ON THE MEDIANS OF A TRIANGLE IN HYPERBOLIC GEOMETRY

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1. In non-Euclidean geometry the three medians of a triangle $A_1A_2A_3$ (each joining a vertex A_i with the internal midpoint G_i of the opposite side) are concurrent; their common point is the centroid G. But the Euclidean theorem

$$\frac{GG_i}{A_iG_i} = \frac{1}{3},$$

which depends on similarity, does not hold. In what follows we make some remarks on this ratio, restricting ourselves to hyperbolic geometry.

In accordance with a procedure recommended by Coxeter (**1**, p. 229), we take $A_1A_2A_3$ as the triangle of reference for projective co-ordinates x_1, x_2, x_3; the equation of the absolute conic Ω then appears in the general form. For our purpose we take, moreover, G as the unit-point. The equation of Ω is now

$$(1)\quad x_1^2 + x_2^2 + x_3^2 + 2\cosh a_1 \,.\, x_2x_3 + 2\cosh a_2 \,.\, x_3x_1 + 2\cosh a_3 \,.\, x_1x_2 = 0,$$

where a_i is the length of the side opposite A_i. The tangential equation of Ω reads

$$(2)\quad \begin{aligned} &\sinh^2 a_1 \,.\, u_1^2 + \sinh^2 a_2 \,.\, u_2^2 + \sinh^2 a_3 \,.\, u_3^2 + 2(\cosh a_1 - \cosh a_2 \,.\, \cosh a_3)u_2u_3 \\ &+ 2(\cosh a_2 - \cosh a_3 \,.\, \cosh a_1)\, u_3u_1 + 2(\cosh a_3 - \cosh a_1 \,.\, \cosh a_2)u_1u_2 \\ &= 0. \end{aligned}$$

From A_i being inside Ω follows the inequality (**1**, p. 239)

$$(3)\quad \gamma \equiv 2\cosh a_1 \,.\, \cosh a_2 \,.\, \cosh a_3 - \cosh^2 a_1 - \cosh^2 a_2 - \cosh^2 a_3 + 1 > 0,$$

which is equivalent with the fact that a side of the triangle is less than the sum of the other two.

2. The median A_3G_3 has the equations $x_1 = x_2 = \lambda$, $x_3 = 1$, where λ is a parameter; for $\lambda = \infty, 0, 1$ we have the points G_3, A_3, G. The points of intersection S_1 and S_2 of the median and the absolute are given by the roots λ_1, λ_2 of the equation

$$(4)\quad 2\lambda^2(1 + b_3) + 2(b_1 + b_2)\lambda + 1 = 0,$$

where b_i is written for $\cosh a_i$. Both roots are negative. We put $\mu_i = -\lambda_i$, $\mu_2 > \mu_1$, $A_iG_i = z_i$, $GG_i = y_i$. Then

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$$z_3 = \tfrac{1}{2}\log(S_1S_2A_3G_3),\ y_3 = \tfrac{1}{2}\log(S_1S_2GG_3)$$

or

$$e^{2z_3} = \frac{\mu_2}{\mu_1}, \qquad e^{2y_3} = \frac{\mu_2+1}{\mu_1+1}.$$

Hence

$$(5) \qquad \sinh z_3 = \frac{\mu_2-\mu_1}{2\sqrt{[\mu_1\mu_2]}}, \qquad \sinh y_3 = \frac{\mu_2-\mu_1}{2\sqrt{[(\mu_2+1)(\mu_1+1)]}},$$
$$\cosh z_3 = \frac{\mu_2+\mu_1}{2\sqrt{[\mu_1\mu_2]}}, \qquad \cosh y_3 = \frac{\mu_2+\mu_1+2}{2\sqrt{[(\mu_2+1)(\mu_1+1)]}}$$

and

$$(6) \qquad \tanh y_3 = \frac{\mu_2-\mu_1}{\mu_2+\mu_1+2} = \frac{\sqrt{[\mu_1\mu_2]}\,.\sinh z_3}{\sqrt{[\mu_1\mu_2]}\cosh z_3+1}.$$

Furthermore,

$$(7) \qquad \mu_1\mu_2 = \lambda_1\lambda_2 = \frac{1}{2(1+b_3)} = \frac{1}{4\cosh^2\frac{1}{2}a_3},$$

and so we get the following formulae:

$$(8) \qquad (\mu_1+1)(\mu_2+1) = \frac{2(b_1+b_2+b_3)+3}{2(1+b_3)},$$

$$(9) \qquad \cosh z_3 = \frac{\cosh a_1 + \cosh a_2}{2\cosh\frac{1}{2}a_3},$$

$$(10) \qquad \frac{\sinh y_3}{\sinh z_3} = \frac{1}{\{2(b_1+b_2+b_3)+3\}^{\frac{1}{2}}},$$

$$(11) \qquad \tanh y_3 = \frac{\sinh z_3}{\cosh z_3 + 2\cosh\frac{1}{2}a_3}.$$

3. In (9) we have the well-known formula giving the length of a median as a function of the sides. From (10) it follows that

If A_iG_i are the medians of the triangle $A_1A_2A_3$, and G is the centroid, then

$$\frac{\sinh GG_1}{\sinh A_1G_1} = \frac{\sinh GG_2}{\sinh A_2G_2} = \frac{\sinh GG_3}{\sinh A_3G_3};$$

the common value of the three ratios is $\{2(\cosh a_1 + \cosh a_2 + \cosh a_3) + 3\}^{-\frac{1}{2}}$.

4. From (11) it is seen that y_3 is a function of z_3 and a_3 only. Therefore:

If for the triangle $A_1A_2A_3$ the base A_1A_2 and the length of the median A_3G_3 are given then GG_3 has a fixed value.

If for abbreviation we denote $p_i = 2\cosh\frac{1}{2}a_i$, we have (suppressing the index i):

$$(12) \qquad \tanh y = \frac{\sinh z}{\cosh z + p}.$$

Obviously $y = 0$ for $z = 0$. Furthermore, differentiating the formula we get

$$\frac{1}{\cosh^2 y} \cdot \frac{dy}{dz} = \frac{1 + p \cosh z}{(\cosh z + p)^2}$$

or

$$\frac{dy}{dz} = \frac{1 + p \cosh z}{1 + p^2 + 2p \cosh z}. \tag{13}$$

Hence dy/dz is an increasing function of z; for $z = 0$ we have

$$\frac{dy}{dz} = \frac{1}{1 + p};$$

its limit for $z \to \infty$ is $\frac{1}{2}$. Therefore:

If the base $A_1A_2 = a_3$ of the triangle is fixed, then GG_3/A_3G_3 increases if A_3G_3 increases and we have the inequality

$$\frac{1}{1 + 2 \cosh \frac{1}{2}a_3} < \frac{GG_3}{A_3G_3} < \tfrac{1}{2}. \tag{14}$$

As a consequence we have *for all triangles* the inequality

$$0 < \frac{GG_3}{A_3G_3} < \tfrac{1}{2}. \tag{15}$$

It follows from the proof that the limits in (14) and (15) cannot be sharpened.

5. The Euclidean value $\frac{1}{3}$ is between the limits given in (15). Therefore there are triangles for which

$$\frac{GG_3}{A_3G_3} = \frac{1}{3}.$$

If in (12) we put $z = 3y$, we get

$$\tanh y = \frac{\sinh 3y}{\cosh 3y + p}.$$

Substituting $\sinh 3y = \sinh y(4 \cosh^2 y - 1)$, $\cosh 3y = \cosh y(4 \cosh^2 y - 3)$, we get

$$\cosh\ y = \tfrac{1}{2}p = \cosh \tfrac{1}{2}a$$

Therefore: *In the triangle $A_1A_2A_3$ we have*

$$\frac{GG_3}{A_3G_3} = \frac{1}{3}$$

if and only if $GG_3 = \frac{1}{2}A_1A_2$; hence in the triangle A_1GA_2 the angle $\angle A_2GA_1$ is the sum of $\angle GA_1A_2$ and $\angle A_1A_2G$.

6. In such a triangle we have

$$z_3 = \frac{3}{2}a_3, \qquad \cosh z_3 = \cosh \frac{a_3}{2}\left(4 \cosh^2 \frac{a_3}{2} - 3\right)$$

and therefore, in view of (9)

(16) $$2\cosh^2 a_3 + \cosh a_3 - \cosh a_1 - \cosh a_2 - 1 = 0.$$

More generally, if

(17) $$k_3 = 2b_3^2 + b_3 - b_1 - b_2 - 1,$$

we have

$$\frac{y_3}{z_3} > \frac{1}{3}, \quad \frac{y_3}{z_3} = \frac{1}{3}, \quad \frac{y_3}{z_3} < \frac{1}{3}$$

according as $k_3 < 0$, $k_3 = 0$, $k_3 > 0$, respectively. If $b_1 = b_2 = b_3$ we have obviously $k_3 > 0$. Hence in an equilateral triangle the ratios y_i/z_i are less than $\frac{1}{3}$. We define k_1 and k_2 analogously to (17). If we put $c_i = b_i + 1$ we get

$$k_1 = 2c_1^2 + 5c_1 - c_2 - c_3.$$

Hence $k_1 + k_2 + k_3 = 2(c_1^2 + c_2^2 + c_3^2) + 3(c_1 + c_2 + c_3) > 0$, since $c_i > 0$. Therefore $k_1 = k_2 = k_3 = 0$ is impossible: There are no triangles for which the three ratios y_i/z_i are $\frac{1}{3}$.

We have

(18) $$k_1 - k_2 = 2(c_1 - c_2)(c_1 + c_2 + 3).$$

If $k_1 = k = 0$, then $c_1 = c_2$, $b_1 = b_2 = b$, $b_3 = 2b^2 - 1$; but then γ is zero and the inequality (3) is not satisfied. Therefore,

There are no triangles for which two ratios y_i/z_i are $\frac{1}{3}$.

From (18) it follows that $k_1 > k_2$ inplies $c_1 > c_2$ (so that $a_1 > a_2$) and conversely.

We have established the existence of triangles for which one of the ratios y_i/z_i is $\frac{1}{3}$. Suppose $k_3 = 0$. Then $c_1 + c_2 = 2c_3^2 + 5c_3$. Moreover, we have

$$\gamma = 2c_1c_2c_3 + 2c_3(c_1 + c_2) - (c_1 + c_2)^2 + 4c_1c_2 - c_3^2 > 0$$

or

$$c_1c_2(c_3 + 2) > 2c_3^4 + 8c_3^3 + 8c_3^2,$$

that is,

$$c_1c_2 > 2c_3^2(c_3 + 2).$$

It follows from this that

$$(c_1 - c_2)^2 < c_3^2(2c_2 + 3)^2.$$

Therefore, assuming $c_1 \geqslant c_2$, we have $c_1 - c_2 < c_3(2c_3 + 3)$, and from $c_1 + c_2 = 2c_3^2 + 5c_3$ it follows that $c_1 \geqslant c_2 > c_3$. Hence

If in a triangle

$$\frac{y_i}{z_i} = \frac{1}{3}$$

then a_i is smaller than each of the two other sides.

In view of all this we have: *In a triangle either all three ratios* y_i/z_i *are less than* $\frac{1}{3}$ *or two of them are* $<\frac{1}{3}$ *and the third* (*belonging to the smallest side*) $\geqslant \frac{1}{3}$.

REFERENCE

1. H. S. M. Coxeter, *Non-Euclidean Geometry* (3rd ed.; Toronto, 1957).

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