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Corrigendum

Around ℓ-independence

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Abstract

We correct the proof of the main ℓ-independence result of the above-mentioned paper by showing that for any smooth and proper variety over an equicharacteristic local field, there exists a globally defined such variety with the same (p-adic and ℓ-adic) cohomology.

1. Introduction

It was pointed out to us by Zheng that the proof of [CL18, Theorem 6.1] is invalid. The problem is in the final step of the proof on p. 237, where we showed that there was an exact sequence

$$0 \to H^{i+n}_{\ell}(X) \to H^{i+n}_{\ell}(X_0) \to H^{i+n}_{\ell}(X_1) \to \cdots$$

and claimed to deduce ℓ-independence of $H^i_\ell(X)$ from ℓ-independence of all the other terms $H^{i+n}_\ell(X_n)$. Of course, this deduction does not work, since there might be infinitely many such other terms.

In their paper [LZ19], Lu and Zheng provide (amongst other things) an alternative proof of this ℓ-independence result, at least for $\ell \neq p$, see Theorem 1.4(2). In this corrigendum we will explain how to fix the proof of [CL18, Theorem 6.1] by instead proving a stronger version of [CL18, Corollary 5.5] where the semistable hypothesis is removed. In particular, this includes the case $\ell = p$.

Notation and conventions

We will use notation from [CL18] freely.

2. Log structures

We begin with a general result on semistable reduction and log schemes. Let R be a complete discrete valuation ring (DVR) with perfect residue field k, π a uniformiser for R, and let $\mathcal{X} \to \mathrm{Spec}(R)$ be a strictly semistable scheme. That is, $\mathcal{X}$ is Zariski locally étale over $R[x_1, \ldots, x_n]/(x_1 \cdots x_r - \pi)$ for some n, r. There is a natural log structure $\mathcal{M}_{\mathcal{X}}$ on $\mathcal{X}$ given by functions invertible outside the special fibre X, and we let $\mathcal{M}_X$ denote the pull-back of this log structure to X. We will also write X_i for the reduction of $\mathcal{X}$ modulo π^{i+1}, and $k^\times$ for k equipped with the log structure pulled back from the canonical log structure $R^\times$ on R.

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PROPOSITION 2.1 (Illusie, Nakayama [Nak98, Appendix A.4]). *If $\mathcal{X}, \mathcal{X}'$ are strictly semistable schemes over R, and $g : X_1 \to X_1'$ is an isomorphism between their mod π^2-reductions, then g induces a canonical isomorphism $g : (X, \mathcal{M}_X) \xrightarrow{\sim} (X', \mathcal{M}_{X'})$ of log schemes over $k^\times$.*

Sketch of proof. Use g to identify X_1 and X_1', and thus X and X'. Let $\mathcal{M}_X$ and $\mathcal{M}_X'$ be the log structures on X coming from $\mathcal{X}$ and $\mathcal{X}'$ respectively.

Near a closed point of X let $X^{(1)}, \dots, X^{(r)}$ be the irreducible components of X, and pick $x_1, \dots, x_r \in \mathcal{O}_{\mathcal{X}}$ such that $X^{(i)} = V(x_i)$. Similarly pick $x_1', \dots, x_r' \in \mathcal{O}_{\mathcal{X}'}$ such that $X^{(i)} = V(x_i')$. Let $v \in \mathcal{O}_{\mathcal{X}}^*$ and $v' \in \mathcal{O}_{\mathcal{X}'}^*$ be such that $x_1 \cdots x_r = v\pi$ and $x_1' \cdots x_r' = v'\pi$. Then in a neighbourhood of p the morphisms $(\mathcal{X}, \mathcal{M}_{\mathcal{X}}) \to \mathrm{Spec}(R^\times)$ and $(\mathcal{X}', \mathcal{M}_{\mathcal{X}'}) \to \mathrm{Spec}(R^\times)$ can be described by the following diagrams:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}^* \oplus \mathbb{N}^r & \xrightarrow{(u,a_1,\dots,a_r)\mapsto u x_1^{a_1}\cdots x_r^{a_r}} & \mathcal{O}_{\mathcal{X}} \\ {\scriptstyle (\lambda,a)\mapsto(\lambda v^{-a},a,\dots,a)}\uparrow & & \uparrow \\ R^* \oplus \mathbb{N} & \xrightarrow{(\lambda,a)\mapsto \lambda\pi^a} & R \end{array}$$

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{X}'} \cong \mathcal{O}_{\mathcal{X}'}^* \oplus \mathbb{N}^r & \xrightarrow{(u,a_1,\dots,a_r)\mapsto u x_1'^{a_1}\cdots x_r'^{a_r}} & \mathcal{O}_{\mathcal{X}'} \\ {\scriptstyle (\lambda,a)\mapsto(\lambda v'^{-a},a,\dots,a)}\uparrow & & \uparrow \\ R^* \oplus \mathbb{N} & \xrightarrow{(\lambda,a)\mapsto \lambda\pi^a} & R \end{array}$$

Pulling back to k, we see that the morphisms $(X, \mathcal{M}_X) \to \mathrm{Spec}(k^\times)$ and $(X, \mathcal{M}_X') \to \mathrm{Spec}(k^\times)$ can be described by the diagrams

$$\begin{array}{ccc} \mathcal{M}_X \cong \mathcal{O}_X^* \oplus \mathbb{N}^r & \xrightarrow{(u,a_1,\dots,a_r)\mapsto u x_1^{a_1}\cdots x_r^{a_r}} & \mathcal{O}_X \\ {\scriptstyle (\lambda,a)\mapsto(\lambda v^{-a},a,\dots,a)}\uparrow & & \uparrow \\ k^* \oplus \mathbb{N} & \xrightarrow{(\lambda,a)\mapsto \lambda 0^a} & k \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}_X' \cong \mathcal{O}_X^* \oplus \mathbb{N}^r & \xrightarrow{(u,a_1,\dots,a_r)\mapsto u x_1'^{a_1}\cdots x_r'^{a_r}} & \mathcal{O}_X \\ {\scriptstyle (\lambda,a)\mapsto(\lambda v'^{-a},a,\dots,a)}\uparrow & & \uparrow \\ k^* \oplus \mathbb{N} & \xrightarrow{(\lambda,a)\mapsto \lambda 0^a} & k \end{array}$$

respectively, again in a neighbourhood of p. Since $V(x_i) = V(x_i')$ inside X_1, we must have $x_i = u_i x_i'$ for some $u_i \in \mathcal{O}_{X_1}^*$, and so we can define an isomorphism

$$\mathcal{M}_X \xrightarrow{\sim} \mathcal{M}_X'$$

of log structures by mapping

$$(u, a_1, \dots, a_r) \mapsto (u u_1^{a_1} \cdots u_r^{a_r}, a_1, \dots, a_r).$$

This is checked to be a morphism of log structures over $k^\times$ by using the above local descriptions. Note that any other choice u_i' must satisfy $(u_i - u_i')x_i' = 0$ in $\mathcal{O}_{X_1}$, and hence we must have $u_i - u_i' \in (\pi)$. In particular, the above isomorphism does not depend on the choice of u_i. By a similar argument, neither does it depend on the choice of x_i and x_i', and so it glues to give a global isomorphism $(X, \mathcal{M}_X) \cong (X, \mathcal{M}_X')$ of log schemes over $k^\times$. □

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We will need to extend this result to cover morphisms between strictly semistable schemes over different bases. So suppose that $R \to S$ is a finite morphism of complete DVRs, with induced residue field extension $k \to k_S$. Let π_S be a uniformiser for S, and let $e = v_{\pi_S}(\pi)$. We do not assume that the induced extension $Q(R) \to Q(S)$ of fraction fields is separable.

Suppose that we have strictly semistable schemes $\mathcal{X}, \mathcal{X}'$ over R and $\mathcal{Y}, \mathcal{Y}'$ over S, and a pair of commutative diagrams

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array} \qquad \begin{array}{ccc} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

As before, let us write Y_j for the reduction of $\mathcal{Y}$ modulo π_S^{j+1}. Suppose that we have isomorphisms

$$g_Y : Y_e \xrightarrow{\sim} Y'_e, \quad g_X : X_1 \xrightarrow{\sim} X'_1$$

of S- and R-schemes respectively such that the diagram

$$\begin{array}{ccc} Y_e & \xrightarrow{f} & X_1 \\ {\scriptstyle g_Y}\downarrow & & \downarrow{\scriptstyle g_X} \\ Y'_e & \xrightarrow{f'} & X'_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

commutes. Then by Proposition 2.1 we obtain isomorphisms

$$g_Y : (Y, \mathcal{M}_Y) \xrightarrow{\sim} (Y', \mathcal{M}_{Y'})$$

of log schemes over $k_S^\times$, as well as

$$g_X : (X, \mathcal{M}_X) \xrightarrow{\sim} (X', \mathcal{M}_{X'})$$

of log schemes over $k^\times$. The above commutative diagrams of strictly semistable schemes induce commutative diagrams

$$\begin{array}{ccc} (Y, \mathcal{M}_Y) & \xrightarrow{f} & (X, \mathcal{M}_X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k^\times) \end{array} \qquad \begin{array}{ccc} (Y', \mathcal{M}_{Y'}) & \xrightarrow{f'} & (X, \mathcal{M}_{X'}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

of log schemes. Note that the morphism of punctured points along the bottom of each square is given by

$$\begin{aligned} k^* \oplus \mathbb{N} &\to k_S^* \oplus \mathbb{N} \\ (\lambda, a) &\mapsto (\lambda u^a, ea), \end{aligned}$$

where $u \in S^*$ is such that $\pi = u\pi_S^e$.

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PROPOSITION 2.2. *The diagram*

$$\begin{array}{ccc} (Y, \mathcal{M}_Y) & \xrightarrow{f} & (X, \mathcal{M}_X) \\ \cong \downarrow g_Y & & \cong \downarrow g_X \\ (Y', \mathcal{M}_{Y'}) & \xrightarrow{f'} & (X', \mathcal{M}_{X'}) \end{array}$$

of log schemes commutes.

Proof. Let us use g to identify $Y_e = Y'_e$ and $Y = Y'$, and let $\mathcal{M}_Y$ and $\mathcal{M}'_Y$ be the log structures on Y coming from $\mathcal{Y}$ and $\mathcal{Y}'$ respectively. Similarly identify $X_1 = X'_1$ and $X = X'$, and let $\mathcal{M}_X$ and $\mathcal{M}'_X$ be the log structures on X coming from $\mathcal{X}$ and $\mathcal{X}'$ respectively.

Locally on X and Y, choose functions $y_1, \ldots, y_s \in \mathcal{O}_{\mathcal{Y}}$, $y'_1, \ldots, y'_s \in \mathcal{O}_{\mathcal{Y}'}$ cutting out the irreducible components of Y, and functions $x_1, \ldots, x_r \in \mathcal{O}_{\mathcal{X}}$ and $x'_1, \ldots, x'_r \in \mathcal{O}_{\mathcal{X}'}$ cutting out the irreducible components of X. Write

$$f^*(x_i) = \alpha_i y_1^{d_{i1}} \cdots y_s^{d_{is}}, \quad f'^*(x'_i) = \alpha'_i {y'_1}^{d'_{i1}} \cdots {y'_s}^{d'_{is}},$$

since both d_{ij} and d'_{ij} are given by the multiplicity of the jth irreducible component of Y in the scheme theoretic preimage of the ith irreducible component of X inside Y_e, we must have $d_{ij} = d'_{ij}$. Moreover, since $V(f^*(x_i)) \subset V(\pi_S^e) = V(y_1^e \cdots y_s^e)$ we must have $d_{ij} \leqslant e$ for all i, j.

Now choose $u_i \in \mathcal{O}^*_{X_1}$ such that $x_i = u_i x'_i$, and $v_j \in \mathcal{O}^*_{Y_e}$ such that $y_j = v_j y'_j$. Then the isomorphisms of log structures induced by g_Y and g_X are given by

$$\begin{aligned} \mathcal{M}_Y = \mathcal{O}^*_Y \oplus \mathbb{N}^s &\to \mathcal{M}'_Y = \mathcal{O}^*_Y \oplus \mathbb{N}^s \\ (v, b_1, \ldots, b_s) &\mapsto (v v_1^{b_1} \cdots v_s^{b_s}, b_1, \ldots, b_s) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_X = \mathcal{O}^*_X \oplus \mathbb{N}^r &\to \mathcal{M}'_X = \mathcal{O}^*_X \oplus \mathbb{N}^r \\ (u, a_1, \ldots, a_r) &\mapsto (u u_1^{a_1} \cdots u_r^{a_r}, a_1, \ldots, a_r) \end{aligned}$$

respectively, and the morphisms $\mathcal{M}_X \to \mathcal{M}_Y$ and $\mathcal{M}'_X \to \mathcal{M}'_Y$ are defined by

$$(u, a_1, \ldots, a_r) \mapsto \left(f^*(u) \alpha_1^{a_1} \cdots \alpha_r^{a_r}, \sum_i d_{i1} a_i, \ldots, \sum_i d_{is} a_i \right)$$

and

$$(u, a_1, \ldots, a_r) \mapsto \left(f^*(u) {\alpha'_1}^{a_1} \cdots {\alpha'_r}^{a_r}, \sum_i d_{i1} a_i, \ldots, \sum_i d_{is} a_i \right)$$

respectively. Hence in the diagram

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{g_X} & \mathcal{M}'_X \\ f \downarrow & & \downarrow f \\ \mathcal{M}_Y & \xrightarrow{g_Y} & \mathcal{M}'_Y \end{array}$$

the composite $f \circ g_X$ is given by

$$(u, a_1, \ldots, a_r) \mapsto \left(f^*(u) (\alpha'_1 f^*(u_1))^{a_1} \cdots (\alpha'_r f^*(u_r))^{a_r}, \sum_i d_{i1} a_i, \ldots, \sum_i d_{is} a_i \right)$$

and the composite $g_Y \circ f$ is given by

$$(u, a_1, \ldots, a_r) \mapsto \left(f^*(u)(\alpha_1 v_1^{d_{11}} \cdots v_s^{d_{1s}})^{a_1} \cdots (\alpha_r v_1^{d_{r1}} \cdots v_s^{d_{rs}})^{a_r}, \sum_i d_{i1} a_i, \ldots, \sum_i d_{is} a_i \right).$$

We thus need to show that $\alpha_i' f^*(u_i) = \alpha_i v_1^{d_{i1}} \cdots v_s^{d_{is}}$ in $\mathcal{O}_Y^*$ for all i. But now we write

$$\alpha_i y_1^{d_{i1}} \cdots y_s^{d_{is}} = f^*(x_i) = f^*(u_i x_i') = f^*(u_i) \alpha_i' {y_1'}^{d_{i1}} \cdots {y_s'}^{d_{is}}$$

in $\mathcal{O}_{Y_e}$ and so deduce that

$$\alpha_i v_1^{d_{i1}} \cdots v_s^{d_{is}} {y_1'}^{d_{i1}} \cdots {y_s'}^{d_{is}} = f^*(u_i) \alpha_i' {y_1'}^{d_{i1}} \cdots {y_s'}^{d_{is}}.$$

We deduce that the difference $\beta_i = \alpha_i' f^*(u_i) - \alpha_i v_1^{d_{i1}} \cdots v_s^{d_{is}}$ annihilates ${y_1'}^{d_{i1}} \cdots {y_s'}^{d_{is}}$ inside $\mathcal{O}_{Y_e}$, and since each $d_{ij} \leqslant e$ we deduce that in fact β_i annihilates π_S^e, and therefore must lie in (π_S). Hence $\beta_i = 0$ in $\mathcal{O}_Y$ and the proof is complete. □

3. Functoriality of comparison isomorphisms

We will also need to know that the comparison isomorphisms [CL18, Propositions 5.3, 5.4] are compatible with morphisms of semistable schemes over different bases. So let us suppose that we are again in the above set-up, where we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

of strictly semistable schemes $\mathcal{Y}$ and $\mathcal{X}$ over S and R respectively, with S the integral closure of R in some finite extension of its fraction field. Let us assume that R, and hence S, is of equicharacteristic $p > 0$, with fraction fields F and F_S respectively, whose absolute Galois groups we will denote by G_F and G_{F_S}. Fix an embedding $F^{\mathrm{sep}} \hookrightarrow F_S^{\mathrm{sep}}$ of separable closures; note that this sends F^{tame} into F_S^{tame} and induces an injective homomorphism $G_{F_S} \to G_F$ with finite cokernel.

Let $\mathcal{X}^\times$ and $\mathcal{Y}^\times$ denote these semistable schemes endowed with their canonical log structures, and $X^\times$ and $Y^\times$ the corresponding log special fibres. We therefore have a commutative diagram

$$\begin{array}{ccc} Y^\times & \longrightarrow & X^\times \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k^\times) \end{array}$$

of log schemes. For every finite subextension $F \subset L \subset F^{\mathrm{tame}}$, let $X_L^\times$ denote the corresponding base change of $X^\times$, and $X^{\times,\mathrm{tame}}$ the inverse limit of the étale topoi of all such $X_L^\times$; we have $Y^{\times,\mathrm{tame}}$ defined entirely similarly. Via the embedding $F^{\mathrm{tame}} \hookrightarrow F_S^{\mathrm{tame}}$ this induces a G_{F_S}-equivariant morphism of topoi

$$Y^{\times,\mathrm{tame}} \to X^{\times,\mathrm{tame}}$$

and hence a G_{F_S}-equivariant morphism

$$H^i_{\text{ét}}(X^{\times,\text{tame}},\mathbb{Q}_\ell)\to H^i_{\text{ét}}(Y^{\times,\text{tame}},\mathbb{Q}_\ell)$$

in cohomology, for any $\ell\neq p$. On the other hand we have a natural G_{F_S}-equivariant map

$$H^i_{\text{ét}}(\mathcal{X}\times_R F^{\text{sep}},\mathbb{Q}_\ell)\to H^i_{\text{ét}}(\mathcal{Y}\times_S F_S^{\text{sep}},\mathbb{Q}_\ell),$$

and by [Nak98, Proposition 4.2] equivariant isomorphisms

$$\begin{aligned}H^i_{\text{ét}}(X^{\times,\text{tame}},\mathbb{Q}_\ell)&\xrightarrow{\sim} H^i_{\text{ét}}(\mathcal{X}\times_R F^{\text{sep}},\mathbb{Q}_\ell),\\ H^i_{\text{ét}}(Y^{\times,\text{tame}},\mathbb{Q}_\ell)&\xrightarrow{\sim} H^i_{\text{ét}}(\mathcal{Y}\times_S F_S^{\text{sep}},\mathbb{Q}_\ell).\end{aligned}$$

PROPOSITION 3.1. *The diagram*

$$\begin{array}{ccc}H^i_{\text{ét}}(X^{\times,\text{tame}},\mathbb{Q}_\ell) & \longrightarrow & H^i_{\text{ét}}(\mathcal{X}\times_R F^{\text{sep}},\mathbb{Q}_\ell)\\ \downarrow & & \downarrow\\ H^i_{\text{ét}}(Y^{\times,\text{tame}},\mathbb{Q}_\ell) & \longrightarrow & H^i_{\text{ét}}(\mathcal{Y}\times_S F_S^{\text{sep}},\mathbb{Q}_\ell)\end{array}$$

commutes.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}Y^{\times,\text{tame}} & \xrightarrow{i_Y} & \mathcal{Y}^{\times,\text{tame}} & \xleftarrow{j_{\mathcal{Y}}} & \mathcal{Y}_{F_S^{\text{sep}}}\\ \downarrow f & & \downarrow f & & \downarrow f\\ X^{\times,\text{tame}} & \xrightarrow{i_X} & \mathcal{X}^{\times,\text{tame}} & \xleftarrow{j_{\mathcal{X}}} & \mathcal{X}_{F^{\text{sep}}}\end{array}$$

of topoi as in [Nak98, § 3], where $\mathcal{Y}^{\times,\text{tame}}$ and $\mathcal{X}^{\times,\text{tame}}$ are defined by 'base change' along $F_S\to F_S^{\text{tame}}$ and $F\to F^{\text{tame}}$ respectively. Then the isomorphism

$$H^i_{\text{ét}}(Y^{\times,\text{tame}},\mathbb{Q}_\ell)\xrightarrow{\sim} H^i_{\text{ét}}(\mathcal{Y}\times_S F_S^{\text{sep}},\mathbb{Q}_\ell)$$

is given as the composite

$$H^i_{\text{ét}}(Y^{\times,\text{tame}},\mathbb{Q}_\ell)\xleftarrow{\sim} H^i_{\text{ét}}(\mathcal{Y}^{\times,\text{tame}},\mathbb{Q}_\ell)\to H^i_{\text{ét}}(\mathcal{Y}\times_S F_S^{\text{sep}},\mathbb{Q}_\ell)$$

using the proper base change theorem in log-étale cohomology [Nak97, Theorem 5.1], and there is a similar statement for $\mathcal{X}$. The claim then follows simply from commutativity of the above diagram of log schemes. □

We will also need a version of this result for p-adic cohomology. Write $W=W(k)$, $W_S=W(k_S)$, let $K=W[1/p]$, $K_S=W_S[1/p]$, and let $\mathcal{R}_K\supset\mathcal{E}_K^\dagger\subset\mathcal{E}_K$, and $\mathcal{R}_{K_S}\supset\mathcal{E}_{K_S}^\dagger\subset\mathcal{E}_{K_S}$ denote copies of the Robba ring, the bounded Robba ring and the Amice ring over K and K_S respectively. Lift the extension $F\to F_S$ to a finite flat morphism $\mathcal{E}_K^\dagger\to\mathcal{E}_{K_S}^\dagger$ which extends to finite flat morphisms $\mathcal{R}_K\to\mathcal{R}_{K_S}$ and $\mathcal{E}_K\to\mathcal{E}_{K_S}$. Then, as above, the morphism of log schemes $Y^\times\to X^\times$ induces a morphism

$$H^i_{\text{log-cris}}(X^\times/K^\times)\to H^i_{\text{log-cris}}(Y^\times/K_S^\times)$$

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in log crystalline cohomology, and the morphism $\mathcal{Y}_{F_S} \to \mathcal{X}_F$ induces a morphism

$$H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K) \to H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$$

in Robba-ring valued rigid cohomology. Then following [CL18, Proposition 5.4] we can construct isomorphisms

$$H^i_{\text{log-cris}}(X^\times/K^\times) \otimes_K \mathcal{R}_K \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K),$$
$$H^i_{\text{log-cris}}(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$$

as follows. Let t denote a co-ordinate on $\mathcal{E}_K^\dagger$ and t_S a co-ordinate on $\mathcal{E}_{K_S}^\dagger$ such that $t \in W_S[\![t_S]\!]$. Write $S_K = K \otimes W[\![t]\!]$ and $S_{K_S} = K_S \otimes W_S[\![t_S]\!]$. Equip $W[\![t]\!]$ (respectively $W_S[\![t_S]\!]$) with the log structure defined by the ideal $(t) \subset W[\![t]\!]$ (respectively $(t_S) \subset W[\![t_S]\!]$) and define the log-crystalline cohomology groups

$$H^i_{\text{log-cris}}(\mathcal{X}^\times/S_K) := H^i_{\text{log-cris}}(\mathcal{X}^\times/W[\![t]\!]) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$H^i_{\text{log-cris}}(\mathcal{Y}^\times/S_{K_S}) := H^i_{\text{log-cris}}(\mathcal{Y}^\times/W_S[\![t_S]\!]) \otimes_{\mathbb{Z}} \mathbb{Q};$$

these are naturally endowed with the extra structure of log-(φ, ∇)-modules over S_K and S_{K_S} respectively. Moreover, we have isomorphisms of φ-modules

$$H^i_{\text{log-cris}}(\mathcal{X}^\times/S_K) \otimes_{S_K, t \mapsto 0} K \xrightarrow{\sim} H^i_{\text{log-cris}}(Y^\times/K_S^\times),$$
$$H^i_{\text{log-cris}}(\mathcal{Y}^\times/S_{K_S}) \otimes_{S_{K_S}, t_S \mapsto 0} K_S \xrightarrow{\sim} H^i_{\text{log-cris}}(Y^\times/K_S^\times),$$

by smooth and proper base change in log-crystalline cohomology, as well as isomorphisms of (φ, ∇)-modules

$$H^i_{\text{log-cris}}(\mathcal{X}^\times/S_K) \otimes_{S_K} \mathcal{R}_K \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K),$$
$$H^i_{\text{log-cris}}(\mathcal{Y}^\times/S_{K_S}) \otimes_{S_{K_S}} \mathcal{R}_{K_S} \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}),$$

by [LP16, Proposition 5.45]. It therefore follows from the logarithmic form of Dwork's trick [Ked10, Corollary 17.2.4] that the (φ, ∇)-modules $H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K)$ and $H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$ are unipotent, that there are isomorphisms

$$\left(H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K)[\log t]\right)^{\nabla=0} \cong H^i_{\text{log-cris}}(X^\times/K^\times),$$
$$\left(H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})[\log t_S]\right)^{\nabla=0} \cong H^i_{\text{log-cris}}(Y^\times/K_S^\times)$$

and moreover the connection ∇ on the rigid cohomology groups appearing on the left-hand side can be completely recovered from the monodromy operator N on the right-hand side. This allows us to construct isomorphisms of (φ, ∇)-modules

$$H^i_{\text{log-cris}}(X^\times/K^\times) \otimes_K \mathcal{R}_K \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K),$$
$$H^i_{\text{log-cris}}(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} \xrightarrow{\sim} H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$$

where the left-hand side is endowed a natural connection coming from N; for more details see, for example, [Mar08, § 3.2].

Proposition 3.2. *The diagram*

$$\begin{array}{ccc} H^i_{\text{log-cris}}(X^\times/K^\times) \otimes_K \mathcal{R}_K & \longrightarrow & H^i_{\mathrm{rig}}(\mathcal{X}_F/\mathcal{R}_K) \\ \downarrow & & \downarrow \\ H^i_{\text{log-cris}}(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} & \longrightarrow & H^i_{\mathrm{rig}}(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}) \end{array}$$

commutes.

Proof. Given the construction of the horizontal isomorphisms outlined above, it suffices to show that the diagram

$$\begin{array}{ccc} H^i_{\text{log-cris}}(\mathcal{X}^\times/S_K) & \longrightarrow & H^i_{\text{log-cris}}(X^\times/K^\times) \\ \downarrow & & \downarrow \\ H^i_{\text{log-cris}}(\mathcal{Y}^\times/S_{K_S}) & \longrightarrow & H^i_{\text{log-cris}}(Y^\times/K_S^\times) \end{array}$$

of log-crystalline cohomology groups commutes, which as in Proposition 3.1 simply follows from functoriality of log-crystalline cohomology. □

4. Cohomology and global approximations

Now suppose that k is a finite field, $F = k(\!(t)\!)$, and X/F is a smooth and proper variety.

DEFINITION 4.1. We say that X is globally defined if there exist a smooth curve C/k, a k-valued point $c \in C(k)$, a smooth and proper morphism $\mathbf{X} \to (C \setminus \{c\})$ and an isomorphism $F \cong \widehat{k(C)}_c$ such that $\mathbf{X}_F \cong X$.

We will prove the following strengthened version of [CL18, Corollary 5.5].

THEOREM 4.2. *For any smooth and proper variety X/F there exists a globally defined smooth and proper variety Z/F such that*

$$H^i_\ell(X) \cong H^i_\ell(Z)$$

for all ℓ (including $\ell = p$).

Once we have shown this, the proof of [CL18, Theorem 6.1] can then be completed using [CL18, Proposition 5.8], exactly as in the proof of [CL18, Theorem 5.1].

To prove Theorem 4.2, first of all choose a proper and flat model $\mathcal{X}$ for X over the ring of integers $\mathcal{O}_F$. By [dJ96, Theorem 6.5] we may choose an alteration $\mathcal{X}_0 \to \mathcal{X}$ and a finite extension F_0/F such that $\mathcal{X}_0$ is strictly semistable over $\mathcal{O}_{F_0}$.

Next, we take the fibre product $\mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$, and let $\mathcal{X}_1'$ denote the disjoint union of the reduced, irreducible components of $\mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$ which are flat over $\mathcal{O}_{F_0}$, or equivalently which map surjectively to $\mathrm{Spec}(\mathcal{O}_{F_0})$. Once more applying [dJ96, Theorem 6.5] to each of the connected components of $\mathcal{X}_1'$ in turn enables us to produce:

– a 2-truncated augmented simplicial scheme

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \to \mathcal{X}$$

which is a proper hypercover after base changing to F;

– a collection $F_{1,1}, \ldots, F_{1,s}$ of finite field extensions of F_0

such that $\mathcal{X}_1$ is a disjoint union of schemes $\mathcal{X}_{1,j}$, for $1 \leqslant j \leqslant s$, proper and strictly semistable over $\mathrm{Spec}(\mathcal{O}_{F_{1,j}})$.

Let k_0 denote the residue field of F_0, $k_{1,j}$ the residue field of $F_{1,j}$, and consider the intermediate extensions

$$F \subset F_0^{\mathrm{un}} \subset F_0^s \subset F_0 \subset F_{1,j}^{\mathrm{un}} \subset F_{1,j}^s \subset F_{1,j},$$

where F_0^{un}/F and $F_{1,j}^{\mathrm{un}}/F_0$ are separable and unramified, F_0^s/F_0^{un} and $F_{1,j}^s/F_{1,j}^{\mathrm{un}}$ are separable and totally ramified, and F_0/F_0^s and $F_{1,j}/F_{1,j}^s$ are totally inseparable, of degree p^{d_0} and $p^{d_{1,j}}$ respectively. Let t denote a uniformiser for F, t_0 one for F_0^s, and let P_0 be the minimal polynomial of t_0 over F_0^{un}. Then $t_0' := t_0^{1/p^{d_0}}$ is a uniformiser for $\mathcal{O}_{F_0}$. Similarly, let $t_{1,j}$ be a uniformiser for $F_{1,j}^s$, and $P_{1,j}$ the minimal polynomial of $t_{1,j}$ over $F_{1,j}^{\mathrm{un}}$. Then $t_{1,j}' := t_{1,j}^{1/p^{d_{1,j}}}$ is a uniformiser for $\mathcal{O}_{F_{1,j}}$.

Now choose a finitely generated sub-k-algebra $R \subset \mathcal{O}_F$, containing t, such that there exists a proper, flat scheme $\mathcal{Y} \to \mathrm{Spec}(R)$ whose base change to $\mathcal{O}_F$ is exactly $\mathcal{X}$. By [Spi99, Theorem 10.1], we may at any point increase R to ensure that it is in fact smooth over k. Next, enlarge R so that $R_0^{\mathrm{un}} := R \otimes_k k_0 \subset \mathcal{O}_{F_0^{\mathrm{un}}}$ contains all the coefficients of the minimal (Eisenstein) polynomial P_0 of t_0, and let R_0^s denote the corresponding finite flat extension $R_0^{\mathrm{un}}[x]/(P_0)$ of R_0^{un}. We can thus consider $R_0^s \subset \mathcal{O}_{F_0^s}$ as a subring containing t_0, and we set $R_0 = R_0^s[t_0']$. Hence we have $R_0 \subset \mathcal{O}_{F_0}$ such that

$$R_0 \otimes_R \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_0}.$$

Note also that R_0 is finite and flat over R; after localising R within $\mathcal{O}_F$ we may in fact assume that R_0 is finite free over R.

Next we enlarge R so that there exists a proper and flat morphism $\mathcal{Y}_0 \to \mathrm{Spec}(R_0)$ whose base change to $\mathcal{O}_{F_0}$ is $\mathcal{X}_0$. Again, by further enlarging R we may in addition assume that the map $\mathcal{X}_0 \to \mathcal{X}$ arises from a proper surjective map

$$\mathcal{Y}_0 \to \mathcal{Y}$$

of R-schemes, and moreover that there exists an open cover of $\mathcal{Y}_0$ by schemes which are étale over $R_0[x_1, \ldots, x_n]/(x_1 \cdots x_r - t_0')$ for some n, r. In other words, $\mathcal{Y}_0$ is 'strictly t_0'-semistable'.

We now repeat this process to produce further finite free extensions $R_0 \to R_{1,j}^{\mathrm{un}} \to R_{1,j}^s \to R_{1,j}$ for all j, and an injection $R_{1,j} \subset \mathcal{O}_{F_{1,j}}$ containing the image of $t_{1,j}'$ such that

$$R_{1,j} \otimes_R \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_{1,j}}.$$

We can also find proper, strictly $t_{1,j}'$-semistable schemes $\mathcal{Y}_{1,j} \to \mathrm{Spec}(R_{1,j})$ whose base change to $\mathcal{O}_{F_{1,j}}$ is $\mathcal{X}_{1,j}$, so that setting $\mathcal{Y}_1 := \coprod_j \mathcal{Y}_{1,j}$ (and again, possibly increasing R), we obtain a 2-truncated augmented simplicial scheme

$$\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0 \to \mathcal{Y}$$

which becomes a proper hypercover over a dense open subscheme of $\mathrm{Spec}(R)$, and whose base change to $\mathcal{O}_F$ is exactly our original 2-truncated augmented simplicial scheme

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \to \mathcal{X}.$$

Let $\iota : R \hookrightarrow \mathcal{O}_F$ denote the canonical inclusion, and $\iota^* : \mathrm{Spec}(\mathcal{O}_F) \to \mathrm{Spec}(R)$ the induced morphism of schemes. Note that since ι^* maps the generic point of $\mathrm{Spec}(\mathcal{O}_F)$ to that of $\mathrm{Spec}(R)$, the map $\mathcal{Y} \to \mathrm{Spec}(R)$ is generically smooth. We may thus choose an open subset $U \subset \mathrm{Spec}(R)$ such that $\mathcal{Y}_U \to U$ is smooth, and such that the base change of $[\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0 \to \mathcal{Y}]$ to U is a proper hypercover.

Lemma 4.3. *For any $n \geqslant 0$ there exists a smooth curve C/k, a rational point $c \in C(k)$, a uniformiser t_c at c, and a locally closed immersion $C \to \mathrm{Spec}(R)$ such that $C \setminus \{c\} \subset U$, and the induced map*

$$\mathrm{Spec}(\mathcal{O}_{C,c}/\mathfrak{m}_c^n) \to \mathrm{Spec}(R)$$

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agrees with the modulo t^n-reduction of ι^* via the isomorphism

$$\widehat{\mathcal{O}}_{C,c} \xrightarrow{\sim} \mathcal{O}_F$$

sending t_c to t.

Proof. Since R is smooth, we may choose étale co-ordinates around the image $\iota^*(s)$ of the closed point of $\mathrm{Spec}(\mathcal{O}_F)$ under ι^*. This induces an étale map $\mathrm{Spec}(R) \to \mathbb{A}^n_k$ for some n, and it is a simple exercise to prove the corresponding claim for $\mathbb{A}^n_k$. We then just take the pull-back to $\mathrm{Spec}(R)$. □

The canonical inclusion ι induces similar inclusions

$$\iota_0^{\#} : R_0^{\#} \hookrightarrow R_0^{\#} \otimes_R \mathcal{O}_F = \mathcal{O}_{F_0^{\#}}$$

for $\# \in \{\mathrm{un}, s, \emptyset\}$, as well as

$$\iota_{1,j}^{\#} : R_{1,j}^{\#} \hookrightarrow R_{1,j}^{\#} \otimes_R \mathcal{O}_F = \mathcal{O}_{F_{1,j}^{\#}}$$

for all j, and again for $\# \in \{\mathrm{un}, s, \emptyset\}$. We will need the following form of Krasner's lemma [Sta18, § 0BU9].

LEMMA 4.4. *Let K be a local field, with ring of integers $\mathcal{O}_K$, and let $P(x)$ be an Eisenstein polynomial over $\mathcal{O}_K$. Let L be the corresponding finite totally ramified extension, and let α be a root of P in L. Then for any $m \geqslant 1$ there exists an $n \geqslant 2$ such that any $Q(x) \in \mathcal{O}_K[x]$ congruent to P modulo $\mathfrak{m}_K^n$ is Eisenstein, and L contains a root β of Q such that $L = K(\beta)$ and $\alpha \equiv \beta$ modulo $\mathfrak{m}_L^m$.*

We will use this as follows: given $n_1 \geqslant \max_j\{[F_{1,j} : F]\}$ Lemma 4.4 shows that there exists some $n_0 \geqslant \max\{2, [F_0 : F]\}$ such that any polynomial $Q_{1,j}$ with coefficients in $\mathcal{O}_{F_{1,j}^{\mathrm{un}}}$ which agrees with the minimal polynomial $P_{1,j}$ of $t_{1,j}$ modulo $(t_0')^{n_0}$ is Eisenstein, and has a root in $\mathcal{O}_{F_{1,j}^s}$ which agrees with $t_{1,j}$ modulo $t_{1,j}^{n_1}$. Applying the lemma again shows the existence of some $n \geqslant 2$ such that any polynomial Q_0 with coefficients in $\mathcal{O}_{F_0^{\mathrm{un}}}$ which agrees with P_0 modulo t^n is Eisenstein, and has a root in $\mathcal{O}_{F_0^s}$ which agrees with t_0 modulo $t_0^{n_0}$. Now choose a k-algebra homomorphism $\lambda : R \to \mathcal{O}_F$ as provided by Lemma 4.3, that is, factoring through the local ring of some smooth point on a curve inside $\mathrm{Spec}(R)$ and agreeing with ι modulo t^n.

Since λ is a k-algebra homomorphism, we have a canonical isomorphism $R_0^{\mathrm{un}} \otimes_{R,\lambda} \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_0^{\mathrm{un}}}$, which therefore induces a homomorphism

$$\lambda_0^{\mathrm{un}} : R_0^{\mathrm{un}} \to \mathcal{O}_{F_0^{\mathrm{un}}}$$

extending λ and which agrees with ι_0^{un} modulo t^n. Now let $Q_0 = \lambda_0^{\mathrm{un}}(P_0)$ denote the image under λ_0^{un} of the minimal polynomial P_0 of t_0; this is therefore a monic polynomial with coefficients in $\mathcal{O}_{F_0^{\mathrm{un}}}$, which agrees with P_0 modulo t^n. Thus it is also Eisenstein, and by the choice of n we know that $\mathcal{O}_{F_0^s}$ contains a root of $\lambda_0^{\mathrm{un}}(P_0)$ which is congruent to t_0 modulo $t_0^{n_0}$ and generates $\mathcal{O}_{F_0^s}$ as an $\mathcal{O}_{F_0^{\mathrm{un}}}$-algebra. This then allows us to extend λ_0^{un} to a homomorphism

$$\lambda_0^s : R_0^s \to \mathcal{O}_{F_0^s}$$

which agrees with ι_0^s modulo $t_0^{n_0}$, and since $\lambda_0^s(t_0)$ generates $\mathcal{O}_{F_0^s}$ as an $\mathcal{O}_{F_0^{\mathrm{un}}}$-algebra, we deduce that the diagram

$$\begin{array}{ccc} R_0^s & \xrightarrow{\lambda_0^s} & \mathcal{O}_{F_0^s} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\lambda} & \mathcal{O}_F \end{array}$$

is coCartesian. We can then extend this to a homomorphism

$$\lambda_0 : R_0 \to \mathcal{O}_{F_0}$$

agreeing with ι_0 modulo $(t_0')^{n_0}$, and forming a similar coCartesian diagram to λ_0^s. We now play exactly the same game for all of the $R_{1,j}$, to produce $\lambda_{1,j} : R_{1,j} \to \mathcal{O}_{F_{1,j}}$ extending all other λ_0 and all previous $\lambda_{1,j}^{\#}$, which agree with $\iota_{1,j}$ modulo $(t_{1,j}')^{n_{1,j}}$, and which form coCartesian diagrams

$$\begin{array}{ccc} R_{1,j} & \xrightarrow{\lambda_{1,j}} & \mathcal{O}_{F_{1,j}} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\lambda} & \mathcal{O}_F \end{array}$$

Now let $\mathcal{Z}$ be the base change of $\mathcal{Y}$ to $\mathcal{O}_F$ via λ; note that the generic fibre $\mathcal{Z}_F$ is globally defined by construction. Similarly define $\mathcal{Z}_0$ to be the base change of $\mathcal{Y}_0$ to $\mathcal{O}_{F_0}$ via λ_0, $\mathcal{Z}_{1,j}$ the base change of $\mathcal{Y}_{1,j}$ to $\mathcal{O}_{F_{1,j}}$ via $\lambda_{1,j}$, and $\mathcal{Z}_1 := \coprod_j \mathcal{Z}_{1,j}$, so we have a 2-truncated augmented simplicial scheme

$$\mathcal{Z}_1 \rightrightarrows \mathcal{Z}_0 \to \mathcal{Z}$$

over $\mathcal{O}_F$, which gives a proper hypercover after base changing to F. For any $m \geqslant 2$ we can therefore take $n_1 \geqslant m \max_j\{[F_{1,j} : F]\}$ to ensure:

- $\mathcal{Z}_0$ is a proper and strictly semistable scheme over $\mathcal{O}_{F_0}$, and each $\mathcal{Z}_{1,j}$ is a proper and strictly semistable scheme over $\mathcal{O}_{F_{1,j}}$;
- there is an isomorphism

$$[\mathcal{X}_1 \rightrightarrows \mathcal{X}_0] \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m \overset{\sim}{\to} [\mathcal{Z}_1 \rightrightarrows \mathcal{Z}_0] \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m$$

  of 2-truncated simplicial schemes, such that

$$\mathcal{X}_0 \otimes \mathcal{O}_F/t^m \overset{\sim}{\to} \mathcal{Z}_0 \otimes \mathcal{O}_F/t^m$$

  is in fact an isomorphism of $\mathcal{O}_{F_0}/(t^m)$-schemes, and

$$\mathcal{X}_1 \otimes \mathcal{O}_F/t^m \overset{\sim}{\to} \mathcal{Z}_1 \otimes \mathcal{O}_F/t^m$$

  is obtained as a disjoint union of isomorphisms

$$\mathcal{X}_{1,j} \otimes \mathcal{O}_F/t^m \overset{\sim}{\to} \mathcal{Z}_{1,j} \otimes \mathcal{O}_F/t^m$$

  of $\mathcal{O}_{F_{1,j}}/(t^m)$-schemes.

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Thus if we let $\mathcal{X}_{0,s}^{\times}$ and $\mathcal{Z}_{0,s}^{\times}$ denote the log schemes over $k_0^{\times}$ given by the special fibres of $\mathcal{X}_0$ and $\mathcal{Z}_0$, and $\mathcal{X}_{1,s}^{\times}$ and $\mathcal{Z}_{1,s}^{\times}$ the log schemes over $\coprod_{j=1}^{s}\operatorname{Spec}(k_{1,j}^{\times})$ given by the special fibres of $\mathcal{X}_1$ and $\mathcal{Z}_1$, then by Proposition 2.2 there is an isomorphism

$$[\mathcal{Z}_{1,s}^{\times} \rightrightarrows \mathcal{Z}_{0,s}^{\times}] \cong [\mathcal{X}_{1,s}^{\times} \rightrightarrows \mathcal{X}_{0,s}^{\times}]$$

of 2-truncated simplicial log schemes over $k^{\times}$. Now by [CL18, Propositions 5.3, 5.4] there are isomorphisms

$$H_{\ell}^{i}(\mathcal{X}_{0,F_0}) \cong H_{\ell}^{i}(\mathcal{Z}_{0,F_0}),$$
$$H_{\ell}^{i}(\mathcal{X}_{1,F_{1,j}}) \cong H_{\ell}^{i}(\mathcal{Z}_{1,j,F_{1,j}})$$

between the cohomology of the generic fibres of $\mathcal{X}_0, \mathcal{X}_{1,j}$ and $\mathcal{Z}_0, \mathcal{Z}_{1,j}$, as Weil–Deligne representations over F_0 and $F_{1,j}$ respectively. If we define the category

$$\operatorname{Rep}_{\mathbb{Q}_\ell'}(\mathrm{WD}_{F_1}) := \prod_{j=1}^{s} \operatorname{Rep}_{\mathbb{Q}_\ell'}(\mathrm{WD}_{F_{1,j}})$$

of Weil–Deligne representations over $F_1 := \prod_j F_{1,j}$ to be the product of the categories of Weil–Deligne representations over each $F_{1,j}$, then by Propositions 3.1 and 3.2, the diagram

$$\begin{array}{ccc} H_{\ell}^{i}(\mathcal{X}_{0,F_0}) & \longrightarrow & H_{\ell}^{i}(\mathcal{X}_{1,F_1}) \\ \cong\downarrow & & \downarrow\cong \\ H_{\ell}^{i}(\mathcal{Z}_{0,F_0}) & \longrightarrow & H_{\ell}^{i}(\mathcal{Z}_{1,F_1}) \end{array}$$

(with horizontal arrows given by the differences of the two pullback maps) commutes via the restriction functor from Weil–Deligne representations over F_0 to Weil–Deligne representations over F_1.

Let $\operatorname{Ind}_{F_i}^{F}$ denote a right adjoint to the restriction functor from Weil–Deligne representations over F to those over F_i: on the separable part this is the normal induction of representations, on the inseparable part it is a quasi-inverse to Frobenius pull-back, and $\operatorname{Ind}_{F_1}^{F} = \bigoplus_j \operatorname{Ind}_{F_{1,j}}^{F}$. We therefore have a commutative diagram

$$\begin{array}{ccc} \operatorname{Ind}_{F_0}^{F} H_{\ell}^{i}(\mathcal{X}_{0,F_0}) & \longrightarrow & \operatorname{Ind}_{F_1}^{F} H_{\ell}^{i}(\mathcal{X}_{1,F_1}) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{Ind}_{F_0}^{F} H_{\ell}^{i}(\mathcal{Z}_{0,F_0}) & \longrightarrow & \operatorname{Ind}_{F_1}^{F} H_{\ell}^{i}(\mathcal{Z}_{1,F_1}) \end{array}$$

and, in particular, the kernels of both horizontal maps are isomorphic as Weil–Deligne representations over F. The proof of Theorem 4.2 now boils down to the following claim.

Proposition 4.5. *Let $X_1 \rightrightarrows X_0 \to X$ be a 2-truncated semisimplicial proper hypercover of a smooth and proper F-variety X, such that there exist finite field extensions F_0/F and $F_{1,j}/F_0$ for $1 \leqslant j \leqslant s$, with X_0 smooth over F_0, and $X_1 = \coprod_j X_{1,j}$ with $X_{1,j}$ smooth over $F_{1,j}$. If we set $F_1 = \prod_{j=1}^{s} F_{1,j}$, then*

$$H_{\ell}^{i}(X) \cong \ker(\operatorname{Ind}_{F_0}^{F} H_{\ell}^{i}(X_0) \to \operatorname{Ind}_{F_1}^{F} H_{\ell}^{i}(X_1))$$

for all primes ℓ.

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Proof. By taking $\widetilde{F}_1/F$ a sufficiently large finite extension such that all of the $F_{1,j}$ embed into $\widetilde{F}_1$ and applying [dJ96, Theorem 4.1], we can extend $X_1 \rightrightarrows X_0 \to X$ to a full proper hypercover $X_\bullet \to X$ such that for $n \geqslant 2$ there exists a finite extension $F_n/\widetilde{F}_1$ with X_n smooth over F_n. Now applying [CL18, Lemma 6.4] we can see that the terms in ith column of the resulting spectral sequence have to be 'quasi-pure' of weight i. Therefore the spectral sequence degenerates exactly as in the proof of [CL18, Theorem 6.1], and the proposition follows. □

We now deduce from the proposition that $H^i_\ell(X) \cong H^i_\ell(\mathcal{Z}_F)$ as Weil–Deligne representations for all i, ℓ, and by construction $\mathcal{Z}_F$ is globally defined. This completes the proof of Theorem 4.2

Remark 4.6. Note the use of the finite field hypothesis (via a weight argument) in the proof of Proposition 4.5. It might be possible to relax the assumption to k perfect using a more sophisticated argument.

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