CONFORMALLY FLAT HYPERSURFACES
OF SYMMETRIC SPACES

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(Received 18 April 1983; revised 26 July 1983)

Communicated by K. Mackenzie

Abstract

In this paper we consider how much we can say about an irreducible symmetric space $M$ which admits a single hypersurface with at most two distinct principal curvatures. Then we prove that if $N$ is conformally flat, then $N$ is quasiumbilical and $M$ must be a sphere, a real projective space or the noncompact dual of a sphere or a real projective space.


Recently, the following problem was proposed by B. Y. Chen and L. Verstraelen [3]: if we assume that an irreducible symmetric space $M$ admits a single submanifold with a particular property, how much can we say about the ambient space? With respect to this problem, the author showed in [4] the following: (1) If $M$ admits a (connected) locally symmetric hypersurface $N$ ($\dim N \geq 3$) with at most two distinct principal curvatures, then $M$ must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space. (2) If an irreducible symmetric space $M$ admits an Einstein hypersurface $N$ ($\dim N \geq 3$) with at most two distinct principal curvatures, then $M$ must be of rank 1.

The purpose of this paper is to prove the following:

**Theorem.** If an irreducible symmetric space $M$ admits a conformally flat hypersurfaced $N$ ($\dim N \geq 4$) with at most two distinct principal curvatures, then $M$ must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.
It is well-known that an $n$-dimensional ($n \geq 4$) hypersurface $N$ in a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space is conformally flat if and only if it is quasiumbilical (see [1] for instance). Hence, we know that: A conformally flat hypersurface $N$ ($\dim N \geq 4$) with at most two distinct principal curvatures in an irreducible symmetric space is quasiumbilical (see Theorem 8.1 of [3]).

1. Symmetric spaces and basic formulas

Let $M$ be a connected Riemannian symmetric space. As usual if $G$ denotes the closure of the group of isometries generated by an involutive isometry for each point of $M$, then $G$ acts transitively on $M$; hence the isotropy subgroup $H$, say at 0, is compact and $M = G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$ denote the Lie algebras corresponding to $G$, $H$, respectively. Then we call $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ by the Cartan decomposition. It is well-known the space $\mathfrak{m}$ consists of the Killing vector field $X$ whose covariant derivative vanishes at 0; in particular, the evaluation map at 0 gives a linear isomorphism of $\mathfrak{m}$ onto $T_0M$: $X \mapsto X(0)$. Hence we have

**Lemma 1.1.** For the curvature tensor $R$ at 0

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for } X, Y, Z \in \mathfrak{m}.$$

**Lemma 1.2.** A linear subspace $L$ of the tangent space $T_0M$ to a symmetric space $M$ is the tangent space to some totally geodesic submanifold $N$ of $M$ if and only if $L$ satisfies the condition $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$, where

$$\mathfrak{m} = \{ X \in \mathfrak{m}; X(0) \in L \}.$$

Next, let $N$ be a hypersurface of an $(n + 1)$-dimensional Riemannian manifold $M$. And let $\nabla$ and $\nabla'$ be the covariant differentiations on $M$ and $N$, respectively. Then the second fundamental form $A$ of the immersion is given by

$$\nabla_X Y = \nabla'_X Y + g(A X, Y) \xi,$$

$$\nabla_X \xi = -A X,$$

for vector fields $X, Y$ tangent to $N$ and a unit vector field $\xi$ normal to $N$, where $g$ is the metric tensor of $N$ induced by the immersion from the metric tensor $g$ of $M$. The equations of Gauss and Codazzi are then given respectively

$$R'(X, Y; Z, W) = R(X, Y; Z, W) + g(A Y, Z)g(A X, W) - g(A X, Z)g(A Y, W),$$

https://doi.org/10.1017/S1446788700023065 Published online by Cambridge University Press
(1.4) \[ R(X, Y; Z, \xi) = g((\nabla_X A) Y, Z) - g((\nabla_Y A) X, Z), \]
for vector fields \( X, Y, Z, W \) tangent to \( N \) and \( \xi \) normal to \( N \), where \( R \) and \( R' \) are the curvature tensors of \( M \) and \( N \), respectively, and \( R(X, Y; Z, W) = g(R(X, Y)Z, W) \).

The following result is basic:

**Lemma 1.3 (Chen & Nagano [2]).** If an irreducible symmetric space \( M \) admits a totally geodesic hypersurface, then \( M \) must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.

## 2. Proof of Theorem

Let \( N \) be a hypersurface in \( M \) and \( E_1, \ldots, E_n \) be an orthonormal basis of \( T_x N \), \( x \in N \). Then the Ricci tensor \( S' \) of \( N \) satisfies

\[
(2.1) \quad S'(Y, Z) = \sum_{i=1}^{n} R'(E_i, Y; Z, E_i) \\
= S(Y, Z) - R(\xi, Y; Z, \xi) + \text{trace} g(AY, Z) - g(A^2 Y, Z)
\]

for \( Y, Z \in T_x N \), where \( S \) denotes the Ricci tensor of \( M \).

We suppose that there is a point \( x_0 \) at which two principal curvatures \( \alpha, \beta \) are exactly distinct. Then we can choose a neighborhood \( U \) of \( x_0 \) on which \( \alpha \neq \beta \). We put \( T_\alpha = \{ X \in TU | AX = \alpha X \} \) and \( T_\beta = \{ X \in TU | AX = \beta X \} \). Then the equation (2.1) gives

\[
(2.1)' \quad S'(Y, Z) = S(Y, Z) - R(\xi, Y; Z, \xi) \\
+ (p\alpha + (n - p)\beta) g(AY, Z) - g(A^2 Y, Z),
\]

where \( p \) denotes the multiplicity of \( \alpha \). Thus the scalar curvatures \( \rho' \) and \( \rho \) of \( N \) and \( M \) satisfy

\[
(2.2) \quad \rho' = \sum_{i=1}^{n} S'(E_i, E_i) \\
= \rho - 2S(\xi, \xi) + (p\alpha + (n - p)\beta)^2 - (p\alpha^2 + (n - p)\beta^2) \\
= \frac{n - 1}{n + 1} \rho + p(p - 1)\alpha^2 + 2p(n - p)\alpha \beta + (n - p)(n - p - 1)\beta^2,
\]

where the last equality holds since \( M \) is Einsteinian. Now, by the assumption that \( N \) is conformally flat, the Weyl conformal curvature tensor of \( N \) vanishes. Thus by (2.1)' and (2.2), we see that the curvature tensor \( R \) of \( M \) satisfies
\[ (n - 2) \left\{ R(X, Y; Z, W) + g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) \right\} = g(Y, W) \left\{ R(\xi, X; Z, \xi) - (p\alpha + (n - p)\beta)g(AX, Z) + g(A^2X, Z) \right\} \\
- g(X, W) \left\{ R(\xi, Y; Z, \xi)X - (p\alpha + (n - p)\beta)g(AY, Z) + g(A^2Y, Z) \right\} \\
+ g(X, Z) \left\{ R(\xi, Y; W, \xi) - (p\alpha + (n - p)\beta)g(AX, W) + g(A^2Y, W) \right\} \\
- g(Y, Z) \left\{ R(\xi, X; W, \xi) - (p\alpha + (n - p)\beta)g(AX, W) + g(A^2X, W) \right\} \\
+ \frac{p}{n+1} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} \\
- \frac{1}{n-1} \left( p(p-1)\alpha^2 + 2p(n-p)\alpha\beta + (n-p)(n-p-1)\beta^2 \right) \\
\cdot \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} \\
\] for \( X, Y, Z, W \) tangent to \( N \).

Let \( X, Y, Z, W \) and \( T \) be vector fields tangent to \( N \). By differentiation of (2.3) with respect to \( T \), we may obtain, after a straightforward computation, that

\[ (n - 2) \left\{ g(AX, X)R(W, Z; Y, \xi) + g(AX, Y)R(Z, W; X, \xi) \right\} \\
+ g(AX, Z)R(Y, X; W, \xi) + g(AX, W)R(X, Y; Z, \xi) \\
+ g(AX, W)g((\nabla^T_A Y, Z) - g(AX, Z)g((\nabla^T_A Y, W)) \\
= g(Y, W) \left\{ -R(AX, X; Z, \xi) \\
- (p\alpha + (n - p)\beta)g(AY, Z) + g((\nabla^T_A)^2 X, Z) \right\} \\
- g(AX, W) \left\{ -R(AX, Y; Z, \xi) - R(\xi, Y; Z, \xi) \\
- (p\alpha + (n - p)\beta)g((\nabla^T_A) Y, Z) + g((\nabla^T_A)^2 Y, Z) \right\} \\
+ g(AX, W) \left\{ -R(AX, Y; W, \xi) \\
- (p\alpha + (n - p)\beta)g((\nabla^T_A Y, W) + g((\nabla^T_A)^2 Y, W) \right\} \\
- g(AX, W) \left\{ -R(\xi, X; W, \xi) \\
- (p\alpha + (n - p)\beta)g((\nabla^T_A) X, W) + g((\nabla^T_A)^2 X, W) \right\} \\
- \frac{1}{n-1} \left( p(p-1)\alpha^2 + 2p(n-p)\alpha\beta + (n-p)(n-p-1)\beta^2 \right) \\
\cdot \left\{ g(AX, W)g(Y, Z) - g(AX, Z)g(Y, W) \right\}. \]
If \( X, Y, Z, W \) are vectors in \( T_a \) such that \( X = W, Y = Z \) and \( X, Y \) are orthonormal, then by (1.4) and (2.4) we find

\[
(2.5) \quad (n - 2)\{2\alpha Xa(T, X) + 2\alpha Ya(T, Y) + 2\alpha Ta\} = 
-2\{-2ATa + \alpha Ya(T, Y) + g((\alpha I - A)\nabla'_Y Y, AT) + \alpha Xa(T, X) + g((\alpha I - A)\nabla'_X X, AT) \\
- \( pTa + (n - p)TB \)a - (\( p\alpha + (n - p)\beta \))Ta + Ta^2 \}
- \frac{1}{n - 1} \{p(p - 1)Ta^2 + 2p(n - p)Ta\beta + (n - p)(n - p - 1)TB^2 \}.
\]

In particular, for \( X = T \), (2.5) implies

\[
(2.6) \quad 4(n - 2)\alpha Xa = -2\{-2(2p - 1)\alpha Xa - (n - p)\beta Xa - (n - p)\alpha X\beta \} \\
- \frac{1}{n - 1} \{2p(p - 1)\alpha Xa + 2p(n - p)\beta Xa \\
+ 2p(n - p)\alpha X\beta + 2(n - p)(n - p - 1)\beta X\beta \}.
\]

Let \( T = X, W = \omega \) in \( T_\beta \) and \( Y, Z \) in \( T_a \) be orthonormal vectors. Then (2.4) gives

\[
(2.7) \quad -(n - 2)\beta(\beta - \alpha)g(\nabla'_Z \omega, Y) = 0
\]
for orthonormal vectors \( Y, Z \) in \( T_a \). By linearization, we find

\[
(2.8) \quad \beta_i\{g(\nabla'_Y \omega, Y) - g(\nabla'_Z \omega, Z)\} = 0
\]
for orthonormal vectors \( Y, Z \) in \( T_a \). Similarly, we have

\[
(2.9) \quad \alpha g(\nabla'_\omega_1 X, \omega_2) = 0,
\]
\[
(2.10) \quad \alpha\{g(\nabla'_\omega_1 X, \omega_1) - g(\nabla'_\omega_2 X, \omega_2)\} = 0
\]
for \( X \) in \( T_a \) and orthonormal vectors \( \omega_1, \omega_2 \) in \( T_\beta \).

Let \( Y = W, Z \) in \( T_a \) be orthonormal vectors and \( T = \omega_1, X = \omega_2 \) unit vectors in \( T_\beta \). Then (2.4) gives

\[
(2.11) \quad (n - 2)\{-\beta g(\omega_1, \omega_2)Z\alpha - \alpha(\alpha - \beta)g(\nabla'_\omega_1 Z, \omega_2) \\
= -\beta(\alpha - \beta)g(\nabla'_\omega_1 Z, \omega_2) + \beta(\alpha - \beta)g(\nabla'_\omega_2 Z, \omega_1) \}
- \beta(\alpha - \beta)g(\nabla'_\omega_1 Z, \omega_2) + \beta g(\omega_1, \omega_2)Z\beta \\
- (p\alpha + (n - p)\beta)(\alpha - \beta)g(\nabla'_\omega_1 Z, \omega_2) + (\alpha^2 - \beta^2)g(\nabla'_\omega_1 Z, \omega_2).
\]
For unit vectors \( Y = W = \omega_0 \) in \( T_\beta, \) \( Z \) in \( T_a, \) and \( T = \omega_1, \) \( X = \omega_2 \) in \( T_\beta \) which are perpendicular to \( \omega_0 \)

(2.12)

\[
(n - 2) \left\{ -\beta g(\omega_1, \omega_2) Z\beta + \beta (\alpha - \beta) g(\omega_1, \omega_2) g(\nabla_{\omega_1} Z, \omega_0) - \beta (\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_2) \right\}
\]

\[
= -\beta (\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_2) + \beta g(\omega_1, \omega_2) Z\beta
\]

\[
- \beta (\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_2) + \beta (\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_1)
\]

\[
- (p \alpha + (n - p) \beta)(\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_2) + (\alpha^2 - \beta^2) g(\nabla_{\omega_1} Z, \omega_2).
\]

Subtracting (2.12) from (2.11), we obtain

(2.13)

\[
\alpha \{-\beta Z\alpha + \beta Z\beta\} g(\omega_1, \omega_2) - \alpha (\alpha - \beta) g(\nabla_{\omega_1} Z, \omega_2)\}
\]

\[
= \alpha \beta (\alpha - \beta) \{ g(\omega_1, \omega_2) g(\nabla_{\omega_1} Z, \omega_0) - g(\nabla_{\omega_1} Z, \omega_2) \}
\]

Putting \( \omega_1 = \omega_2 \) and using (2.10), we find

(2.13)'

\[
\alpha \{-\beta Z\alpha + \beta Z\beta - \alpha \beta \} g(\nabla_{\omega_1} Z, \omega_1) \} = 0
\]

Let \( X_1, \ldots, X_p, \omega_1, \ldots, \omega_{n-p} \) be an orthonormal basis of \( T_x N \) such that \( X_1, \ldots, X_p \) (resp. \( \omega_1, \ldots, \omega_{n-p} \)) forms an orthonormal basis of \( T_a \) (resp. \( T_\beta \)). Since \( M \) is Einstein, we have

\[
0 = S(X_i, \xi)
\]

(2.14)

\[
= \sum_{j=1}^{p} R(X_i, X_j; X_j, \xi) + \sum_{k=1}^{n-p} R(X_i, \omega_k; \omega_k, \xi)
\]

\[
= pX_i \alpha + (n - p)X_i \beta - (n - p)(\alpha - \beta) g(\nabla_{\omega_k} X_i, \omega_k),
\]

using (2.10) for all \( i, k. \) From (2.13)' and (2.14) we obtain

(2.15)

\[
\alpha \{ ( p \alpha + (n - p) \beta) X_i \alpha + (n - p)(\alpha - \beta) X_i \beta \} = 0.
\]

Now, we assume that \( \dim T_a \geq 3. \) Let \( X, Y = Z, T = W \) be orthonormal vectors in \( T_a. \) Then (2.4) gives

(2.16)

\[
(n - 1) a X\alpha = 0.
\]

If \( a \neq 0, \) then from (2.6) we obtain \( (n - p - 1)(\alpha - \beta) X\beta = 0. \) Since we may assume \( p \neq n - 1, \) we have \( X\beta = 0. \) Therefore from (2.9), (2.10) and (2.13)' we obtain \( g(\nabla_{\omega_1} Z, \omega_2) = 0 \) for all \( \omega_1, \omega_2 \) in \( T_\beta. \) If \( \alpha \equiv 0, \) then (2.6) gives \( X\beta = 0. \) Then (2.11) and (2.12) imply

(2.11)'

\[
\beta^2 \{ (n - p + 1) g(\nabla_{\omega_1} Z, \omega_2) - g(\nabla_{\omega_2} Z, \omega_1) \} = 0
\]
Putting \( \omega_1 = \omega_2 \) in (2.12)', we have

\[
\beta^2 (n - 2) \left\{-g(\omega_1, \omega_2) g(\nabla'_{\omega_0} Z, \omega_0) + g(\nabla'_{\omega_1} Z, \omega_1)\right\} = 0.
\]

for orthonormal vectors \( \omega_0, \omega_1 \) in \( T_\beta \). Combining (2.14) and (2.17), we obtain

\[
g(\nabla'_{\omega_1} Z, \omega_1) = 0.
\]

By linearization, we find

\[
(n - p + 2) g(\nabla'_{\omega_1} Z, \omega_2) = 0,
\]

that is,

\[
g(\nabla'_{\omega_1} Z, \omega_2) = 0
\]

for all \( \omega_1, \omega_2 \) in \( T_\beta \). If \( \dim T_\alpha = 2 \), then we have only to show \( X_\alpha = X_\beta = 0 \) for all unit vectors \( X \) in \( T_\alpha \), since we can make use of the above argument. Then from

(2.6) and (2.15)

\[
\alpha \{ (2\alpha + (n - 2)\beta) X_\alpha + (n - 2)(\alpha - \beta) X_\beta \} = 0
\]

and

\[
(2n^2 - 9n + 9) \alpha - (n - 2)(n - 3)\beta \} X_\alpha - (n - 2)(n - 3)(\alpha - \beta) X_\beta = 0.
\]

Hence we obtain \( X_\alpha = X_\beta = 0 \). Therefore we have \( R(X, Y; Z, \xi) = 0 \) for all \( X, Y, Z \) in \( TU \). From Lemmas 1.1, 1.2 and 1.3 we obtain the conclusion.

References