Report of the Working Group 'Relativity for Celestial Mechanics and Astrometry'

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1. Introduction

This is a brief report on the work done by the IAU working group 'Relativity for Celestial Mechanics and Astrometry' that is well documented on the Working Group (WG) web-site: http://rcswww.urz.tu-dresden.de/~lohrmobs/iauwg.html. There one finds a list of members, the circulars of the WG, related material and references to relevant publications as well as some online documents containing important formulas and explanations.

The first central task of the WG was to recommend some specific form of the metric tensor $g_{\alpha\beta}$ that is related to the distance ds of two neighboring points in space-time with coordinates $x^\alpha = (ct, x^i)$ and $x^\alpha + dx^\alpha$ by

$$ds^2 = g_{\alpha\beta}(t, x^i)\, dx^\alpha\, dx^\beta.$$

The metric tensor allows one to derive translational and rotational equations of motion of bodies, to describe the propagation of light, the rates of atomic clocks and to model the processes of observation. Meanwhile it is widely accepted that in order to describe adequately modern astronomical observations one has to use several relativistic reference systems. The barycentric celestial reference system (BCRS) can be used to model the light propagation from distant celestial objects as well as the motion of bodies within the solar system. The geocentric celestial reference system (GCRS) is physically adequate to describe processes occurring in the vicinity of the Earth (Earth's rotation, motion of Earth's satellites).

The necessity to use several reference systems can be understood from the following. If we were to characterize terrestrial observers by the difference between their BCRS coordinates and the BCRS coordinates of the geocenter, the positions of the observers relative to the geocenter would change with time also due to purely relativistic coordinate effects (such as Lorentz contraction, *etc.*) which have nothing to do with the Earth's rotation or geophysical factors and vanish if one employs suitable GCRS coordinates instead. On the other hand, the coordinate positions derived with VLBI observations are used to investigate local geophysical processes and some adequate Geocentric RS allows one to simplify their description. For these reasons the central task of the working group is to specify the metric tensors both in the BCRS and in the GCRS and the corresponding space-time coordinate transformations between these two systems.

Two advanced relativistic formalisms have been elaborated to tackle the problem of astronomical reference frames in the first post-Newtonian approximation of general relativity. One formalism is due to Brumberg and Kopeikin

(Brumberg and Kopeikin, 1989; Kopeikin, 1988; Brumberg, 1991; see also Klioner and Voinov, 1993) and another one is due to Damour, Soffel and Xu (Damour, Soffel, Xu, 1991, 1992, 1993, 1994, referred to as DSX I-IV). Although the formalisms look rather different at first glance, it is mainly the concept of mass multipole moments (potential coefficients) at the first post-Newtonian level that differs in the two formalisms. The new recommendations of the WG improve and extend those from the IAU 1991 framework that will be recalled in the next section.

2. The IAU 1991 framework

The IAU resolution A4(1991) contains nine recommendations, the first five of which are directly relevant to our discussion.

In the first recommendation, the metric tensor in space-time coordinates $(t, \mathbf{x})$ centered at the barycenter of an ensemble of masses is recommended to be written in the form

$$\begin{aligned} g_{00} &= -1 + \frac{2U(t,\mathbf{x})}{c^2} + \mathcal{O}(c^{-4}), \\ g_{0i} &= \mathcal{O}(c^{-3}), \\ g_{ij} &= \delta_{ij}\left(1 + \frac{2U(t,\mathbf{x})}{c^2}\right) + \mathcal{O}(c^{-4}), \end{aligned} \tag{1}$$

where c is the speed of light in vacuum U is the sum of the gravitational potentials of the ensemble of masses, and of a tidal potential generated by bodies external to the ensemble, the latter vanishing at the barycenter. The algebraic sign of U is taken to be positive.

This recommendation recognizes that space-time cannot be described by a single coordinate system. The recommended form of the metric tensor can be used not only to describe the barycentric celestial reference system of the whole solar system resulting in the BCRS, but also to define the geocentric celestial reference system (GCRS) centered at the center of mass of the Earth with a suitable function U, now depending upon geocentric coordinates. In analogy to the GCRS a corresponding celestial reference system can be constructed for any other body of the solar system.

In the second recommendation, the origin and orientation of the spatial coordinate grids for the solar system (BCRS) and for the Earth (GCRS) are defined. Notably it is specified that the spatial coordinates of these systems should show no global rotation with respect to a set of distant extragalactic objects. It also specifies that the SI (International System of Units) second and the SI meter should be the physical units of proper time and proper length in all coordinate systems. It states in addition that the time coordinates should be derived from an Earth atomic time scale.

The third recommendation defines TCB (Barycentric Coordinate Time) and TCG (Geocentric Coordinate Time) — the time coordinates of the BCRS and GCRS, respectively. Here we write ($t = TCB, x^i$) and ($T = TCG, X^i$) for the respective coordinates. The recommendation also defines the origin of the time scales (their reading on 1977 January 1, $0^h 0^m 0^s$ TAI ($JD = 2443144.5\, TAI$) must be 1977 January 1, $0^h 0^m 32.184^s$) and declares that the units of measurements of the coordinate times of all reference systems must coincide with the SI second and SI meter. The relationship between TCB and TCG is given by a full 4-dimensional transformation

$$TCB - TCG = c^{-2} \left[\int_{t_0}^{t} \left(\frac{v_E^2}{2} + \overline{U}(t, \mathbf{x}_E(t)) \right) dt + v_E^i r_E^i \right] + \mathcal{O}(c^{-4}), \qquad (2)$$

where x_E^i and v_E^i are the barycentric coordinate position and velocity of the geocenter, $r_E^i = x^i - x_E^i$ with x^i the barycentric position of the observer, and $\overline{U}(t, \mathbf{x}_E(t))$ is the Newtonian potential of all solar system bodies apart from the Earth evaluated at the geocenter.

In the fourth recommendation another time coordinate, Terrestrial Time (TT), is defined for the GRS. It differs from TCG by a constant rate only

$$TCG - TT = L_G \times (JD - 2443144.5) \times 86400, \qquad L_G \approx 6.969291 \times 10^{-10}, \quad (3)$$

so that the unit of measurement of TT agrees with the SI second on the geoid. TT represents an ideal form of TAI, the divergence between them being a consequence of the physical defects of atomic clocks. It is also recognized that the TT is nothing else than a rescaling of the GRS coordinate time TCG.

The fifth recommendation states that the old barycentric time TDB may still be used where discontinuity with previous work is deemed to be undesirable. In the notes to the third recommendation the relation of the TCB with TDB given as

$$TCB - TDB = L_B \times (JD - 2443144.5) \times 86400, \qquad L_B \approx 1.550505 \times 10^{-8}. \quad (4)$$

3. General framework for new conventions

3.1. New conventions for the Barycentric Celestial Reference System

The metric tensor of the Barycentric Celestial Reference System (BCRS) in the first post-Newtonian approximation should be written in the form

$$\begin{aligned} g_{00} &= -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + \mathcal{O}(c^{-5}), \\ g_{0i} &= -\frac{4}{c^3} w_i + \mathcal{O}(c^{-5}), \\ g_{ij} &= \delta_{ij}\left(1 + \frac{2}{c^2} w\right) + \mathcal{O}(c^{-4}). \end{aligned} \qquad (5)$$

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Here, the post-Newtonian gravitational potential w generalizes the usual Newtonian potential U and w^i is the vector potential related with gravito-magnetic type effects. The best reason for writing the metric in that form is a simplicity argument. This is the most compact form to write the metric tensor to first post-Newtonian order. Let us also note here that the potential w contains also explicit post-Newtonian terms and, therefore, does *not* coincide with the Newtonian potential. As is well known it makes no sense to distinguish between the Newtonian potential and explicit post-Newtonian terms. It is the post-Newtonian potential w as a whole that plays a role in observations.

Note that this form (5) of the barycentric metric tensor implies the barycentric spatial coordinates x^i to satisfy *the harmonic gauge condition. We recommend also to use the harmonic gauge for the barycentric coordinate time* $t = TCB$. The main arguments in favour of the harmonic gauge are:

- tremendous work on General Relativity has been done with the harmonic gauge that was found to be a useful and simplifying gauge for all kinds of applications;
- in contrast to the standard PN-gauge the harmonic gauge can be defined to higher PN-orders, and in fact for the exact Einstein theory of gravity.

Assuming space-time to be asymptotically flat (no gravitational fields far from the system), *i.e.*,

$$\lim_{\substack{r\to\infty \\ t=\mathrm{const}}} g_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$$

in the standard harmonic gauge, the post-Newtonian field equations of General Relativity are solved by

$$\begin{aligned} w(t,\mathbf{x}) &= G\int d^3x'\,\frac{\sigma(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{2c^2}G\frac{\partial^2}{\partial t^2}\int d^3x'\,\sigma(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|\,, \\ w^i(t,\mathbf{x}) &= G\int d^3x'\frac{\sigma^i(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \end{aligned} \tag{6}$$

where

$$\sigma(t,\mathbf{x}) = \frac{T^{00}+T^{ss}}{c^2}, \qquad \sigma^i(t,\mathbf{x}) = \frac{T^{0i}}{c}\,.$$

$T^{\mu\nu} = T^{\mu\nu}(t,x^i)$ are the components of the energy-momentum tensor in the barycentric coordinate system and $T^{ss} = T^{11}+T^{22}+T^{33}$. For many applications explicit expressions for the gravitational mass and mass-current density, σ and σ^i, will *not* be needed.

Since the integrations in eqs. (6) have to be taken over all the massive solar system bodies A, the metric potentials, w and w^i, can be written as a sum of the form

$$w = \sum_A w_A; \qquad w^i = \sum_A w^i_A\,.$$

Here the potentials with index A are obtained from relations (6) with integrals taken over the support of body A only. In the following we will use

$$w_{\rm ext} = \sum_{A\neq E} w_A, \qquad w^i_{\rm ext} = \sum_{A\neq E} w^i_A\,,$$

where E stands for the Earth.

3.2. New conventions for the Geocentric Celestial Reference System

The metric tensor of the Geocentric Celestial Reference System (GCRS) (T, X^i) should be written in the same form as that of the BCRS:

$$\begin{aligned} G_{00} &= -1 + \frac{2W}{c^2} - \frac{2W^2}{c^4} + \mathcal{O}(c^{-5}), \\ G_{0a} &= -\frac{4}{c^3} W_a, \\ G_{ab} &= \delta_{ab}\left(1 + \frac{2}{c^2}W\right) + \mathcal{O}(c^{-4})\,. \end{aligned} \tag{7}$$

Here $W = W(T, \mathbf{X})$ is the post-Newtonian gravitational potential in the geocentric system and $W^a(T, \mathbf{X})$ is the corresponding vector potential. These geocentric potentials should be split into two parts: potentials W_E and W^a_E arising from the gravitational action of the Earth and external parts $W_{\rm ext}$ and $W^a_{\rm ext}$ due to tidal and inertial effects. The external parts are assumed to vanish at the geocenter and admit an expansion into positive powers of $\mathbf{X}$. Explicitly,

$$\begin{aligned} W(T, \mathbf{X}) &= W_E(T, \mathbf{X}) + W_{\rm ext}(T, \mathbf{X})\,, \\ W^a(T, \mathbf{X}) &= W^a_E(T, \mathbf{X}) + W^a_{\rm ext}(T, \mathbf{X})\,. \end{aligned} \tag{8}$$

The Earth's potentials W_E and W^a_E are defined in the same way as w_E and w^i_E but with quantities calculated in the GCRS. We may write

$$\begin{aligned} W_{\rm ext} &= W_{\rm iner} + W_{\rm tidal}\,, \\ W^a_{\rm ext} &= W^a_{\rm iner} + W^a_{\rm tidal}\,. \end{aligned} \tag{9}$$

Here, $W_{\rm iner}$ and $W^a_{\rm iner}$ are inertial contributions that are linear in X^a

$$\begin{aligned} W_{\rm iner} &= Q_a X^a, \\ W^a_{\rm iner} &= \frac{1}{4} c^2 \varepsilon_{abc}(\Omega^b - \Omega^b_{\rm iner}) X^c + \mathcal{O}(c^{-2})\,. \end{aligned} \tag{10}$$

Q_a characterizes the deviation of the actual worldline of the origin of the GCRS from geodesic motion in the external gravitational field that is determined mainly by the coupling of the Earth's nonsphericity to the external potential. To Newtonian order Q_a is given by

$$Q_a = R^a_{\ i}\left(\frac{\partial}{\partial x^i} w^{\rm ext} - a^i_E\right) + \mathcal{O}(c^{-2}). \tag{11}$$

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Here, $x^i_E(t), v^i_E(t) = dx^i_E/dt$ and $a^i_E = dv^i_E/dt$ are the barycentric coordinate position, velocity and acceleration of the origin of the GCRS (geocenter). R^a_i is a rotation matrix that determines the relative orientation of barycentric and geocentric spatial coordinate lines. Ω^a is a slowly varying function of time related to R^a_i by

$$\epsilon_{abc}\Omega^b = \dot{R}^a_k R^c{}_k \,, \tag{12}$$

and $\Omega^a_{\rm iner}$ mainly describes the geodetic precession of inertial axes with respect to remote objects. One sees that for $\Omega^a = \Omega^a_{\rm iner}$ the vector potential $W^a_{\rm iner}$ vanishes. This implies that dynamical equations of motion, *e.g.*, for a satellite around the Earth, do not contain Coriolis and centrifugal terms, *i.e.*, the local geocentric spatial coordinates X^a are *dynamically non-rotating.* Recommended, however, is the use of *kinematically non-rotating* geocentric coordinates defined by $\Omega^a = 0$, *i.e.*, by $R^a{}_i = \delta_{ai}$.

$W^{\rm tidal}_a$ is a generalization of the Newtonian tidal potential

$$W^{\rm Newton}_{\rm tidal}(T, \mathbf{X}) = w_{\rm ext}(\mathbf{x}_E + \mathbf{X}) - w_{\rm ext}(\mathbf{x}_E) - X^a \frac{\partial}{\partial X^a} w_{\rm ext}(\mathbf{x}_E) \,. \tag{13}$$

Full post-Newtonian expressions for $W_{\rm tidal}$ and $W^a_{\rm tidal}$ can be found in DSX II, IV.

Finally, the local gravitational potentials W_E and W^a_E of the Earth are related to the barycentric gravitational potentials w_E and w^i_E by

$$\begin{aligned} W_E(T, \mathbf{X}) &= w_E(t, \mathbf{x}) \left(1 + \frac{1}{c^2}\, 2v_E^2\right) - \frac{4}{c^2}\, v^i_E\, w^i_E(t, \mathbf{x}) + \mathcal{O}(c^{-4}), \\ W^a_E(T, \mathbf{X}) &= R^a_i \left(w^i_E(t, \mathbf{x}) - v^i_E\, w_E(t, \mathbf{x})\right) + \mathcal{O}(c^{-2}). \end{aligned} \tag{14}$$

3.3. Transformations between the reference systems

The coordinate transformations between the BCRS and GCRS can be written as

$$\begin{aligned} T &= t - \frac{1}{c^2}\left(A(t) + v^i_E\, r^i_E\right) \\ &\quad + \frac{1}{c^4}\left(B(t) + B^i(t)\, r^i_E + B^{ij}(t)\, r^i_E\, r^j_E + C(t, \mathbf{x})\right) \\ &\quad + \mathcal{O}(c^{-5}), \end{aligned} \tag{15}$$

$$\begin{aligned} X^a &= \delta_{ai}\left[r^i_E + \frac{1}{c^2}\left(\frac{1}{2} v^i_E v^j_E r^j_E + w_{\rm ext}(\mathbf{x}_E) r^i_E + r^i_E a^j_E r^j_E - \frac{1}{2} a^i_E r^2_E\right)\right] \\ &\quad + \mathcal{O}(c^{-4}), \end{aligned} \tag{16}$$

where

$$\frac{d}{dt} A(t) = \frac{1}{2}\, v^2_E + w_{\rm ext}(\mathbf{x}_E),$$

$$\frac{d}{dt}\,B(t) = -\frac{1}{8}\,v_E^4 - \frac{3}{2}\,v_E^2\,w_{\rm ext}(\mathbf{x}_E) + 4\,v_E^i\,w_{\rm ext}^i + \frac{1}{2}\,w_{\rm ext}^2(\mathbf{x}_E),$$

$$B^i(t) = -\frac{1}{2}\,v_E^2\,v_E^i + 4\,w_{\rm ext}^i(\mathbf{x}_E) - 3\,v_E^i\,w_{\rm ext}(\mathbf{x}_E),$$

$$B^{ij}(t) = -v_E^i\,R_j^a\,Q^a + 2\frac{\partial}{\partial x^j}w_{\rm ext}^i(\mathbf{x}_E) - v_E^i\,\frac{\partial}{\partial x^j}w_{\rm ext}(\mathbf{x}_E) + \frac{1}{2}\,\delta^{ij}\,\dot{w}_{\rm ext}(\mathbf{x}_E),$$

$$C(t,\mathbf{x}) = -\frac{1}{10}\,r_E^2\,(\dot{a}_E^i\,r_E^i)\,.$$

Let us remark that the harmonic gauge condition does not fix the function C uniquely. Here we have indicated the simplest solution. Though the theoretical domain of validity of the GCRS coordinates is quite large (larger than the size of the solar system) the spatial region where the GCRS coordinates are to be used in practice, should be restricted to the immediate vicinity of the Earth. We suggest here that the GCRS coordinates be used only to about the geostationary orbit, that is, for $|\mathbf{X}| < 50000$ km. The size of this region allows one to estimate the terms in the transformations and to neglect those terms which are smaller than the targeted accuracy. Thus, it is easy to see that in the transformations between coordinate times $c^{-4}\,C < 0.1$ ps for $|\mathbf{X}| < 0.1$ AU and can be neglected for most applications. In the same way, one can show that $c^{-4}B^{ij}\,r_E^i\,r_E^j < 0.1$ ps for $|\mathbf{X}| < 0.01$ AU and also can be neglected for most purposes. However, to avoid ambiguities one should remember that the terms $\mathcal{O}(r_E^2)$ are **fixed** in the time transfromations and neglected only because of their small numerical values in the considered region of space.

3.4. Multipole expansions of the local gravitational potentials

For many problems it is advantageous to present the local gravitational potentials of the Earth as multipole series that usually converge everywhere outside the Earth. The definition of corresponding post-Newtonian multipole moments or *potential coefficients* is not obvious from the very beginning. However, a certain set of potential coefficients, called Blanchet-Damour moments, defined to first post-Newtonian order has especially attractive features. Moreover, by using such Blanchet-Damour potential coefficients we get the simplest possible form of the multipole expansion of the post-Newtonian potentials (these expansions take an almost Newtonian form). Basically two sets of BD-moments occur in the formalism: mass-multipole moments and (mass) current multipole moments. Expressed in terms of (symmetric and trace-free) Cartesian tensors they are denoted by $\mathcal{M}_L$ and $\mathcal{S}_L$. Here L stands for a Cartesian multi-index, $L = i_1 \ldots i_l$ and each index i runs over the three spatial indices. The set $\mathcal{M}_L$ is equivalent to a set of potential coefficients C_{lm} and S_{lm} that appear in a spherical harmonic expansion of the potentials. The first spin-moment of a body agrees with its spin or total angular momentum. The multipole expansion of W_E and W_E^a reads (a dot indicates the time derivative):

$$
\begin{aligned}
W_E &= G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[\mathcal{M}_L\,\partial_L\frac{1}{|\mathbf{X}|}+\frac{1}{2c^2}\ddot{\mathcal{M}}_L\,\partial_L|\mathbf{X}|\right]+\frac{4}{c^2}\Lambda_{,T}+\mathcal{O}(c^{-4}),\\
W_E^a &= -G\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\left[\dot{\mathcal{M}}_{aL-1}\partial_{L-1}\frac{1}{|\mathbf{X}|}+\frac{l}{l+1}\varepsilon_{abc}\mathcal{S}_{cL-1}\partial_{bL-1}\frac{1}{|\mathbf{X}|}\right]-\Lambda_{,a}\\
&\quad+\mathcal{O}(c^{-2}),
\end{aligned}
\tag{17}
$$

where

$$
\Lambda = G\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\frac{2l+1}{2l+3}\mathcal{P}_L\,\partial_L\frac{1}{|\mathbf{X}|},
$$

$$
\mathcal{P}_L = \int_V \Sigma^a\,\hat{X}^{aL}\,d^3X,
$$

and

$$
\Sigma(T,\mathbf{X}) = \mathcal{T}^{00}+\frac{1}{c^2}\mathcal{T}^{ss}, \qquad \Sigma^a(T,\mathbf{X}) = \frac{1}{c}\mathcal{T}^{0a},
$$

$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu\nu}(t,x^i)$ are the components of the energy-momentum tensor in the GRS.

The function Λ does not enter the post-Newtonian equations of motion. The latter contains only the BD multipole moments $\mathcal{M}_L$ and $\mathcal{S}_L$. The only place where the function Λ should be accounted for is the transformation between the proper time of an observer and the coordinate time of the GRS. First estimates indicate that the Λ-terms will be negligible in the foreseeable future. For these reasons the gauge function Λ will not be mentioned in the recommendations.

A spherical harmonic expansion of W_E reads ($R = |\mathbf{X}|$)

$$
\begin{aligned}
W^{\mathrm{E}}(T,\mathbf{X}) &= \frac{GM_{\mathrm{E}}}{R}\Big[1+\sum_{l=2}^{\infty}\sum_{m=0}^{+l}\left(\frac{R_{\mathrm{E}}}{R}\right)^l P_{lm}(\cos\theta)(\mathcal{C}_{lm}(T,R)\cos m\phi\\
&\quad+\mathcal{S}_{lm}(T,R)\sin m\phi)\Big]+\frac{4}{c^2}\partial_T\Lambda+\mathcal{O}(c^{-4})
\end{aligned}
\tag{18}
$$

with

$$
\begin{aligned}
\mathcal{C}^E_{lm}(T,R) &= C^E_{lm}(T)-\frac{1}{2(2l-1)}\frac{R^2}{c^2}\frac{d^2}{dT^2}C^E_{lm}(T),\\
\mathcal{S}^E_{lm}(T,R) &= S^E_{lm}(T)-\frac{1}{2(2l-1)}\frac{R^2}{c^2}\frac{d^2}{dT^2}S^E_{lm}(T).
\end{aligned}
$$

Here the relativistic time derivative terms are not expected to play some interesting role in the near future; they will not be mentioned in the corresponding recommendation. The gravitomagnetic vector potential of the Earth, W_E^a, is dominated by the Earth's spin-vector $\mathbf{S}_\mathrm{E}$ (total angular momentum), *i.e.*, to a good approximation

$$W_a^\mathrm{E}(T,\mathbf{X}) \simeq -\frac{G}{2}\frac{(\mathbf{X}\times\mathbf{S}_\mathrm{E})^a}{R^3} - \Lambda_{,a}\,.$$

Note that this representation of the geocentric metric correctly yields the Schwarzschild and Lense-Thirring accelerations in satellite motion to first PN-order as recommended *e.g.*, by the IERS Conventions 2000.

3.5. The barycentric metric in the mass-monopole approximation

For many applications it is sufficient to keep the mass-monopoles of the various bodies only, *i.e.* to put

$$\mathcal{M}_L = 0 \quad \text{for} \quad l \geq 1, \qquad \mathcal{S}_L = 0 \quad \text{for} \quad l \geq 1$$

for all bodies and to keep the masses $M_A \equiv \mathcal{M}_A$ only. Furthermore, we will assume all moments $\mathcal{P}_L$ to vanish. From the transformation rules for the metric potentials one derives the metric in the barycentric coordinate system in the form

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}w_0(t,\mathbf{x}) - \frac{2}{c^4}(w_0^2(t,\mathbf{x}) + \Delta(t,\mathbf{x})), \\ g_{0i} &= -\frac{4}{c^3}w_i(t,\mathbf{x}), \\ g_{ij} &= \left(1 + \frac{2w_0(t,\mathbf{x})}{c^2}\right)\delta_{ij}\,, \end{aligned} \tag{19}$$

where

$$w_0(t,\mathbf{x}) \equiv \sum_A \frac{GM_A}{r_A}, \tag{20}$$

and

$$\Delta(t,\mathbf{x}) = \sum_B \Delta_B(t,\mathbf{x}), \tag{21}$$

with

$$\begin{aligned} \Delta_B(t,\mathbf{x}) = &\ \frac{GM_B}{r_B}\left(-\frac{3}{2}v_B^2 + \sum_{C\neq B}\frac{GM_C}{r_{CB}}\right) - \frac{1}{2}GM_B\, r_{B,tt} \\ = &\ \frac{GM_B}{r_B}\left[-2v_B^2 + \sum_{C\neq B}\frac{GM_C}{r_{CB}} + \frac{1}{2}\left(\frac{(r_B^k v_B^k)^2}{r_B^2} + r_B^k a_B^k\right)\right]\,. \end{aligned} \tag{22}$$

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Futhermore, in our approximation

$$w_i(t, \mathbf{x}) = \sum_B \frac{GM_B}{r_B} v_B^i \,. \tag{23}$$

Note, that the post-Newtonian Einstein-Infeld-Hoffmann equations of motion for a system of mass-monopoles that form the basis of modern solar system ephemerides can be derived from that form of the barycentric metric. Thus, the barycentric mass-monopole metric given above is already in use for the description of solar system dynamics.

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