CONFORMALLY FLAT HYPERSURFACES
OF SYMMETRIC SPACES

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Abstract

In this paper we consider how much we can say about an irreducible symmetric space \( M \) which admits a single hypersurface with at most two distinct principal curvatures. Then we prove that if \( N \) is conformally flat, then \( N \) is quasiumbilical and \( M \) must be a sphere, a real projective space or the noncompact dual of a sphere or a real projective space.


Recently, the following problem was proposed by B. Y. Chen and L. Verstraelen [3]: if we assume that an irreducible symmetric space \( M \) admits a single submanifold with a particular property, how much can we say about the ambient space? With respect to this problem, the author showed in [4] the following: (1) If \( M \) admits a (connected) locally symmetric hypersurface \( N \) (\( \dim N \geq 3 \)) with at most two distinct principal curvatures, then \( M \) must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space. (2) If an irreducible symmetric space \( M \) admits an Einstein hypersurface \( N \) (\( \dim N \geq 3 \)) with at most two distinct principal curvatures, then \( M \) must be of rank 1.

The purpose of this paper is to prove the following:

**Theorem.** If an irreducible symmetric space \( M \) admits a conformally flat hypersurfaced \( N \) (\( \dim N \geq 4 \)) with at most two distinct principal curvatures, then \( M \) must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.

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It is well-known that an $n$-dimensional ($n \geq 4$) hypersurface $N$ in a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space is conformally flat if and only if it is quasiumbilical (see [1] for instance). Hence, we know that: A conformally flat hypersurface $N$ (dim $N \geq 4$) with at most two distinct principal curvatures in an irreducible symmetric space is quasiumbilical (see Theorem 8.1 of [3]).

1. Symmetric spaces and basic formulas

Let $M$ be a connected Riemannian symmetric space. As usual if $G$ denotes the closure of the group of isometries generated by an involutive isometry for each point of $M$, then $G$ acts transitively on $M$; hence the isotropy subgroup $H$, say at 0, is compact and $M = G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$ denote the Lie algebras corresponding to $G$, $H$, respectively. Then we call

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \text{and} \quad \mathfrak{h} = \{ [\mathfrak{h}, \mathfrak{h}], \mathfrak{h} \}$$

by the Cartan decomposition. It is well-known the space $\mathfrak{m}$ consists of the Killing vector field $X$ whose covariant derivative vanishes at 0; in particular, the evaluation map at 0 gives a linear isomorphism of $\mathfrak{m}$ onto $T_0M$: $X \mapsto X(0)$. Hence we have

**Lemma 1.1.** For the curvature tensor $R$ at 0

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for} \ X, Y, Z \in \mathfrak{m}. \quad (1.1)$$

**Lemma 1.2.** A linear subspace $L$ of the tangent space $T_0M$ to a symmetric space $M$ is the tangent space to some totally geodesic submanifold $N$ of $M$ if and only if $L$ satisfies the condition $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{n}$, where

$$\mathfrak{n} = \{ X \in \mathfrak{m}; X(0) \in L \}.$$

Next, let $N$ be a hypersurface of an $(n + 1)$-dimensional Riemannian manifold $M$. And let $\nabla$ and $\nabla'$ be the covariant differentiations on $M$ and $N$, respectively. Then the second fundamental form $A$ of the immersion is given by

$$\nabla_X Y = \nabla'_X Y + g(A X, Y) \xi, \quad (1.1)$$

$$\nabla_X \xi = -A X, \quad (1.2)$$

for vector fields $X, Y$ tangent to $N$ and a unit vector field $\xi$ normal to $N$, where $g$ is the metric tensor of $N$ induced by the immersion from the metric tensor $g$ of $M$. The equations of Gauss and Codazzi are then given respectively

$$R'(X, Y; Z, W) = R(X, Y; Z, W) + g(A Y, Z) g(A X, W)$$

$$- g(A X, Z) g(A Y, W), \quad (1.3)$$
(1.4) \[ R(X, Y; Z, \xi) = g((\nabla_X A) Y, Z) - g((\nabla_Y A) X, Z), \]

for vector fields \( X, Y, Z, W \) tangent to \( N \) and \( \xi \) normal to \( N \), where \( R \) and \( R' \) are the curvature tensors of \( M \) and \( N \), respectively, and \( R(X, Y; Z, W) = g(R(X, Y)Z, W) \).

The following result is basic:

**Lemma 1.3 (Chen & Nagano [2]).** If an irreducible symmetric space \( M \) admits a totally geodesic hypersurface, then \( M \) must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.

2. Proof of Theorem

Let \( N \) be a hypersurface in \( M \) and \( E_1, \ldots, E_n \) be an orthonormal basis of \( T_x N \), \( x \in N \). Then the Ricci tensor \( S' \) of \( N \) satisfies

\[
S'(Y, Z) = \sum_{i=1}^{n} R'(E_i, Y; Z, E_i) = S(Y, Z) - R(\xi, Y; Z, \xi) + \text{trace} g(AY, Z) - g(A^2 Y, Z)
\]

for \( Y, Z \in T_x N \), where \( S \) denotes the Ricci tensor of \( M \).

We suppose that there is a point \( x_0 \) at which two principal curvatures \( \alpha, \beta \) are exactly distinct. Then we can choose a neighborhood \( U \) of \( x_0 \) on which \( \alpha \neq \beta \). We put \( T_\alpha = \{ X \in TU | AX = \alpha X \} \) and \( T_\beta = \{ X \in TU | AX = \beta X \} \). Then the equation (2.1) gives

\[
(2.1)' \quad S'(Y, Z) = S(Y, Z) - R(\xi, Y; Z, \xi) + (p\alpha + (n-p)\beta) g(AY, Z) - g(A^2 Y, Z),
\]

where \( p \) denotes the multiplicity of \( \alpha \). Thus the scalar curvatures \( \rho' \) and \( \rho \) of \( N \) and \( M \) satisfy

\[
(2.2) \quad \rho' = \sum_{i=1}^{n} S'(E_i, E_i) = \rho - 2S(\xi, \xi) + (p\alpha + (n-p)\beta)^2 - (p\alpha^2 + (n-p)\beta^2)
\]

\[
= \frac{n - 1}{n + 1} \rho + p(p - 1)\alpha^2 + 2p(n-p)\alpha\beta + (n-p)(n-p-1)\beta^2,
\]

where the last equality holds since \( M \) is Einsteinian. Now, by the assumption that \( N \) is conformally flat, the Weyl conformal curvature tensor of \( N \) vanishes. Thus by (2.1)' and (2.2), we see that the curvature tensor \( R \) of \( M \) satisfies
(2.3)
\[(n - 2)\{ R(X, Y; Z, W) + g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) \}
\]
\[= g(Y, W)\{ R(\xi, X; Z, \xi) - (p\alpha - (n - p)\beta)g(AX, Z) + g(A^2X, Z) \}
\]
\[+ g(X, W)\{ R(\xi, Y; Z, \xi)X - (p\alpha - (n - p)\beta)g(AY, Z) + g(AX, Z)g(A^2Y, W) \}
\]
\[+ g(Y, Z)\{ R(\xi, X; W, \xi) - (p\alpha - (n - p)\beta)g(AX, W) + g(A^2X, W) \}
\]
\[+ \frac{n}{n + 1} \{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \} \]
\[- \frac{1}{n - 1} \{ p(p - 1)\alpha^2 + 2p(n - p)\alpha\beta + (n - p)(n - p - 1)\beta^2 \}
\]
\cdot \{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \} \]
for \( X, Y, Z, W \) tangent to \( N \).

Let \( X, Y, Z, W \) and \( T \) be vector fields tangent to \( N \). By differentiation of (2.3) with respect to \( T \), we may obtain, after a straightforward computation, that
\[(2.4) \quad (n - 2)\{ g(AT, X)R(W, Z; Y, \xi) + g(AT, Y)R(Z, W; X, \xi) \]
\[+ g(AT, Z)R(Y, X; W, \xi) + g(AT, W)R(X, Y; Z, \xi) \]
\[+ g((\nabla_T A)g(AY, Z) - g((v^T A)X, Z)g(AY, W) \]
\[+ g(AX, W)g((\nabla_T A)Y, Z) - g(AX, Z)g((\nabla_T A)Y, W) \}
\[= g(Y, W)\{ -R(AT, X; Z, \xi) \]
\[+ R(\xi, X; Z, AT) - (pT\alpha + (n - p)T\beta)g(AX, Z) \]
\[+ g(Y, W)\{ -R(AT, Y; Z, \xi) - R(\xi, Y; Z, AT) \]
\[+ g(AT, Z)R(Y, X; W, \xi) + g(AT, W)R(X, Y; Z, \xi) \]
\[+ g(AT, Z)R(Y, X; W, \xi) + g(AT, W)R(X, Y; Z, \xi) \]
\[+ \frac{n}{n + 1} \{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \} \]
\[- \frac{1}{n - 1} \{ p(p - 1)T\alpha^2 + 2p(n - p)T\alpha\beta + (n - p)(n - p - 1)T\beta^2 \}
\]
\cdot \{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \} .
Conformally flat hypersurfaces of symmetric spaces

If \( X, Y, Z, W \) are vectors in \( T_a \) such that \( X = W, \ Y = Z \) and \( X, Y \) are orthonormal, then by (1.4) and (2.4) we find

(2.5)

\[
(n - 2)\{2\alpha X\alpha g(T, X) + 2\alpha X\alpha g(T, Y) + 2\alpha T\alpha \} = -2\{-2AT\alpha + \alpha Y\alpha g(T, Y) + g((\alpha I - A)\nabla_T Y, AT) + \alpha X\alpha g(T, X) + g((\alpha I - A)\nabla_T X, AT) - (pT\alpha + (n - p)TB)\alpha - (p\alpha + (n - p)\beta)T\alpha + T\alpha^2 \} \\
- \frac{1}{n - 1} \{p(p - 1)T\alpha^2 + 2p(n - p)T\alpha\beta + (n - p)(n - p - 1)T\beta^2 \}.
\]

In particular, for \( X = T \), (2.5) implies

(2.6) 4(n - 2)\alpha X\alpha = -2\{-2(p - 1)\alpha X\alpha - (n - p)\beta X\alpha - (n - p)\alpha X\beta \} \\
- \frac{1}{n - 1} \{2p(p - 1)\alpha X\alpha + 2p(n - p)\beta X\alpha \\
+ 2p(n - p)\alpha X\beta + 2(n - p)(n - p - 1)\beta X\beta \}.

Let \( T = X, W = \omega \) in \( T_B \) and \( Y, Z \) in \( T_a \) be orthonormal vectors. Then (2.4) gives

(2.7) \( -(n - 2)\beta(\beta - \alpha)g(\nabla_T^2 \omega, Y) = 0 \)

for orthonormal vectors \( Y, Z \) in \( T_a \). By linearization, we find

(2.8) \( \beta\{g(\nabla_T^2 \omega, Y) - g(\nabla_T^2 \omega, Z) \} = 0 \)

for orthonormal vectors \( Y, Z \) in \( T_a \). Similarly, we have

(2.9) \( \alpha g(\nabla_{\omega_1} X, \omega_2) = 0, \)

(2.10) \( \alpha\{g(\nabla_{\omega_1} X, \omega_1) - g(\nabla_{\omega_2} X, \omega_2) \} = 0 \)

for \( X \) in \( T_a \) and orthonormal vectors \( \omega_1, \omega_2 \) in \( T_B \).

Let \( Y = W, Z \) in \( T_a \) be orthonormal vectors and \( T = \omega_1, X = \omega_2 \) unit vectors in \( T_B \). Then (2.4) gives

(2.11) \( (n - 2)\{-\beta g(\omega_1, \omega_2)Z\alpha - \alpha(\alpha - \beta)g(\nabla_{\omega_1} Z, \omega_2) \} \\
= -\beta(\alpha - \beta)g(\nabla_{\omega_1} Z, \omega_2) + \beta(\alpha - \beta)g(\nabla_{\omega_2} Z, \omega_1) \} \\
- \beta(\alpha - \beta)g(\nabla_{\omega_1} Z, \omega_2) + \beta g(\omega_1, \omega_2)Z\beta \\
- (p\alpha + (n - p)\beta)(\alpha - \beta)g(\nabla_{\omega_1} Z, \omega_2) + (\alpha^2 - \beta^2)g(\nabla_{\omega_1} Z, \omega_2).\)
For unit vectors $Y = W = \omega_0$ in $T_\beta$, $Z$ in $T_\alpha$, and $T = \omega_1$, $X = \omega_2$ in $T_\beta$ which are perpendicular to $\omega_0$

\[(2.12)\]
\[(n - 2)\left\{-\beta g(\omega_1, \omega_2)Z\beta + \beta(\alpha - \beta)g(\omega_1, \omega_2)g(\nabla_{\omega_0}Z, \omega_0)\right.\]
\[\left. - \beta(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_2)\right\}\]
\[= -\beta(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_2) + \beta g(\omega_1, \omega_2)Z\beta\]
\[- \beta(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_2) + \beta(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_1)\]
\[- (p\alpha + (n - p)\beta)(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_2) + (\alpha^2 - \beta^2)g(\nabla_{\omega_1}Z, \omega_2).\]

Subtracting (2.12) from (2.11), we obtain

\[(2.13)\]
\[\alpha\{-\beta Z\alpha + \beta Z\beta\}g(\omega_1, \omega_2) - \alpha(\alpha - \beta)g(\nabla_{\omega_1}Z, \omega_2)\]
\[= a\beta(\alpha - \beta)\{g(\omega_1, \omega_2)g(\nabla_{\omega_0}Z, \omega_0) - g(\nabla_{\omega_1}Z, \omega_2)\}\]

Putting $\omega_1 = \omega_2$ and using (2.10), we find

\[(2.13)'\]
\[\alpha\{-\beta Z\alpha + \beta Z\beta - a\beta g(\nabla_{\omega_1}Z, \omega_1)\} = 0\]

Let $X_1, \ldots, X_p, \omega_1, \ldots, \omega_{n-p}$ be an orthonormal basis of $T_xN$ such that $X_1, \ldots, X_p$ (resp. $\omega_1, \ldots, \omega_{n-p}$) forms an orthonormal basis of $T_\alpha$ (resp. $T_\beta$). Since $M$ is Einstein, we have

\[0 = S(X_i, \xi)\]

\[(2.14)\]
\[= \sum_{j=1}^{p} R(X_i, X_j; X_j, \xi) + \sum_{k=1}^{n-p} R(X_i, \omega_k; \omega_k, \xi)\]
\[= pX_i\alpha + (n - p)X_i\beta - (n - p)(\alpha - \beta)g(\nabla_{\omega_k}X_i, \omega_k),\]

using (2.10) for all $i, k$. From (2.13)' and (2.14) we obtain

\[(2.15)\]
\[\alpha\{ (p\alpha + (n - p)\beta)X_i\alpha + (n - p)(\alpha - \beta)X_i\beta \} = 0.\]

Now, we assume that $\dim T_\alpha \geq 3$. Let $X$, $Y = Z$, $T = W$ be orthonormal vectors in $T_\alpha$. Then (2.4) gives

\[(2.16)\]
\[(n - 1)\alpha X\alpha = 0.\]

If $\alpha \neq 0$, then from (2.6) we obtain $(n - p - 1)(\alpha - \beta)X\beta = 0$. Since we may assume $p \neq n - 1$, we have $X\beta = 0$. Therefore from (2.9), (2.10) and (2.13)' we obtain $g(\nabla_{\omega_1}Z, \omega_2) = 0$ for all $\omega_1, \omega_2$ in $T_\beta$. If $\alpha = 0$, then (2.6) gives $X\beta = 0$. Then (2.11) and (2.12) imply

\[(2.11)\]
\[\beta^2\left\{(n - p + 1)g(\nabla_{\omega_1}Z, \omega_2) - g(\nabla_{\omega_2}Z, \omega_1)\right\} = 0\]
Putting $\omega_1 = \omega_2$ in (2.12)', we have
\[ (n - p + 1)g(\nabla_{\omega_1}Z, \omega_2) - g(\nabla_{\omega_2}Z, \omega_1) = 0. \]

By linearization, we find
\[ g(\nabla_{\omega_0}Z, \omega_0) = g(\nabla_{\omega_1}Z, \omega_1) \]
for orthonormal vectors $\omega_0, \omega_1$ in $T_B$. Combining (2.14) and (2.17), we obtain
\[ g(\nabla_{\omega}Z, \omega) = 0 \]
for all $\omega$ in $T_B$. By linearization, we find
\[ g(\nabla_{\omega_1}Z, \omega_2) + g(\nabla_{\omega_2}Z, \omega_1) = 0. \]

Summing up (2.11)' and (2.18), we have
\[ (n - p + 2)g(\nabla_{\omega_1}Z, \omega_2) = 0, \]
that is,
\[ g(\nabla_{\omega_1}Z, \omega_2) = 0 \]
for all $\omega_1, \omega_2$ in $T_B$. If $\dim T_B = 2$, then we have only to show $X\alpha = X\beta = 0$ for all unit vectors $X$ in $T_B$, since we can make use of the above argument. Then from (2.6) and (2.15)
\[ \alpha \{(2\alpha + (n - 2)\beta)X_\alpha + (n - 2)(\alpha - \beta)X_\beta \} = 0 \]
\[ \{(2n^2 - 9n + 9)\alpha - (n - 2)(n - 3)\beta \} X_\alpha - (n - 2)(n - 3)(\alpha - \beta)X_\beta = 0. \]
Hence we obtain $X_\alpha = X_\beta = 0$. Therefore we have $R(X, Y; Z, \xi) = 0$ for all $X, Y, Z$ in $TU$. From Lemmas 1.1, 1.2 and 1.3 we obtain the conclusion.

References