7] A.G. Miamee and M. Pourahmadi, 'Best approximations in $L^p(d\mu)$, and prediction problems of Szegö, Kolmogorov, Yaglom and Nakazi', *J. London Math. Soc.* **38** (1988), 133–145.

[8] G. Szegö, 'Beiträge zur Theorie der Toeplitzchen Formen', *Math. Z.* **6** (1920), 167–202.

Department of Mathematics
University of Louisville
Louisville KY 40292
United States of America