GROUPS FIXING GRAPHS IN SWITCHING CLASSES

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Abstract

A permutation group $G$ on a finite set $\Omega$ is *always exposable* if whenever $G$ stabilises a switching class of graphs on $\Omega$, $G$ fixes a graph in the switching class. Here we consider the problem: given a finite group $G$, which permutation representations of $G$ are always exposable? We present solutions to the problem for (i) 2-generator abelian groups, (ii) all abelian groups in semiregular representations, (iii) generalised quaternion groups and (iv) some representations of the symmetric group $S_n$.


Introduction

According to Harries and Liebeck (1978), a permutation group $G$ on a finite set $\Omega$ is *always exposable* if whenever $G$ stabilises a switching class of graphs on $\Omega$, $G$ fixes a graph in the switching class. Equivalently, in the notation of Cameron (1977), $G$ is always exposable if the first invariant $\gamma$ of $G$ and $\tau$ is zero for every 2-graph $\tau$ on $\Omega$ on which $G$ acts. Here we consider the following problem:

**Problem.** Given a finite group $G$, which permutation representations of $G$ are always exposable?

The problem has been solved when $G$ is cyclic by Mallows and Sloane (1975) and when $G$ is dihedral by Harries and Liebeck (1978). Here we present solutions for the following groups: (i) 2-generator abelian groups, (ii) all abelian groups in semiregular representations, (iii) generalised quaternion groups, (iv) some representations of the symmetric group $S_n$. The methods used for (i), (ii) and (iii) are...
based on those introduced by Harries and Liebeck (1978); it will readily be seen that they suffice to solve the problem for any group \( G \), given a presentation for \( G \) and a very large supply of patience. For (iv) we use a different technique.

One application of the results is given by the observation that if \( X \) is a 2-transitive automorphism group of a nontrivial 2-graph and \( P \) is a Sylow 2-subgroup of \( X \) then \( P \) is not always exposable (see Corollary 3.7 of Harries and Liebeck (1978) or Proposition 2.5 of Cameron (1977)). Hence solutions to the above problem give restrictions on possible permutation representations of Sylow 2-subgroups of 2-transitive automorphism groups of nontrivial 2-graphs (the fact that such 2-graphs must be regular gives further restrictions).

For notation and an introduction to switching classes see Harries and Liebeck (1978); for connections with 2-graphs (and other things) see Sections 2, 3 of Cameron (1977).

1. Abelian groups in semiregular representations

If \( \Gamma \) is a graph on a finite set \( \Omega \) we denote the switching class of \( \Gamma \) by \( S(\Gamma) \); if \( s \) is the switch with respect to the subset \( \Phi \) of \( \Omega \) then \( s\Gamma \) is the graph obtained from \( \Gamma \) by switching with respect to \( \Phi \). For a permutation \( \alpha \) of \( \Omega \), \( \alpha(\Phi) \) is the image of \( \Phi \) under \( \alpha \), and \( \alpha_s \) is the switch with respect to \( \alpha(\Phi) \). We say that \( \Phi \) is compatible with \( \alpha \) if each cycle of \( \alpha \) involves an even number of elements of \( \Phi \). Write \( \Phi_\alpha \) for the symmetric difference \( \Phi \Delta \alpha(\Phi) \); note that if \( s \) is the switch with respect to \( \Phi \) then \( s_\alpha s \) is the switch with respect to \( \Phi_\alpha \). As observed in Section 3 of Harries and Liebeck (1978), the graphs on \( \Omega \) are permuted by switches \( s \), by permutations \( \alpha \) and by compositions \( s_\alpha \) of these operations, which are called switch-permutations. These satisfy the rule \( \alpha s = \alpha_\Phi s \alpha \). If \( \Psi \) is a fixed set of \( \langle \alpha \rangle \) then \( \alpha^\Psi \) denotes the action of \( \alpha \) on \( \Psi \). Let \( Z_n \) denote a cyclic group of order \( n \).

**Theorem 1.1.** Let \( G \) be a finite, abelian, semiregular permutation group. Then \( G \) is always exposable if and only if \( G \) has no subgroup isomorphic to \( Z_2 \times Z_4 \).

**Proof.** First suppose that \( Z_2 \times Z_4 \leq G \) and write \( G = Z_{r_1} \times Z_{r_2} \times \cdots \times Z_{r_n} \) where \( 4 \mid r_1 \) and \( 2 \mid r_2 \). We show that \( G \) is not always exposable. Let \( \alpha_1, \ldots, \alpha_n \) be generators for the cyclic factors \( Z_{r_1}, \ldots, Z_{r_n} \) respectively. For simplicity we suppose that \( G \) is regular (the proof extends readily to the semiregular case). We may identify \( \Omega \) with \( \{1, 2, \ldots, r_1 r_2 \cdots r_n\} \) and take

\[
\alpha_1 = (1 2 \cdots r_1)(r_1 + 1 \cdots 2r_1) \cdots,
\]

\[
\alpha_i = (1 r_1 \cdots r_{i-1} + 1 \cdots (r_i - 1)r_1 \cdots r_{i-1} + 1)(2 r_1 \cdots r_{i-1} + 2 \cdots) \cdots
\]
for \( i \geq 2 \). Let \( \Phi = \{ r \in \Omega \mid r \text{ odd} \} \) and let \( s \) be the switch with respect to \( \Phi \). For \( i = 3, \ldots, n \) define

\[
\beta_i = \begin{cases} 
\alpha_i & \text{if } r_i \text{ is even}, \\
\alpha_2 \alpha_i & \text{if } r_i \text{ is odd}, 
\end{cases}
\]

so that each \( \beta_i \) has even order and \( \Phi \) is compatible with \( \alpha_1, \alpha_2, \beta_3, \ldots, \beta_n \) (since \( 4 \mid r_1 \)). Note that \( \alpha_i(\Phi) = \Omega \setminus \Phi \), \( \alpha_2(\Phi) = \Phi \) and \( \beta_i(\Phi) = \Phi \) for \( i \geq 3 \). Let \( Q \) be the group \( \langle s\alpha_1, s\alpha_2, s\beta_3, \ldots, s\beta_n \rangle \) of switch-permutations. It is easy to verify, in the notation of Harries and Liebeck (1978), (i) that \( Q = \langle \alpha_1, \alpha_2, \beta_3, \ldots, \beta_n \rangle \) \( (= G) \), hence that no element \( Q \) involves switch 1-cycles \( (a) \) \( (b) \), and (ii) that no involution of \( Q \) involves a switch-transposition \( (ab) \). Hence by Theorem 3.8 of Harries and Liebeck (1978), \( Q \) fixes a graph \( \Gamma \) on \( \Omega \). Then \( G \) stabilises \( S(\Gamma) \); however if \( G \) fixes the graph \( s' \Gamma \in S(\Gamma) \) then \( s = s_{\alpha_1} s' = s_{\alpha_2} s' \). It is not hard to see that no such switch \( s' \) can exist, so \( G \) is not always exposable.

Conversely, suppose that \( G \) has no subgroup \( Z_2 \times Z_4 \) and let \( P \) be a Sylow 2-subgroup of \( G \); then \( P \) is cyclic or elementary abelian. If \( P \) is cyclic then \( G \) is always exposable (Theorem 4.6 of Harries and Liebeck (1978) or the remark after Theorem 3.4 of Cameron (1977)). So suppose that \( P = \langle \alpha_1, \ldots, \alpha_n \rangle \equiv (Z_2)^n \). We show that \( P \), and hence \( G \), is always exposable by induction on \( n \). Let \( P \) stabilise a switching class \( S(\Gamma) \). By induction, the subgroup \( Q = \langle \alpha_1 \alpha_2, \alpha_1 \alpha_3, \ldots, \alpha_1 \alpha_n \rangle \) fixes a graph \( \Gamma_{\alpha_1} \in S(\Gamma) \), so \( \alpha_1 \Gamma_{\alpha_1} = \alpha_2 \Gamma_{\alpha_1} = \cdots = \alpha_n \Gamma_{\alpha_1} = s \Gamma_{\alpha_1} \), say, and \( \langle s \alpha_1, \ldots, s \alpha_n \rangle \) fixes \( \Gamma_{\alpha_1} \). If \( \Phi \) is the subset switched by \( s \) then \( \Phi \) is compatible with each \( \alpha_i \) (Lemma 4.4 of Harries and Liebeck (1978)). Hence from the action of \( P \) on an orbit we see that \( \Phi \) must be a union of \( P \)-orbits. For each \( P \)-orbit \( \Psi \subseteq \Phi \), let \( \Psi' \) be a \( Q \)-orbit contained in \( \Psi \). Then since \( P = Q \langle \alpha_i \rangle \), we have \( \Psi_{\alpha_i} = \Psi \) for \( i = 1, \ldots, n \) (recall that \( \Psi_{\alpha_i} = \Psi' \Delta \alpha_i(\Psi') \)). Hence if \( s' \) is the switch with respect to a union of \( Q \)-orbits, one for each \( P \)-orbit in \( \Phi \), then \( s = s_{\alpha_i} s' \) for all \( i \) and so \( P \) fixes the graph \( s' \Gamma_{\alpha_1} \). By induction then, \( P \), and so \( G \), is always exposable.

### 2. Two-generator abelian groups

In this section we obtain a necessary and sufficient condition for a 2-generator abelian permutation group \( X \) to be always exposable. We shall see that \( X \) is always exposable if and only if a Sylow 2-subgroup of \( X \) is always exposable (Theorem 2.7), so we first restrict our attention to 2-generator abelian 2-groups. Throughout this section the group \( G \) is defined by

\[
G = \left\langle \alpha, \beta \mid \alpha^{2^m} = \beta^{2^n} = [\alpha, \beta] = 1 \right\rangle \quad (m, n \geq 1).
\]

Let \( G \) act on a finite set \( \Omega \). Then \( \Omega \) is a union of \( G \)-orbits and since \( G \) is abelian, each orbit is determined by the kernel of the action of \( G \) on it. We shall see that

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the exposability of $G$ depends closely on the nature of the intersections of the kernels of the actions of $G$ on its various orbits. First we list the proper subgroups of $G$ in four classes:

(A) $\langle \alpha^{2^j} \rangle, \langle \beta^{2^j} \rangle, \langle \alpha^{2^i}\beta^{2^j} \rangle, \langle \alpha^{2^{k+1}}\beta^{2^{k+1}} \rangle$ ($i \geq 1$, $j \geq 1$, $k > l \geq 1$, any $r$);

(B) $\langle \alpha \rangle, \langle \alpha^{2^i}\beta \rangle, \langle \alpha^{2^{k+1}}\beta^{2^{k+1}} \rangle$ ($i \geq 1$, $k > l \geq 1$);

(C) subgroups in (B) with $\alpha, \beta$ interchanged;

(D) $\langle \alpha^{2^i}\beta^{2^{k+1}} \rangle, \langle \alpha^{2^i}\alpha^{2^j}\beta^{2^{k+1}} \rangle$ ($i > 1$).

Write $\Psi_{\gamma_1,\ldots,\gamma_r}$ or $\Psi_H$ for an orbit of $G$ on $\Omega$ with kernel $H = \langle \gamma_1, \ldots, \gamma_r \rangle$ and $n_{\gamma_1,\ldots,\gamma_r}$ or $n_H$ for the number of such orbits. Denote by $\Psi_{(A)}(\Psi_{(B)}, \Psi_{(C)}, \Psi_{(D)})$ the union of the orbits with kernels in $(A)$ ($(B), (C), (D)$) and by $n_{(A)}(n_{(B)}, n_{(C)}, n_{(D)})$ the number of these orbits.

Our strategy in considering the exposability of $G$ is as follows: if $G$ stabilises the switching class $S(T)$ then there is a switch $s$ such that $\langle sa, s\beta \rangle$ fixes a graph $\Gamma_1 \in S(T)$ (Lemma 5.1 of Harries and Liebeck (1978)). The group $G$ fixes the graph $s\Gamma_1 \in S(T)$ if and only if

\[ s = s'_a s' = s'_\beta s'. \]

Thus we seek all switches $s$ such that $\langle sa, s\beta \rangle$ fixes a graph and determine whether or not there is a switch $s'$ satisfying ($\ast$). The following definition is useful in this strategy.

**Definition 2.1.** Let $\Psi$ be an orbit of $G$ on $\Omega$ and let $\Sigma \subseteq \Psi$.

(i) The subset $\Sigma'$ of $\Sigma$ is an $(\alpha, \beta)$-subset of $\Sigma$ (respectively $(\alpha, \bar{\beta})$-subset; $(\bar{\alpha}, \beta)$-subset) if $\Sigma'_\alpha (= \Sigma' \Delta a(\Sigma')) = \Sigma$ and $\Sigma'_\beta = \Sigma$ (respectively $\Sigma'_{\bar{\alpha}} = \Sigma$ and $\Sigma'_{\beta} = \Psi \setminus \Sigma$; $\Sigma'_{\alpha} = \Psi \setminus \Sigma$ and $\Sigma'_{\beta} = \Sigma$). We say that $\Sigma$ is amenable if it has $(\alpha, \beta)$-, $(\alpha, \bar{\beta})$- and $(\bar{\alpha}, \beta)$-subsets.

(ii) The orbit $\Psi$ is strictly $(\alpha, \beta)$ if it has an $(\alpha, \beta)$-subset but no $(\alpha, \bar{\beta})$- or $(\bar{\alpha}, \beta)$-subset (similar definitions apply for a strictly $(\alpha, \bar{\beta})$ and a strictly $(\bar{\alpha}, \beta)$ orbit).

**Example 2.2.** Let $m = 3$, $n = 2$ and let $\Delta_1, \Delta_2$ be the orbits of $G$ with kernels $K_1 = \langle \alpha^{2^2} \beta \rangle$, $K_2 = \langle \alpha^{2^3} \beta \rangle$ respectively. We may write

\[ a^{\Delta_1 \cup \Delta_2} = (1 2 3 4)(5 6 7 8)(9 10 11 12), \]

\[ \beta^{\Delta_1 \cup \Delta_2} = (1 5 3 7)(2 6 4 8)(9 11)(10 12). \]

It is easy to see that $\Delta_1$ is amenable, whereas $\Delta_2$ is strictly $(\alpha, \bar{\beta})$.

**Lemma 2.3.** The orbits with kernels in class $(A), (B), (C), (D)$ are, respectively, amenable, strictly $(\alpha, \bar{\beta})$, strictly $(\bar{\alpha}, \beta)$, strictly $(\alpha, \beta)$ orbits.
PROOF. We consider only an orbit \( \Psi = \Psi_{\alpha^r \beta^j} \) (r odd); other cases are similar. Write \( H = \langle \alpha^2 \beta^2 \rangle \). There exists an odd integer \( k \) such that \( \beta^{2j} \equiv \alpha^{2k} \) (mod \( H \)) and assuming that \( m + j \leq n + i \), we may write

\[
\alpha^\Psi = (1 2 \cdots 2^m) \cdots ((2^j - 1)2^m + 1 \cdots 2^{m+j}) \cdots ,
\]
\[
\beta^\Psi = (1 2^m + 1 2 2^m + 1 \cdots (2^j - 1)2^m + 1 2^m + 2^j + 1 \cdots ) \cdots .
\]

Let \( \delta_1, \ldots, \delta_2 \), be the orbits of \( \langle \alpha^r \rangle \) in the order written above. If \( i \geq 1, j \geq 1 \) then \( \{ \text{r odd} \mid 1 \leq r \leq 2^{m+j} \} \), \( \delta_1 \cup \delta_3 \cup \cdots \cup \delta_{2j-1} \), \( \{ \text{r odd} \mid r \in \delta_{2j+1}, \ l = 0, 1, \ldots \} \cup \{ \text{r even} \mid r \in \delta_{2j}, \ l = 1, 2, \ldots \} \) are, respectively, \( (\alpha, \beta^-), (\alpha, \beta^+), (\alpha, \beta^-) \)-subsets of \( \Psi \), so \( \Psi \) is amenable. If \( j = 0, i \geq 1 \) then \( \Psi \) is strictly \( (\alpha, \beta^-) \) and if \( i = 0, j \geq 1 \) then \( \Psi \) is strictly \( (\alpha, \beta^-) \). Finally if \( i = j = 0 \) then \( \beta^\Psi = (\alpha^\Psi)^k \) so \( \Psi \) is strictly \( (\alpha, \beta^-) \).

**Lemma 2.4.** Let \( H_\lambda (\lambda \in \Lambda) \) be subgroups of \( G \) and let \( s \) be the switch with respect to a union \( \Psi = \bigcup_{\lambda \in \Lambda} \Psi_{H_\lambda} \) of \( G \)-orbits. Then \( \langle s\alpha, s\beta \rangle \) does not fix a graph on \( \Omega \) if and only if there is a subgroup \( K \) of \( G \) such that (i) \( \Omega \setminus \Phi \) contains an orbit \( \Psi_K \), and (ii) for some \( \lambda, H_\lambda \cap K \) contains an element of odd length in \( \alpha, \beta \) (that is, an element \( \alpha^t \beta^u \) where \( t + u \) is odd).

**Proof.** We use Theorem 3.8 of Harries and Liebeck (1978). Since \( s \) is a switch with respect to a union of \( G \)-orbits, \( \langle s\alpha, s\beta \rangle \) contains no elements which involve a switch-transposition \( (\bar{a} \ b) \). Also an element \( (\alpha^x) \gamma(s\beta)^{\gamma} \) involves switch 1-cycles \( (\bar{a}) \ (b) \) if and only if for some \( \lambda, a \in \Psi_{H_\lambda} \subseteq \Phi \) and \( b \in \Psi_K \subseteq \Omega \setminus \Phi \) where \( \alpha^t \beta^u \in H_\lambda \cap K \) and \( x + y \) is odd.

In view of Lemma 2.4 we make the following definition.

**Definition 2.5.** (i) The **B-graph** of \( G \) is defined as follows: its vertices are those subgroups \( H \) in \( (B) \) with \( n_H > 0 \), and \( H \) is joined to \( K \) if and only if \( H \cap K \) contains an element of odd length in \( \alpha, \beta \). The **C-graph** is similarly defined.

(ii) The **D-graph** of \( G \) has vertex set \( \{H \in (D) \mid n_H > 0\} \) and \( H \) is joined to \( K \) if and only if \( H \cap K \) contains \( \alpha^t \beta^u \) for some odd \( t, u \).

By Lemma 2.4, the subsets \( \Phi \subseteq \Psi_{(B)} \), which are unions of \( G \)-orbits such that \( \langle s\alpha, s\beta \rangle \) fixes a graph (where \( s \) is the switch with respect to \( \Phi \), are in 1-1 correspondence with unions of components of the **B-graph**; if \( \mathcal{C} \) is the set of vertices in a union of components, the corresponding subset \( \Phi \subseteq \bigcup_{H \in \mathcal{C}} \Psi_{H^\prime} \).
Theorem 2.6. The 2-generator abelian 2-group $G$ acting on $\Omega$ is not always exposable if and only if $\text{fix } G = \emptyset$ and one of the following holds:

(i) all orbits of $G$ have their kernels in class $(A)$ and $n_{a_2B_2} = 0$;

(ii) $n_{(B)}n_{(C)}n_{(D)} > 0$;

(iii) the $B$-graph is disconnected;

(iv) the $C$-graph is disconnected;

(v) the $D$-graph is disconnected.

Proof. If $\text{fix } G \neq \emptyset$ then $G$ is always exposable (every switching class contains a unique graph in which a given vertex is isolated), so suppose that $\text{fix } G = \emptyset$. Let $G$ stabilise $S(\Gamma)$; then there is a switch $s$ with respect to $\Phi$ such that $\langle sa, s\beta \rangle$ fixes a graph $\Gamma_1 \in S(\Gamma)$ (Lemma 5.1 of Harries and Liebeck (1978)). As explained before, we seek all such switches $s$ and determine whether or not there is a switch $s'$ with $s = s's = s'\beta$. Certainly $sas = s\beta s$, so $sas = s\beta s$. Consequently $\alpha(\Phi)$ is either $\beta(\Phi)$ or $\Omega \setminus \beta(\Phi)$.

Case 1. $\Phi$ is a union of $G$-orbits. Since orbits with kernels in $(A)$ are amenable (Lemma 2.3) we may assume that $\Phi \subseteq \Psi_{(B)} \cup \Psi_{(C)} \cup \Psi_{(D)}$. As noted after Definition 2.5 the sets $\Phi \subseteq \Psi_{(B)}$ for which $\langle sa, s\beta \rangle$ fixes a graph are in 1-1 correspondence with unions of components of the $B$-graph. Since $\Psi_{(B)}$ consists of strictly $(a, \bar{\beta})$ orbits, the required switch $s'$ will not exist if and only if $\Omega \setminus \Phi$ contains either a strictly $(a, \bar{\beta})$ orbit or both a strictly $(\bar{a}, \beta)$ and a strictly $(a, \beta)$ orbit. Hence $G$ is not always exposable in cases (ii), (iii) of the theorem. The case $\Phi \subseteq \Psi_{(C)}$ yields (iv) of the theorem; other possibilities for $\Phi$ give no further cases where $s'$ does not exist.

Case 2. $\Phi$ is not a union of $G$-orbits. Recall that $\alpha(\Phi)$ is either $\beta(\Phi)$ or $\Omega \setminus \beta(\Phi)$. Suppose first that $\alpha(\Phi) = \beta(\Phi)$. By considering the relevant permutation representations we find that if $\emptyset \neq H \subseteq (A) \cup (B) \cup (C)$ then $\Phi \cap \Psi_H$ is amenable. Now consider an orbit $\Psi = \Psi_{(a^r, \bar{a}^r, \beta)}$ (r odd) in $(D)$. We have (assuming $i \leq n$)

$$\alpha^\Psi = (12 \cdots 2^r), \quad \beta^\Psi = (\alpha^\Psi)^r.$$ 

If $\alpha^{-1}\beta \not\in \langle a^{2^r}, \alpha^{-1}\beta \rangle$ (so that $r \neq 1$) and $r - 1 = 2^j k$ where $k$ is odd, then $\Phi^\Psi = \{1, 2^j + 1, 2^j + 1, \ldots, 2^j - 2^j + 1\}$ is an orbit of $\langle \alpha^{-1}\beta \rangle$ and $\Phi^\Psi$ has $(a, \bar{\beta})$- and $(\bar{a}, \beta)$-subsets but no $(a, \beta)$-subset. Further, if $s_1$ is the switch with respect to $\Phi^\Psi$ then $(s_1 \alpha)^r(s_1 \beta)^{-1} = s_{\Psi} \alpha' \beta^{-1}$ where $s_{\Psi}$ is the switch with respect to $\Psi$. Hence if

$$\Phi = \bigcup \{ \Phi^\Psi \mid \Psi = \Psi_H, H \text{ ranges over a component of the } D\text{-graph} \}$$
then \(\langle s\alpha, s\beta \rangle\) fixes a graph (Theorem 3.8 of Harries and Liebeck (1978)). Consequently \(s'\) does not exist in case (v) of the theorem; this is the only case where \(\alpha(\Phi) = \beta(\Phi)\) and no switch \(s'\) exists.

Finally, if \(\alpha(\Phi) = \Omega(\beta(\Phi))\) then, using Theorem 3.8 of Harries and Liebeck (1978) we see that \(n_{(B)} = n_{(C)} = n_{(D)} = n_{\alpha^2, \beta^2} = 0\), since \(\langle s\alpha, s\beta \rangle\) fixes a graph; and if this is the case it is easy to construct a switch \(s\) for which there is no \(s'\). For instance, for an orbit \(\Psi = \Psi_{\alpha^{2i}, \beta^{2j}}\) \((i \geq 2, j \geq 1)\) we have
\[
\alpha^\Psi = (1 2 \cdots 2^i) \cdots ((2^i - 1)2^i + 1 \cdots 2^{i+j}),
\]
\[
\beta^\Psi = (1 2^i + 1 2.2^i + 1 \cdots (2^j - 1)2^i + 1) \cdots
\]
and we take \(\Phi \cap \Psi\) to be \(\{r\ text{odd} | 1 \leq r \leq 2^{i+j}\}\).

**Theorem 2.7.** Let \(X\) be a 2-generator abelian permutation group on a finite set \(\Omega\) and let \(G\) be a Sylow 2-subgroup of \(X\). Then \(X\) is always exposable if and only if \(G\) is always exposable.

**Proof.** Write \(X = \langle \alpha, \beta | \alpha^m = \beta^n = [\alpha, \beta] = 1 \rangle\). The result is clearly true if \(m\) or \(n\) is odd (for then \(G\) is cyclic), so suppose that \(m, n\) are even. It is not difficult to see that if \(H\) is a subgroup of \(X\) containing elements \(\alpha^{t_i} \beta^{w_i}, \alpha^{t_2} \beta^{w_2}\) where \(t_1, u_2\) are odd and \(t_2, u_1\) are even then \(G \leq H\). Hence we may partition the subgroups \(K = \langle \alpha^{t_i} \beta^{w_i}, \alpha^{t_2} \beta^{w_2} \rangle\) of \(X\) into five classes: (A) subgroups \(K\) with \(v_i, w_i\) even \((i = 1, 2)\); (B) subgroups with \(v_i, v_2\) even, \(w_i\) odd; (C) subgroups with \(w_1, w_2\) even, \(v_1\) odd; (D) subgroups with \(v_i, w_i\) odd \((i = 1, 2)\); (E) subgroups \(K\) containing \(G\). Note that this agrees with the previous use of (A), (B), (C), (D).

As before we write \(n_{(A)}\) for the number of orbits of \(X\) with kernel in class (A), and so on. An orbit of \(X\) with kernel \(K\) breaks up into isomorphic orbits of \(G\), each having kernel \(G \cap K\). And if \(K\) belongs to class (A) ((B), (C), (D)) then, as a subgroup of \(G\), \(G \cap K\) belongs to class (A) ((B), (C), (D)) respectively. If \(X\) has an orbit \(\Psi\) with kernel in class (E) then \(\Psi\) has odd size and so \(X\) is always exposable by Corollary 3.6 of Harries and Liebeck (1978); also \(\Psi \subseteq \text{fix} \ G\). The \(B-, C-\) and \(Z-\)graphs of \(X\) are defined in the same way as in Definition 2.5.

The method of proof of Theorem 2.6 shows that \(X\) is not always exposable if and only if \(n_{(E)} = 0\) and one of the following holds: (i) all orbits of \(X\) have kernel in (A) and \(n_{\alpha^{2q}, \beta^{-r}} = 0\) for any odd \(q, r\); (ii) \(n_{(B)} n_{(C)} n_{(D)} > 0\); (iii) the \(B-, C-\) or \(D-\)graph is disconnected. It is easy to see that connectedness of the \(B-\)graph (\(C-\)graph, \(D-\)graph) of \(X\) is equivalent to connectedness of the \(B-\)graph (\(C-\)graph, \(D-\)graph) of \(G\). Hence by Theorem 2.6, \(X\) is always exposable if and only if \(G\) is.
3. Generalised quaternion groups

Let \( G \) be the generalised quaternion group of order \( 2^{2n+1} \geq 8 \) defined by
\[
G = \langle \alpha, \beta \mid \alpha^{2^n} = 1, \beta^{2^n} = \alpha^{-1}, \beta^{-1} \alpha \beta = \alpha^{-1} \rangle.
\]
For \( j = 1, \ldots, a - 1 \) denote by \( \Psi_j \) the set of right cosets of \( \langle \alpha^{2^n-1}, \alpha \beta \rangle \) in \( G \). The methods of the proof of Theorem 2.6 yield

**Theorem 3.1.** The generalised quaternion group \( G \) acting on a finite set \( \Omega \) is not always exposable if and only if one of the following holds:

(i) \( G \) is semiregular;

(ii) \( \text{fix } G = \emptyset, \text{fix } \alpha \neq \emptyset, \text{fix } \beta \neq \emptyset \) and for some \( j \), \( G \) has an orbit isomorphic to \( \Psi_j \).

4. Some representations of \( S_n \)

The methods used in the previous two sections to solve the problem of the Introduction are only really efficient when the group \( G \) is easily presented on few generators. We now introduce a different technique which applies to any permutation group; we apply it only to certain representations of the symmetric group \( S_n \).

**Theorem 4.1.** Let \( S_n \) act naturally on \( \Sigma = \{1, 2, \ldots, n\} \), let \( k \) be a positive integer and denote by \( \Sigma^{(k)} \) the set of \( k \)-subsets of \( \Sigma \). If \( n \geq 4k - 2 \) then the action of \( S_n \) on \( \Sigma^{(k)} \) is always exposable.

**Proof.** Write \( \Omega = \Sigma^{(k)} \). Our strategy is as follows: firstly, by looking at the orbits of \( S_n \) on \( \Omega^{(2)} \) we classify all the switching classes on \( \Omega \) in which \( S_n \) fixes a graph; then by considering the orbits of \( S_n \) on \( \Omega^{(3)} \) we show that any 2-graph on \( \Omega \) on which \( S_n \) acts corresponds to one of these switching classes.

**Step 1. Orbits on \( \Omega^{(3)} \).** An orbit of \( S_n \) on \( \Omega^{(3)} \) is uniquely determined by a 4-tuple \((r, s, t, u)\) of nonnegative integers with \( u \leq r \leq s \leq t \), where \( \{A, B, C\} \) is in the orbit \((r, s, t, u)\) \((A, B, C \in \Omega)\) if and only if \(|A \cap B| = r, |A \cap C| = s, |B \cap C| = t\) and \(|A \cap B \cap C| = u\). Clearly (i) \( u \leq r \leq s \leq t < k \) and (ii) \( s + t \leq k + u \); and if \((r, s, t, u)\) satisfies (i) and (ii) then it corresponds to an orbit of \( S_n \) on \( \Omega^{(3)} \).
Step 2. Orbits on $\Omega^{(2)}$ and corresponding 2-graphs. Denote the orbits of $S_n$ on $\Omega^{(2)}$ by $(0), (1), \ldots, (k - 1)$ where $\{A, B\} \in (r)$ if and only if $|A \cap B| = r$. The graphs on $\Omega$ on which $S_n$ acts are $\Gamma_\mathcal{C}$ (any subset of $\{0, 1, \ldots, k - 1\}$), where the edge-set of $\Gamma_\mathcal{C}$ is $\bigcup_{c \in \mathcal{C}}(c)$. Recall (see Section 2 of Cameron (1977)) that the 2-graph $\Delta_\mathcal{C}$ corresponding to the graph $\Gamma_\mathcal{C}$ is the set of triples of vertices containing an odd number of edges of $\Gamma_\mathcal{C}$. Thus $\Delta_\mathcal{C}$ is the union of all orbits $(r, s, t, u)$ of $S_n$ on $\Omega^{(3)}$ such that $|\mathcal{C} \cap \{r, s, t\}|$ is odd. Clearly any switching class on $\Omega$ in which $S_n$ fixes a graph corresponds to some $\Delta_\mathcal{C}$. So to complete the proof we must show that the $\Delta_\mathcal{C}$ are the only 2-graphs on $\Omega$ on which $S_n$ acts.

Thus let $S_n$ act on a 2-graph $\Delta \subset \Omega^{(3)}$. Put $\mathcal{C} = \{c \mid (c, 0, 0, 0) \subseteq \Delta\}$.

Step 3. We have $\Delta = \Delta_\mathcal{C}$. Let $\{A, B, C\} \in (r, s, t, u)$ where $rs \neq 0$. Then

$$|A \cup B \cup C| = 3k - (r + s + t) + u \leq 3k - (r + s) \leq 3k - 2.$$ 

Since $n \geq 4k - 2$ we may pick $D \in \Omega$ with $D$ disjoint from $A \cup B \cup C$. Consider the 4-set $\{A, B, C, D\}$. We have

$$\{A, B, C\} \in (r, s, t, u), \quad \{A, C, D\} \in (s, 0, 0, 0),$$

$$\{A, B, D\} \in (r, 0, 0, 0), \quad \{B, C, D\} \in (t, 0, 0, 0).$$

Since $\Delta$ is a 2-graph, an even number of these triples lies in $\Delta$. Hence $(r, s, t, u) \subseteq \Delta$ if and only if an odd number of $r, s, t$ lies in $\mathcal{C}$. Thus $\Delta = \Delta_\mathcal{C}$.

Corollary 4.2. If $n \geq 4k - 2$ and $n > 10$ then the action of $A_n$ on $\Sigma^{(k)}$ is always exposable.

Proof. This follows from the fact that $A_n$ is $3k$-transitive and hence has the same orbits on $\Omega^{(3)}$ (where $\Omega = \Sigma^{(k)}$) as $S_n$ (see Remark 2 below).

Remarks. 1. The restriction $n \geq 4k - 2$ was made entirely for convenience in the proof of Theorem 4.1 and can probably be relaxed considerably.

2. The following observations are elementary: let $G$ and $H$ be permutation groups on $\Sigma$ and suppose that $H$ is always exposable. Then $G$ is always exposable if either (i) $H \leq G$ and $G$ has the same orbits as $H$ on $\Sigma^{(2)}$, or (ii) $G \leq H$ and $G$ has the same orbits as $H$ on $\Omega^{(3)}$. 

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