

ON FROBENIUS EXTENSIONS II

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In Part I¹⁾ we introduced the notion of 2. Frobenius extensions of a ring, as a generalization of Kasch's [10] Frobenius extensions and hence of classical Frobenius algebras. We proved, in I, bilinear (or sesqui-linear, rather, to follow Bourbaki's terminology) form and scalar product characterizations of Frobenius extensions in such extended sense, generalizing Kasch's and classical case, and then studied homological dimensions in them, generalizing and refining the results in Eilenberg-Nakayama [4] and Hirata [6]. Dual bases were considered in case of quasi-free (2.) Frobenius extensions. Also the case of a semi-primary or S -ring ground ring was studied.

In the present Part II we continue our study of Frobenius extensions in such generalized sense. Thus we first study relative homological dimensions in them, generalizing the Maschke-Ikeda-Kasch characterization of relatively projective and relatively injective modules as well as Hirata's [6] results. Then in §7 we establish Kasch's [10] theorem on the endomorphism ring of a Frobenius ring for our generalized case. Here the removal of Kasch's S -ring assumption (which we have already discussed in our previous note [13]) and the replacement of free module property with projective or quasi-free ones make our proof more complicated, respectively in substance and in computation, than Kasch's case.

Then, in §8, we transfer to the present case the annihilator relations given in [17] for classical Frobenius algebras, on restricting ourselves to d -ideals (similar to (but slightly more general than) v -ideals in Kasch [10]). Further, in §9 we consider residue-rings of a Frobenius extension, in order to study when they are also Frobenius extensions. In these considerations Frobenius extensions are naturally taken in our generalized sense and thus deviations, some rather essential and some rather formal, from (the classical case and)

Received May 12, 1961.

¹⁾ T. Nakayama-T. Tsuzuku, On Frobenius extensions I, Nagoya Math. J. 17 (1960), 89–110.

Kasch's [10] case are needed. In §10 the case of a quasi-Frobenius ground ring is considered. Thus, Hall's [15] theorem (on the ordinary scalar product in vector-modules over a quasi-Frobenius ring) is observed to be transferable to the case of a regular sesqui-linear scalar product in projective modules, and this is combined with §9 to consider residue-rings of (quasi-Frobenius) Frobenius extensions of a quasi-Frobenius ring.

§ 6. Relative homological dimensions in Frobenius dimensions.

First we proceed in parallel with I, §3, generalizing Hirata [6]. Let A be a ring (with unit element) and B a subring of A (containing the unit element of A).

LEMMA 17. *Let \mathcal{Q} be an A -left-module and m a B -left-module. Then*

$$(1) \quad \text{Ext}_{(A, B)}^q(\mathcal{Q}, A \otimes_B m) = \begin{cases} 0 & q > 0, \\ \text{Hom}_B(\mathcal{Q}, (\beta^{-1}, m)) & q = 0, \end{cases}$$

$$(2) \quad \text{Ext}_{(A, B)}^q(\text{Hom}_B(A, m), \mathcal{Q}) = \begin{cases} 0 & q > 0 \\ \text{Hom}_B((\beta^{-1}, m), \mathcal{Q}) & q = 0, \end{cases}$$

Let further \mathfrak{R} be an A -right-module. Then

$$(3) \quad \text{Tor}_{(A, B)}^q(\mathfrak{R}, \text{Hom}_B(A, m)) = \begin{cases} 0 & q > 0 \\ \mathfrak{R} \otimes_B (\beta, m) & q = 0. \end{cases}$$

Proof of (1), for instance, runs similarly as in Lemma 7. Thus we recall the isomorphism

$$(4) \quad \text{Hom}_A(\mathcal{Q}, A \otimes_B m) \simeq \text{Hom}_B(\mathcal{Q}, (\beta^{-1}, m))$$

derived there from the isomorphism I, (18), i.e. $A \otimes_B m \simeq \text{Hom}_{R, \beta^{-1}}({}_B A, m)$. (4) proves (1) for $q = 0$. On replacing \mathcal{Q} with its (A, B) -projective resolution \mathfrak{X} and passing to homology we obtain

$$\text{Ext}_{(A, B)}^q(\mathcal{Q}, A \otimes_B m) \simeq H^q(\text{Hom}_B(\mathfrak{X}, (\beta^{-1}, m)))$$

and the right-hand side is 0 for $q > 0$, since \mathfrak{X} B -splits.

Analogously, (2), (3) may be proved similarly as Lemmas 7', 7'', again using I, (18).

THEOREM 18. *Let A be a β -Frobenius extension of B . If \mathcal{Q} is an A -left-module and if $1.\dim_{(A, B)} \mathcal{Q} < \infty$, then $1.\dim_{(A, B)} \mathcal{Q} = 0$. The same holds with*

$\text{l.dim}_{(A, B)}$ replaced either by $\text{l.inj.dim}_{(A, B)}$ or $\text{l.w.dim}_{(A, B)}$.

Proof. Let $\text{l.dim}_{(A, B)} \mathcal{Q} = n < \infty$. Let \mathfrak{M} be an A -left-module such that $\text{Ext}_{(A, B)}^n(\mathcal{Q}, \mathfrak{M}) \neq 0$ and

$$0 \rightarrow \mathfrak{N} \rightarrow \mathfrak{F} \rightarrow \mathfrak{M} \rightarrow 0$$

be exact with \mathfrak{F} A -left-free. Since $\text{Ext}_{(A, B)}^{n+1}(\mathcal{Q}, \mathfrak{N}) = 0$ it follows that the induced homomorphism $\text{Ext}_{(A, B)}^n(\mathcal{Q}, \mathfrak{F}) \rightarrow \text{Ext}_{(A, B)}^n(\mathcal{Q}, \mathfrak{M})$ is epimorphic. Since here \mathfrak{F} has a form $\mathfrak{F} = A \otimes_B \mathfrak{M}$ with a B -left(\cdot -free) module \mathfrak{M} , $\text{Ext}_{(A, B)}^n(\mathcal{Q}, \mathfrak{F}) = 0$ if $n > 0$, by our lemma. Hence $\text{Ext}_{(A, B)}^n(\mathcal{Q}, \mathfrak{M}) = 0$ too if $n > 0$. This implies that n must be 0.

The other parts of the theorem can be seen analogously.

COROLLARY 19. *Let A be a 2-Frobenius extension of B . Then*

$$\begin{aligned} \text{l. (resp. r.) gl.dim}(A, B) &= 0 && \text{or } \infty, \\ \text{w. gl.dim}(A, B) &= 0 && \text{or } \infty. \end{aligned}$$

Remark. In analogy to this corollary, we could state the following corollary to Theorems 8, 8', 8'', 9, 9', 9'' in I, § 3:

COROLLARY 9 a. *Let A be 2-Frobenius over B and assume that B is γ -1-isomorphic to a direct summand of the B - B -module A . Then*

$$\begin{aligned} \text{l. (resp. r.) gl.dim } A &= \text{l. (resp. r.) gl.dim } B && \text{or } \infty \\ \text{w. gl.dim } A &= \text{w. gl.dim } B && \text{or } \infty. \end{aligned}$$

The equation $\text{l.dim}_{(A, B)} \mathcal{Q} = 0$ (which we have, in case A is 2-Frobenius over B , seen to be the case whenever $\text{l.dim}_{(A, B)} \mathcal{Q} < \infty$) means that \mathcal{Q} is (relatively) (A, B) -projective. It is well known (cf. Hochschild [7]) that with any B -left-module \mathfrak{M} the A -left-module $A \otimes_B \mathfrak{M}$ is (A, B) -projective and conversely any (A, B) -projective A -left-module \mathcal{Q} is a direct summand of an A -left-module of form $A \otimes_B \mathfrak{M}$, for example $A \otimes_B \mathcal{Q}$. On the other hand, in case A is 2-Frobenius over B , Lemma 17, (1) shows that $A \otimes_B \mathfrak{M}$ is (A, B) -injective and hence any (A, B) -projective A -left-module is (A, B) -injective too. The converse is seen from (a well known general fact on (A, B) -injective modules and) Lemma 17, (2). Thus

THEOREM 20. *A left-module \mathcal{Q} over a 2-Frobenius extension A of B is (A, B) -projective if and only if it is (A, B) -injective.*

The above mentioned fact, that an (A, B) -projective A -left-module \mathcal{Q} is a direct summand of $A \otimes_B \mathcal{Q}$, is seen by considering the A -epimorphism ρ of $A \otimes_B \mathcal{Q}$ to \mathcal{Q} defined in

$$(5) \quad \rho(\sum a_i \otimes l_i) = \sum a_i l_i \quad (a_i \in A, l_i \in \mathcal{Q})$$

and observing that $\text{Ker } \rho$ is a direct summand of $A \otimes_B \mathcal{Q}$ as a B -left-module (as we see by the B -monomorphism $l \rightarrow 1 \otimes l$ of \mathcal{Q} into $A \otimes_B \mathcal{Q}$). The last is the case generally, and \mathcal{Q} is (A, B) -projective if and only if $\text{Ker } \rho$ is a direct summand of the A -left-module $A \otimes_B \mathcal{Q}$. Similarly the (A, B) -injectivity of an A -left-module \mathcal{Q} is reduced to the image of a certain canonical A -monomorphism $\mathcal{Q} \rightarrow \text{Hom}_B(A, \mathcal{Q})$ being a direct summand of the A -left-module $\text{Hom}_B(A, \mathcal{Q})$.

In case A is β -Frobenius over B , we have an A -isomorphism $A \otimes_B(\beta, \mathcal{Q}) \simeq \text{Hom}_{B, \beta^{-1}}(A, (\beta, \mathcal{Q})) = \text{Hom}_B(A, \mathcal{Q})$, by I, (18) (used above repeatedly). In combination with the above remark it follows that the (A, B) -injectivity too can then be characterized by means of a tensor product, $A \otimes_B(\beta, \mathcal{Q})$, and indeed by that the image of a certain canonical A -monomorphism $\mathcal{Q} \rightarrow A \otimes_B(\beta, \mathcal{Q})$ is a direct summand of $A \otimes_B(\beta, \mathcal{Q})$ as A -module.

This last we have seen without making use of Theorem 20 (If we use it, it is clear that the (A, B) -injectivity of \mathcal{Q} is, in case A is 2.Frobenius over B , characterized by means of the tensor product $A \otimes_B \mathcal{Q}$, indeed by that the kernel of a canonical A -epimorphism $A \otimes_B \mathcal{Q} \rightarrow \mathcal{Q}$ is a direct summand of $A \otimes_B \mathcal{Q}$ as A -module.) In fact Theorem 20 may be derived directly from the above remark (and its dual) readily. For, an (A, B) -injective module \mathcal{Q} is a direct summand of $A \otimes_B(\beta, \mathcal{Q})$ as A -module, by the above remark. The last module is, being of form $A \otimes_B m, (A, B)$ -projective, and hence its direct summand \mathcal{Q} is so too. This proves one half of Th.20, and the other half is seen dually.

Now, if A is further quasi-free over B , Theorem 20 can be made more explicit in

THEOREM 21. *Let A be quasi-free β -Frobenius over B and let u_ν, v_ν be as in Prop. 10. Then an A -left-module \mathcal{Q} is (A, B) -projective if and only if there is a (B, β) -epimorphism ι of \mathcal{Q} such that on \mathcal{Q} we have*

$$(6) \quad \sum_\nu v_\nu \iota u_\nu = 1$$

The same condition is necessary and sufficient for the (A, B) -injectivity too.

Proof. Suppose that \mathcal{Q} is (A, B) -projective, i.e. that $\text{Ker } \rho$ (which is always a direct summand of $A \otimes_B \mathcal{Q}$ as B -module) is a direct summand of $A \otimes_B \mathcal{Q}$ as A -module. Then there is a direct summand A -submodule \mathcal{Q}_0 of $A \otimes_B \mathcal{Q}$ which is A -isomorphic to \mathcal{Q} (in fact $A \otimes_B \mathcal{Q} = \mathcal{Q}_0 \oplus \text{Ker } \rho$). Let λ be the projection of $A \otimes_B \mathcal{Q}$ onto \mathcal{Q}_0 , with respect to a direct decomposition of $A \otimes_B \mathcal{Q}$ into \mathcal{Q}_0 and a second A -submodule (which may be $\text{Ker } \rho$, but not necessarily). Let, on the other hand, κ be the (B, β^{-1}) -endomorphism of $A \otimes_B \mathcal{Q}$ defined by

$$(7) \quad \kappa(\sum x_i \otimes l_i) = 1 \otimes \sum (\beta^{-1} \pi x_i) l_i \quad (x_i \in A, l_i \in \mathcal{Q})$$

(i.e. $\kappa = \beta^{-1} \pi \otimes 1$). As an operator on $A \otimes_B \mathcal{Q}$ we have

$$\sum_{\nu} v_{\nu} \kappa u_{\nu} = 1.$$

For, $(\sum_{\nu} v_{\nu} \kappa u_{\nu})(x \otimes l) = \sum_{\nu} v_{\nu} \otimes (\beta^{-1} \pi u_{\nu} x) l = \sum_{\nu} v_{\nu} \beta^{-1} \pi u_{\nu} x \otimes l =$
 $\sum_{\nu} v_{\nu} \beta^{-1} \pi \sum_{\mu} (\beta y_{\nu \mu}) u_{\mu} \otimes l = \sum_{\mu} \sum_{\nu} v_{\nu} \beta^{-1} (\beta y_{\nu \mu}) \pi u_{\mu} \otimes l = \sum_{\mu} \sum_{\nu} v_{\nu} y_{\nu \mu} \beta^{-1} \pi u_{\mu} \otimes l =$
 $\sum_{\mu} x v_{\mu} \beta^{-1} \pi u_{\mu} \otimes l = x \otimes l$

by I, (42), (41) and (40). Since λ is A -homomorphic, it follows that $\sum_{\nu} v_{\nu} \lambda \kappa u_{\nu} = \lambda \sum_{\nu} v_{\nu} \kappa u_{\nu} = \lambda$ and induces the identity map on \mathcal{Q}_0 . On denoting by ι the (B, β) -endomorphism of \mathcal{Q} which corresponds to $\lambda \kappa$ on \mathcal{Q}_0 , by an A -isomorphism of \mathcal{Q} and \mathcal{Q}_0 , we obtain (6).

Now, let (6) be the case. Let \mathfrak{M} be an A -left-module having \mathcal{Q} as an A -submodule such that \mathfrak{M} is the direct sum of \mathcal{Q} and a B -submodule. Let λ_1 be the projection of \mathfrak{M} onto \mathcal{Q} , with respect to this direct decomposition of \mathfrak{M} , and let λ_2 be the map $\sum_{\nu} v_{\nu} \iota \lambda_1 u_{\nu}$ on \mathfrak{M} . λ_2 maps \mathfrak{M} A -homomorphically into \mathcal{Q} , as the relations I, (41), (42) and the (B, β) -homomorphic feature of ι show. Further, λ_2 induces the identity map on \mathcal{Q} , as the relation (6) entails. Thus $\mathfrak{M} = \mathcal{Q} \oplus \text{Ker } \lambda_2$ is a direct A -decomposition and this shows that \mathcal{Q} is (A, B) -injective.

Lastly, assume that \mathcal{Q} is (A, B) -injective. Then \mathcal{Q} is (A, B) -projective, by Theorem 20.

This closes a circuit and proves our Theorem 21. (The last step in the above circuit, concluding the (A, B) -projectivity from the (A, B) -injectivity, is seen also by means of $A \otimes_B (\beta, \mathcal{Q})$, as was remarked above. An A -monomorphism of \mathcal{Q} into $A \otimes_B (\beta, \mathcal{Q})$, used in that argument, may be given, in case A is quasi-free β -Frobenius over B , by

$$(2) \quad \sigma: l \rightarrow \sum_{\nu} v_{\nu} \otimes (u_{\nu} l)' \in A \otimes_B (\beta, \mathcal{Q}) \quad (l \in \mathcal{Q}),$$

where l' denotes the element of (β, \mathcal{Q}) corresponding to an element l of \mathcal{Q} . For, the A -homomorphic feature of σ may be seen in

$$\begin{aligned} \sigma xl &= \sum_{\nu} v_{\nu} \otimes (u_{\nu} xl)' = \sum_{\mu, \nu} v_{\nu} \otimes ((\beta y_{\nu\mu}) u_{\nu})' \\ &= \sum_{\mu, \nu} v_{\nu} \otimes y_{\nu\mu} (u_{\nu} l)' = \sum_{\mu, \nu} v_{\nu} y_{\nu\mu} \otimes (u_{\nu} l)' \\ &= \sum_{\mu} x v_{\mu} \otimes (u_{\nu} l)' = x \sigma l, \end{aligned}$$

$y_{\nu\mu}$ being as in I, (41), (42). That $\text{Im } \sigma$ is a direct summand of $A \otimes_B (\beta, \mathcal{Q})$ as mere B -module, is seen by means of the B -homomorphism σ_1 of $A \otimes_B (\beta, \mathcal{Q})$ to \mathcal{Q} defined by $\sigma_1(\sum a_i \otimes l'_i) = \sum (\pi a_i) l_i$ (observe that π is $(B, 1)$ - (B, β) -homomorphic) and by observing that $\sigma_1 \sigma l = \sigma_1(\sum v_{\nu} \otimes (u_{\nu} l)') = (\pi v_{\nu}) u_{\nu} l = l$ (by I, (40)), i.e. that $\sigma_1 \sigma$ is identity on \mathcal{Q} . This explicit way and the similarly explicit construction in the proof of Theorem 21 form the generalization of the Maschke-Ikeda-Kasch [10] argument to our present case of a quasi-free 2.Frobenius extension.)

§ 7. Endomorphism ring.

With a ring A having a subring B , consider the B -rightendomorphism ring \mathcal{E} of A ; the elements of \mathcal{E} are considered as left-operators on A . The ring A_l of left-multiplications of elements of A , on A , is a subring; with a subset S of A we denote by S_l the set of left-multiplications of elements of S on A .

We prove the following refinement of Kasch [10], Satz 5 and Nakayama-Tsuzuku [13], Theorem :

THEOREM 22. *Let A be an extension ring of a ring B . Let β be an automorphism of B which can be extended to an automorphism, say α , of A , and denote by α' the automorphism of the left-multiplication ring A_l of A corresponding to α^{-1} . If A is quasi-free β -Frobenius over B , then the ring \mathcal{E} of B -right-endomorphisms of A is quasi-free α' -Frobenius over A_l . Conversely, if \mathcal{E} is α' -Frobenius over A_l , then A is quasi-free β -Frobenius over B , provided that A is finitely generated B -left-quasi-free.*

Proof. We first prove the second half of the theorem. Thus, let

$$(9) \quad A = v_1 B + \cdots + v_n B$$

be, as in I, (33), a direct decomposition of A into B -right-submodules $v_\nu B$ such that $v_\nu B \approx e_\nu B$ (B -right), $v_\nu e_\nu = v_\nu$, where e_ν are idempotents in B . Let E_ν be, for each $\nu = 1, \dots, n$, the B -right-endomorphism of A such that

$$(10) \quad E_\nu v_\nu = e_\nu (\in B \subset A), \quad E_\nu v_\mu = 0 \quad (\mu \neq \nu).$$

As v_ν is mapped by any B -right-endomorphism of A onto an element in Ae_ν , we readily see the direct decomposition

$$(11) \quad \mathfrak{E} = (Ae_1)_l E_1 + \dots + (Ae_n)_l E_n.$$

By the assumption that \mathfrak{E} is α' -Frobenius over A_l there exists, according to Prop. 4, an $(A_l, 1)$ - (A_l, α') -homomorphism A of \mathfrak{E} to A_l such that $A\mathfrak{E}\varphi = 0$ ($\varphi \in \mathfrak{E}$) implies $\varphi = 0$.

Set $w_{\nu l} = AE_\nu(w_\nu \in A, \text{ in fact } \in e_\nu A)$. We contend that a relation

$$(12) \quad y_{1l}w_{1l} + \dots + y_{nl}w_{nl} = 0 \quad (y_\nu \in Be_\nu)$$

can hold only when $y_1 = \dots = y_n = 0$. To prove this, set

$$x_\nu = v_1 y_\nu + \dots + v_n y_\nu = (v_1 + \dots + v_n)y_\nu \in Ae_\nu$$

and consider the elements

$$X_\mu = E_\mu(x_{1l}E_1 + \dots + x_{nl}E_n)$$

of \mathfrak{E} . For each μ, ν and $b \in B$ we have

$$\begin{aligned} X_\mu v_\nu b &= E_\mu(x_{1l}E_1 + \dots + x_{nl}E_n)v_\nu b \\ &= E_\mu x_\nu e_\nu b = E_\nu x_\nu b = E_\nu(v_1 + \dots + v_n)y_\nu b = e_\nu y_\nu b. \end{aligned}$$

Since this is the case for every $\nu = 1, \dots, n$, we see

$$X_\mu = (e_\mu y_{1l})_l E_1 + \dots + (e_\mu y_{nl})_l E_n = e_{\mu l}(y_{1l}E_1 + \dots + y_{nl}E_n).$$

Hence

$$AX_\mu = A(y_{1l}E_1 + \dots + y_{nl}E_n)e_{\mu l} = (y_{1l}w_{1l} + \dots + y_{nl}w_{nl})\alpha e_{\mu l}$$

and this is 0 by the assumed relation (12). Then $Ax_l X_\mu = x_l AX_\mu = 0$ for any $x \in A$, i.e.

$$Ax_l E_\mu(x_{1l}E_1 + \dots + x_{nl}E_n) = 0$$

for any $x \in A$. As this holds for every $\mu = 1, \dots, n$, we have, in view of (11),

$$A\mathfrak{E}(x_{1l}E_1 + \dots + x_{nl}E_l) = 0$$

and hence

$$x_{1l}E_1 + \dots + x_{nl}E_n = 0.$$

As $x_\nu \in Ae_\nu$, we have then $x_\nu = 0$, for each $\nu = 1, \dots, n$. Then $y_\nu = 0, \nu = 1, \dots, n$, as was asserted.

So we obtain a direct sum submodule

$$(Be_1)_l w_{1l} + \dots + (Be_n)_l w_{nl}$$

of A_l , where each summand $(Be_\nu)_l w_{\nu l}$ is B_l -left-isomorphic to $B_l e_\nu$; remember that B_l is the set of the left-multiplications of the elements of B onto A (not onto B).

Next we prove that the sum

$$(13) \quad B_l E_1 + \dots + B_l E_n$$

is a right-ideal in $\mathbb{C} = A_l E_1 + \dots + A_l E_n$ ((11)). Namely, with $x = v_1 y_1 + \dots + v_n y_n \in A(y_\nu \in e_\nu B)$ and $y \in B$ we have

$$\begin{aligned} y_l E_\kappa x_l E_\nu v_\mu b &= \delta_{\mu\nu} y_l E_\kappa x_l e_\nu b = \delta_{\mu\nu} y_l E_\kappa (v_1 y_1 + \dots + v_n y_n) e_\nu b \\ &= \delta_{\mu\nu} y_l e_\kappa y_\kappa e_\nu b = \delta_{\mu\nu} y_l y_\kappa e_\nu b \end{aligned}$$

for any $b \in B$ and κ , which means

$$y_l E_\kappa x_l E_\nu = (y y_\kappa)_l E_\nu \in B_l E_\nu.$$

This shows $B_l E_\kappa \mathbb{C} \subseteq B_l E_1 + \dots + B_l E_n$, for any κ , and therefore (13) is a right-ideal of \mathbb{C} .

As Λ is (A_l, α') -right-homomorphic, it follows that $\Lambda(B_l E_1 + \dots + B_l E_n) = B_l w_{1l} + \dots + B_l w_{nl}$ is a right-ideal of A_l and indeed an B_l -left, A_l -right-submodule of A_l .

Another consequence of the right-ideal property of $B_l E_1 + \dots + B_l E_n$ is, as we see by virtue of Prop. 4, ii) (with A, B, β replaced by \mathbb{C}, A_l, α'), that by any (A_l, α'^{-1}) -left-homomorphism of \mathbb{C} into A_l every element in $B_l E_1 + \dots + B_l E_n$ is mapped into $\alpha'^{-1} \Lambda(B_l E_1 + \dots + B_l E_n) = \alpha'^{-1} (B_l w_{1l} + \dots + B_l w_{nl})$. But, on the other hand, we see by (11) the existence of an (A_l, α'^{-1}) -left-homomorphism of \mathbb{C} into A_l mapping E_ν onto an arbitrarily given element of $\alpha'^{-1} (Ae_\nu)_l$. Hence we have $\alpha'^{-1} (B_l w_{1l} + \dots + B_l w_{nl}) \supseteq \alpha'^{-1} (Ae_\nu)_l$, or equivalently,

$$B_l w_{1l} + \dots + B_l w_{nl} \supseteq (Ae_\nu)_l.$$

Since the left-hand side is a right-ideal of A_l , as observed above, we have

$B_l w_{l1} + \dots + B_l w_{ln} \supseteq (Ae_\nu A)_l$, or, what is the same

$$Bw_1 + \dots + Bw_n \supseteq Ae_\nu A.$$

As this holds for every $\nu = 1, \dots, n$ and as $Ae_1 A + \dots + Ae_n A = A$ by (9) (and $v_\nu e_\nu = v_\nu$), we obtain

$$(14) \quad A = Bw_1 + \dots + Bw_n$$

where, as we proved above with respect to A_l in stead of A itself, the sum is direct and each Bw_ν is B -left-isomorphic to Be_ν .

Now we wish to show that the elements $u_\nu = \alpha w_\nu$, v_ν are in the dual relationship as in Prop. 11. Thus, if, with an element x of A , we have I, (41), i.e.

$$xv_\nu = \sum_{\mu=1}^n v_\mu y_{\mu\nu} \quad (y_{\mu\nu} \in e_\mu B e_\nu)$$

then we have $E_\mu x_l = \sum_\nu y_{\mu\nu l} E_\nu$ as is seen from

$$E_\mu x_l v_\nu b = E_\mu x v_\nu b = E_\mu \sum_\nu v_\mu y_{\mu\nu} b = e_\mu y_{\mu\nu} b = y_{\mu\nu} b \quad (b \in B).$$

Applying A we have $w_{\mu l} \alpha' x_l = \sum_\nu y_{\mu\nu l} w_{\nu l}$ or $(\alpha'^{-1} w_{\mu l}) x_l = \sum_\nu (\alpha'^{-1} y_{\mu\nu l}) \alpha'^{-1} w_{\nu l}$.

Setting $u_\mu = \alpha w_\mu$ ($\mu = 1, \dots, n$) we obtain

$$u_\mu x = \sum_{\nu=1}^n (\beta y_{\mu\nu}) u_\nu,$$

i.e. I, (42); observe that β is the restriction to B of the automorphism α of A which in turn corresponds to the automorphism α'^{-1} of A_l . Here $(\beta e_\nu) u_\nu = (\alpha e_\nu) u_\nu = u_\nu$ and Bu_ν is B -left-isomorphic to $B\beta e_\nu$, since $e_\nu w_\nu = u_\nu$ and Bw_ν is B -left-isomorphic to Be_ν . Further we have

$$A = Bu_1 + \dots + Bu_n$$

and the sum is direct, as again the application of α^{-1} reduces the case to that of w_ν .

So A is β -Frobenius over B , and the second half of our theorem is proved. The verification of the first half is rather straight-forward. Thus, assume that A is quasi-free β -Frobenius over B . We have I, (33), (35), in notations described in Prop 10, and I, (41), (42) entailing each other as in Prop. 11. Writing I, (33) in the form (9) (as Prop. 10 tells) and defining $E_\nu \in \mathfrak{E}$ as in (10), we obtain the direct decomposition (11). By the assumption of our theorem β is

extended to an automorphism α of A , and we denote also by α the automorphism of A_l naturally induced by it. To each element X of \mathfrak{E} we associate an element φ_X of $\text{Hom}_{A_l, \alpha}(A_l \mathfrak{E}, A_l A_l)$ defined by

$$(15) \quad \varphi_X Y = \sum_{\nu=1}^n (\alpha Y X v_\nu)_l u_{\nu l} \quad (Y \in \mathfrak{E}).$$

with our “dual basic elements” u_ν, v_ν . Then the map $X \rightarrow \varphi_X$ is an A_l -right-homomorphism as we see from the relations I, (41), (42). The map is evidently \mathfrak{E} -left-homomorphic too.

We wish to show that the map is an isomorphism. Let, for this purpose, $X = x_{1l} E_1 + \dots + x_{nl} E_n$ ($x_\nu \in A e_\nu$) be a non-zero element of \mathfrak{E} . There exists an index, say ν_0 , with $x_{\nu_0} \neq 0$. If we write $x_\nu = \sum_{\mu=1}^n v_\mu z_{\mu\nu}$ ($z_{\mu\nu} \in e_\mu B$) for each ν , then $z_{\mu_0 \nu_0} \neq 0$ for some μ_0 . We have

$$\begin{aligned} \varphi_X E_{\nu_0} &= \sum_{\nu=1}^n (\alpha E_{\nu_0} X v_\nu)_l u_{\nu l} = \sum_{\nu=1}^n (\alpha E_{\nu_0} x_\nu)_l u_{\nu l} = \sum_{\nu=1}^n (\alpha z_{\nu_0 \nu})_l u_{\nu l} \\ &= (\sum_{\nu=1}^n (\beta z_{\nu_0 \nu})_l u_\nu)_l \neq 0. \end{aligned}$$

Hence $\varphi_X \neq 0$, and this proves that our map is monomorphic.

Further, for arbitrarily given elements $b_\nu \in B e_\nu$ ($\nu = 1, \dots, n$), set

$$X = \sum_{\nu=1}^n (v_\nu b_\nu)_l E_\nu.$$

Then

$$(16) \quad \varphi_X E_\mu = \sum_{\nu=1}^n (\alpha E_\mu X v_\nu)_l u_{\nu l} = \sum_{\nu=1}^n (\alpha E_\mu v_\nu b_\nu)_l u_{\nu l} = \begin{cases} \sum_{\nu=1}^n (\alpha e_1 b_\nu)_l u_{\nu l} = ((\beta e_1) \sum_{\nu=1}^n (\beta b_\nu)_l u_\nu)_l & \text{for } \mu = 1, \\ 0 & \text{for } \mu \neq 1 \end{cases}$$

Similarly, for each ν and for every element x of A there exists an X in \mathfrak{E} such that $\varphi_X E_\mu = ((\beta e_\nu) x)_l$ or 0 according as $\mu = \nu$ or $\mu \neq \nu$. This proves that our map $X \rightarrow \varphi_X$ is epimorphic too.

Thus $\mathfrak{E} \mathfrak{E}_{A_l} \approx \text{Hom}_{A_l, \alpha}(A_l \mathfrak{E}, A_l A_l)$ and \mathfrak{E} is (quasi-free) α' -Frobenius over A_l , where α' denotes, as in our theorem, the inverse of the automorphism α of A_l . The first half of our theorem is thus proved too.

Remark. The necessity of the rather strong assumption on the extendability (to α) of the automorphism β , in our theorem, seems to come from

the midway feature of our definition of 2.Frobenius extensions mentioned already in a Remark at the end of I, § 1. We intends to come to this point in a subsequent paper.

Remark. Kasch [10] uses his Satz 5 (which we have generalized to our Theorem 22) to prove that if \mathfrak{G} is a semiregular automorphism group (in the sense of [18]) of a simple ring with minimum condition A (which means that the totality of inner automorphisms of A contained in \mathfrak{G} forms an (invariant) subgroup \mathfrak{G}_0 of finite index in \mathfrak{G} and is indeed the set of inner automorphisms of A induced by (all) the regular elements of a semisimple subring T of A having finite rank over the center C of A) then A is a Frobenius extension of the fixed subring B of \mathfrak{G} . As Kasch [10] briefly remarks, the same holds with a complete automorphism group of (a simple ring with minimum condition) A whose subgroup of inner automorphisms is of finite index and is induced by (all) the regular elements of a subring T of A Frobenius over C , the center of A , provided that the product of T_r and A_l is a tensor product over $C_r (= C_l)$ and A is projective with respect to the endomorphism ring of A generated by \mathfrak{G} and A_l ; Curtis [14] discusses when this last is the case.

One might try to make a further generalization by replacing here "Frobenius" with our "2.Frobenius". But this would be rather meaningless. For, firstly, since C is a field, every 2.Frobenius extension T of C is free and, moreover, since C should naturally be in the center of T , T must be 1.Frobenius. Namely, if A is a β -Frobenius extension of a ring B and if B is contained in the center of A , then necessarily $\beta = 1$. For, there is, by Prop. 4, a $(B, 1)$ - (B, β) -homomorphism π of A into B such that $\pi a A = 0$ ($a \in A$) entails $a = 0$. So, with $b \in B$, $x \in A$, we have, since $b, \beta b$ lie in the center of A , $\pi b x = \pi x b = (\pi x) \beta b = (\beta b) \pi x = \pi(\beta b) x$, $\pi(b - \beta b) x = 0$. As x is arbitrary in A , $\pi(b - \beta b) A = 0$ and this entails $b - \beta b = 0$. It follows that $\beta = 1$.

(If we should consider a Galois theory of non-simple, and perhaps non-semisimple, rings, then it would probably be useful to consider "non-free 1.Frobenius" case. However, as the development of Galois theory for such general rings seems still to be at an unsatisfactory stage (except with the outer Galois theory), we shall not go into such a consideration here).

In this context we want to remark, however, that the field property makes no hindrance for having a non-1.Frobenius 2.Frobenius extension. As the

examples given in I, §2 were with respect to non-field subrings, we thus give here

Example 3. Let K be a field having an involutive automorphism ($\neq 1$) $\xi \rightarrow \bar{\xi}$, and let A be the ring of all matrices of form

$$x = \begin{pmatrix} \xi & 0 \\ \zeta & \bar{\xi} \end{pmatrix} \quad (\xi, \zeta \in K)$$

in K . Let B be the subring of A consisting of all x with $\zeta = 0$. Clearly B is a field isomorphic to K , and has an (involutive) automorphism β mapping $\begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix}$ to $\begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}$. Now, set

$$\pi x = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \quad (\in B).$$

If $x_1 = \begin{pmatrix} \xi_1 & 0 \\ \zeta & \bar{\xi}_1 \end{pmatrix}$, then $xx_1 = \begin{pmatrix} \xi\xi_1 & 0 \\ \zeta\xi_1 + \bar{\xi}\zeta_1 & \bar{\xi}\bar{\xi}_1 \end{pmatrix}$ and hence

$$\pi xx_1 = \begin{pmatrix} \bar{\zeta}\bar{\xi}_1 + \xi\zeta_1 & 0 \\ 0 & \zeta\xi_1 + \bar{\xi}\zeta_1 \end{pmatrix}.$$

From this relation we see readily that π is a $(B, 1)$ - (B, β) -homomorphism of A to B . From the same relation we see also immediately that the condition i_r of Prop. 4 holds. On the other hand, 1 and $u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ form a free B -right-basis of A . Hence $\text{Hom}_{B, \beta}(A_B, B_B)$ is a free B -left-module of rank 2. Hence we have a B - A -monomorphism $a \rightarrow \varphi_a = \pi^a$ of ${}_B A_A$ into $\text{Hom}_{B, \beta}(A_B, B_B)$. Since $(1, u)$ is also a free B -left-basis of A , the map $a \rightarrow \varphi_a = \pi^a$ is epimorphic too. A is thus a 2-Frobenius extension of the field B with respect to β . (In fact, $(1, u)$ and $(u, 1)$ are B -right and B -left free bases of A dual in the sense of Prop. 10, 12). Since the automorphism β of B can not be induced by a transformation by a regular element in A , A is not 1-Frobenius over B . (This example is closely related to Example 2 in I, §2. Indeed, if we consider our A as an algebra over the fixed subfield L of the automorphism $\xi \rightarrow \bar{\xi}$ in K and pass to the tensor product $A \otimes_L K$, then we come back to the case of Example 2).

§ 8. Annihilators of ideals.

Let A be β -Frobenius over B . By means of the $(B, 1)$ - (B, β) -homomorphism $\pi = \theta 1$, as in Prop. 4, we defined in §2

$$(17) \quad R_\pi X = \{x \in A \mid \pi Xx = 0\},$$

$$(18) \quad L_\pi X = \{x \in A \mid \pi xX = 0\}$$

for a subset X of A .

LEMMA 24. $L_\pi AX$ is a right ideal of A and is indeed the largest right ideal of A contained in $L_\pi X$.

Proof. As AX is a left ideal, we see readily that $L_\pi AX$ is a right ideal in A . Clearly $L_\pi AX \subseteq L_\pi X$. Further, if τ is a right ideal of A contained in $L_\pi X$, then

$$\pi \tau AX = \pi \tau X = 0$$

whence $\tau \subseteq L_\pi AX$. (The lemma will be made use of in the next section).

LEMMA 25. $L_\pi XA$ is a left ideal of A and is indeed the largest left ideal of A contained in $L_\pi X$. Moreover, $L_\pi XA$ coincides with the left annihilator (i.e. the set of all left annihilators, to be more precise) ℓXA of XA in A .

Proof. Denote the right ideal XA by τ . As $(\ell \tau)\tau = 0$ we have clearly $\ell \tau \subseteq L_\pi \tau$. Conversely, from $\pi(L_\pi \tau)\tau (= \pi(L_\pi \tau)\tau A) = 0$ we obtain $(L_\pi \tau)\tau = 0$ (i.e. $\ell \tau \supseteq L_\pi \tau$) by virtue of Prop. 4, i_r). Thus $L_\pi \tau = \ell \tau$, proving the second half of the lemma. Further, clearly $L_\pi \tau = L_\pi XA \subseteq L_\pi X$. If ι is a left ideal of A contained in $L_\pi X$, then $\pi \iota X = 0$, $\pi A \iota X = 0$, whence $\iota X = 0$, by Prop. 4, i_l), and therefore, $\iota XA = 0$, $\pi \iota XA = 0$, $\iota \subseteq L_\pi XA$. This shows that $L_\pi XA = L_\pi \tau$ is the largest left ideal contained in $L_\pi X$. Now, we call *right d-ideals* those right ideals of A which are direct B -right-summands of A ; they are similar to, but somewhat more general than, what are called right v -ideals in Kasch [10]. Left d -ideals are defined similarly.

THEOREM 26. Let A be a 2-Frobenius extension of B . For every right d -ideal τ of A the left annihilator $\ell \tau$ is a left d -ideal of A and its right annihilator is τ , that is,

$$(19) \quad \tau \ell \tau = \tau.$$

(Similarly

$$(20) \quad \ell \tau \ell = \ell$$

for every left d -ideal ℓ of A).

Proof. This follows immediately from Prop. 6 and Lemma 25 (and its right-left symmetry).

Assume now that B is semi-primary. For a finitely generated projective (whence quasi-free) B -right (resp. left) module \mathfrak{R} (resp. \mathfrak{Q}) we denote by $(\mathfrak{R} : B)_r$ (resp. $(\mathfrak{Q} : B)_l$) the number of components in a direct decomposition of \mathfrak{R} (resp. \mathfrak{Q}) into directly indecomposable B -right (resp. left) modules. We have $(\mathfrak{R} : B)_r = (\text{Hom}_{B, \beta}(\mathfrak{R}_B, B) : B)_l$ for any automorphism β of B . In particular, we have $(A : B)_r = (A : B)_l$ for a (necessarily quasi-free) 2-Frobenius algebra A , and this common value is denoted by $(A : B)$.

THEOREM 27. *Let B be semi-primary and A be 2-Frobenius over B . For every right d -ideal \mathfrak{r} of A we have*

$$(21) \quad (\mathfrak{r} : B)_r + (l\mathfrak{r} : B)_l = (A : B).$$

Proof. Let, generally, $A = \mathfrak{R} \oplus \mathfrak{R}'$ be a direct B -right decomposition of A . Decomposing $\mathfrak{R}, \mathfrak{R}'$ further into directly indecomposable B -right modules and applying Prop. 6 to the thus obtained direct decomposition of A , we readily find

$$(\mathfrak{R} : B)_r + (L_\pi \mathfrak{R} : B)_l = (A : B).$$

If in particular $\mathfrak{R} = \mathfrak{r}$ is a right d -ideal of A , we obtain (21), by virtue of Lemma 25.

THEOREM 28. *Let B be semi-primary and A be 2-Frobenius over B . For every element a in A such that aA is a right d -ideal of A , the left ideal Aa of A is B -left projective and we have*

$$(22) \quad (Aa : B)_l = (aA : B)_r.$$

Proof. We have $A = aA \oplus \mathfrak{R}'$ with a B -right module \mathfrak{R}' and hence, by Prop. 6, $A = L_\pi aA \oplus L_\pi \mathfrak{R}' = laA \otimes L_\pi \mathfrak{R}'$. But $A/laA \approx Aa$ (A -left). Hence $Aa \approx L_\pi \mathfrak{R}'$ (B -left). Thus Aa is B -left projective and $(Aa : B)_l = (L_\pi \mathfrak{R}' : B)_l = (aA : B)_r$ proving (22).

In Theorem 28 aA is contended merely to be B -left-projective. If B is an S -ring (cf. I, §5), it is a left d -ideal. Thus

PROPOSITION 29. *Let B be an S -ring and A be 2-Frobenius over B . Every B -right projective right ideal of A is a right d -ideal.*

Proof. By Prop. 15.

§ 9. Residue-rings.

With a (two-sided) ideal \mathfrak{a} of an extension A of B satisfying $\mathfrak{a} \cap B = 0$, the residue-ring A/\mathfrak{a} can naturally be looked upon as an extension of B .

PROPOSITION 30. *Let β -Frobenius over B , and \mathfrak{a} be a (two-sided) ideal of A which is a left d -ideal and satisfies $\mathfrak{a} \cap B = 0$. If there is an element a in A such that*

- a) $R_\pi \mathfrak{a} = Aa = aA$, and
- b) $\pi(xya - xay) = 0$ for any $x \in A, y \in B$,

where $\pi = \emptyset 1$ as in Prop. 4, then the residue-ring A/\mathfrak{a} is β -Frobenius over B .

Proof. Consider the map $\pi': x \rightarrow \pi xa$ of A into B . As π is $(B, 1) \cdot (B, \beta)$ -homomorphic and $\pi(xya - xay) = 0$ for any $x \in A, y \in B$, we see that π' is also $(B, 1) \cdot (B, \beta)$ -homomorphic. Since $\mathfrak{a} = L_\pi R_\pi \mathfrak{a} = L_\pi aA \subseteq L_\pi a$, the kernel of π', π' induces a $(B, 1) \cdot (B, \beta)$ -homomorphism of $A' = A/\mathfrak{a}$ into B , which we denote also by π' . The kernel of the last is $L_\pi a/\mathfrak{a}$ and this contains no non-zero left-ideal in A' , since $\mathfrak{a} (= L_\pi R_\pi \mathfrak{a}) = L_\pi aA$ is, by Lemma 25, the largest left-ideal of A contained in $L_\pi a$. Thus the map π' of A' into B has the property i_l) of Prop. 4 for A' instead of A . For every ψ' in $\text{Hom}_{B, \beta^{-1}}({}_B A', {}_B B)$ there is, by Prop. 4, ii_l), an element a_1 in A such that $\psi'(x \text{ mod } \mathfrak{a}) = \beta^{-1} \pi x a_1$. We have $\beta^{-1} \pi x a_1 = 0$ for every $x \in \mathfrak{a}$, and therefore $a_1 \in R_\pi \mathfrak{a} = Aa$. Set thus $a_1 = a_2 a, a_2 \in A$. Then $\psi'(x \text{ mod } \mathfrak{a}) = \beta^{-1} \pi x a_2 a = \beta^{-1} \pi' x a_2$ and this means that the map π' of A' into B satisfies Prop. 4, ii_l) for A' too, besides i_l). Further, since \mathfrak{a} is a left d -ideal, $A' = A/\mathfrak{a}$ is evidently (finite generated) B -left projective. Hence A' is β -Frobenius over B by Prop. 4.

PROPOSITION 31. *Let A be a β -Frobenius extension of an S -ring B , and \mathfrak{a} be a (two-sided) ideal of A which is a left (or right) d -ideal and satisfies $\mathfrak{a} \cap B = 0$. If there is an element a in A such that*

- a') $\mathfrak{a} = L_\pi Aa = L_\pi aA$ and b) of Prop. 30 hold,

where $\pi = \emptyset 1$ as in Prop. 4, then the residue-ring A/\mathfrak{a} is β -Frobenius over B .

Proof. As in our proof to the preceding proposition, we see that the map $\pi': x \rightarrow \pi xa$ of A into B is $(B, 1) \cdot (B, \beta)$ -homomorphic and its kernel is $L_\pi a$ and contains $\mathfrak{a} (= L_\pi aA \subseteq L_\pi a)$. By $\mathfrak{a} = L_\pi Aa = L_\pi aA$ and by Lemmas 24, 25, \mathfrak{a} is both the largest right-ideal and the largest left-ideal of A contained in the

kernel $L_\pi a$ of π' . Hence the kernel $L_\pi a/\mathfrak{a}$ of the induced map of $A' = A/\mathfrak{a}$ into B , denoted again by the same letter π' , contains neither a non-zero right-ideal nor a non-zero left-ideal in A' . The map π' , of A' into B , satisfies Prop. 16, a) (i.e. Prop. 4, i_r) and i_l) for A' . $A' = A/\mathfrak{a}$ is again (finitely generated) B -left (or right) projective, since \mathfrak{a} is a left (or right, resp.) d -ideal. Hence A' is β -Frobenius over B by Prop. 16. (We wish to correct an error in Prop. 16: “ B -left” in the second line in Prop. 16 should read “ B -right” (in order to be in accord with “ 1_r ”) in the third line. However, if we replace this “ 1_r ”) by “ 1_l ”), then “ B -left” goes well (without injuring the validity of the proposition)).

Remark. The application of L_π shows that the condition a') for a left d -ideal \mathfrak{a} is weaker than the condition a).

LEMMA 32. *Let be 2.Frobenius over B . Assume that the B -right-module A has a direct summand, say \mathfrak{R} , which can be (B -right) homomorphically mapped upon B ; this is automatically the case with $\mathfrak{R} = B$ if B is an S -ring. For $\pi = \emptyset 1$ as in Prop. 4 we have then $\pi A\mathfrak{R} = B$ and so $\pi A = B$.*

Proof. There exists, by assumption, a direct B -right decomposition $A = \mathfrak{R} \oplus \mathfrak{R}_1$ of A and \mathfrak{R} has a B -right epimorphism ψ onto B . Mapping $x + x_1$ ($x \in \mathfrak{R}, x_1 \in \mathfrak{R}_1$) to $\beta\psi x \in B$, where β is an automorphism of B belonging to the 2.Frobenius extension A over B , we obtain an element φ of $\text{Hom}_{B,\beta}(A_B, B_B)$. Since ψ is epimorphic, φ is epimorphic and indeed $\varphi\mathfrak{R} = B$. By Prop. 4 $\varphi \in \emptyset A$. So $B = (\emptyset A)\mathfrak{R} = (\emptyset 1)A\mathfrak{R} = \pi A\mathfrak{R}$, proving the lemma.

THEOREM 33. *Let A be a β -Frobenius extension of an S -ring B and \mathfrak{a} be a (two-sided) ideal of A which is a left d -ideal of A and satisfies $\mathfrak{a} \cap B = 0$. The residue-ring A/\mathfrak{a} is β -Frobenius over B if and only if there is an element a in A such that a') of Prop. 31 and b) of Prop. 30 hold, i.e. $\mathfrak{a} = L_\pi Aa = L_\pi aA$ and $\pi(xya - xay) = 0$ for all $x \in A, y \in B$, where π is as in Prop. 4.*

Proof. The “if” was proved in Proposition 31. To prove “only if”, assume that the residue-ring $A' = A/\mathfrak{a}$ is β -Frobenius over B . There exists then a $(B, 1)$ - (B, β) -homomorphism π' of A' into B which has the properties i_r), i_l), ii_r), ii_l) of Prop. 4 with respect to A' in place of A . Let \mathfrak{K}' be the kernel of π' , i.e.

$$\mathfrak{K}' = L_\pi 1' = R_\pi 1'$$

(where $1'$ is the unit element (1 mod \mathfrak{a}) of A'). As the residue-module A'/\mathfrak{K}'

is $(B, 1)$ - (B, β) -, whence B -left-, isomorphic to B , by Lemma 32, (and B is B -left projective), there is a B -left submodule \mathcal{Q}' of A' with $A' = \mathfrak{K}' \oplus \mathcal{Q}'$. π' induces a B -left isomorphism of \mathcal{Q}' and B .

Now, since \mathfrak{J} is a left \mathcal{A} -ideal, we have $A = \mathfrak{J} \oplus \mathcal{Q}$ with a B -left submodule \mathcal{Q} of A . By the canonical map $A \rightarrow A' = A/\mathfrak{J}$, \mathcal{Q} is mapped B -left isomorphically onto A' ($= \mathfrak{K}' \oplus \mathcal{Q}'$). Denoting by \mathfrak{K}'' , \mathcal{Q}'' the counter-images in \mathcal{Q} of \mathfrak{K}' , \mathcal{Q}' , we obtain the direct B -left decomposition $A = \mathfrak{J} \oplus \mathfrak{K}'' \oplus \mathcal{Q}''$; $\mathfrak{J} \oplus \mathfrak{K}''$ is the counter-image, in A , of \mathfrak{K}' by the map $A \rightarrow A' = A/\mathfrak{J}$. By Prop. 6 we have the direct B -right decomposition

$$A = R_\pi(\mathfrak{K}'' \oplus \mathcal{Q}'') \oplus R_\pi(\mathfrak{J} \oplus \mathcal{Q}'') \oplus R_\pi(\mathfrak{J} \oplus \mathfrak{K}'').$$

As $R_\pi(\mathfrak{J} \oplus \mathfrak{K}'') \simeq \text{Hom}_{B, \beta^{-1}}(\mathcal{Q}'', {}_B B)$ (B -right) and as $\mathcal{Q}'' \simeq \mathcal{Q}' \simeq B$ (B -left), we see $R_\pi(\mathfrak{J} \oplus \mathfrak{K}'') \simeq \text{Hom}_{B, \beta^{-1}}({}_B B, {}_B B) \simeq B$ (B -right). Hence there is an element a in A with $R_\pi(\mathfrak{J} \oplus \mathfrak{K}'') = aB$. Hence $\mathfrak{J} \oplus \mathfrak{K}'' = L_\pi aB = L_\pi a$, again by Prop. 6. Now, \mathfrak{J} is the largest right-ideal of A contained in $\mathfrak{J} \oplus \mathfrak{K}''$, since otherwise the kernel \mathfrak{K}' of π' would contain a non-zero right-ideal of A' . So, as $\mathfrak{J} \oplus \mathfrak{K}'' = L_\pi aB = L_\pi a$, we obtain $\mathfrak{J} = L_\pi Aa = L_\pi AaB$ by Lemma 24. Similarly \mathfrak{J} is the largest left-ideal of A contained in $\mathfrak{J} \oplus \mathfrak{K}''$ and hence, by Lemma 25, $\mathfrak{J} = L_\pi aA$.

The map $x \rightarrow \pi xa$ is a B -left-homomorphism of A into B . Its kernel is the module $L_\pi a$ which we have just seen to be $\mathfrak{J} \oplus \mathfrak{K}''$. It follows that the map

$$(x \text{ mod } \mathfrak{J} \oplus \mathfrak{K}'') \in A/(\mathfrak{J} \oplus \mathfrak{K}'') \rightarrow \pi xa \in B$$

is a B -left-isomorphism of $A/(\mathfrak{J} \oplus \mathfrak{K}'')$ onto B . On the other hand, $A/(\mathfrak{J} \oplus \mathfrak{K}'') = A'/\mathfrak{K}'$ and we have a B -left (and even $(B, 1)$ - (B, β) -) isomorphism of A'/\mathfrak{K}' onto B given by

$$(x' \text{ mod } \mathfrak{K}') \rightarrow \pi' x' \in B.$$

(which we have made use of before). So the composite

$$\pi xa \rightarrow (x \text{ mod } \mathfrak{J} \oplus \mathfrak{K}'') \rightarrow ((x \text{ mod } \mathfrak{J}) \text{ mod } \mathfrak{K}') \rightarrow \pi'(x \text{ mod } \mathfrak{J})$$

is a B -left isomorphism of $\pi Aa (\subseteq B)$ onto B . Considering B -left lengths, we see that $\pi Aa = B$ and $\pi xa \rightarrow \pi'(x \text{ mod } \mathfrak{J})$ is a B -left automorphism of B . Hence there is a regular element b in B such that

$$(\pi xa)b = \pi'(x \text{ mod } \mathfrak{J}) \quad \text{for all } x \in A.$$

Since $x \rightarrow \pi'(x \text{ mod } \delta)$ is $(B, 1)$ - (B, β) -homomorphic, we have hence $(\pi xy a)b = (\pi xa)b\beta y$, i.e. $\pi xy a\beta^{-1}b = \pi xa(\beta^{-1}b)y$, for all $y \in B$ (and for all $x \in A$). Putting $a_0 = a\beta^{-1}b$, we have

$$\pi xy a_0 = \pi xa_0 y$$

for all $y \in B, x \in A$. As $L_\pi A a_0 = L_\pi A a_0 B = L_\pi A a B = L_\pi A a (= \delta), L_\pi a_0 A = L_\pi a A (= \delta)$, we have the “only if” part of the proposition on denoting a_0 by a anew.

Remark. As our proof shows, the “only if” part of Prop. 33 remains valid when we replace the S -ring condition on B by that B satisfies the minimum condition for left \cdot -ideals and $A' = A/\delta$ has a B -right direct summand (B -right) isomorphic to B .

§ 10. Frobenius extension of a quasi-Frobenius ring.

By homological means we have seen already in Corollary 11 that a 2-Frobenius extension of a quasi-Frobenius ring B is a quasi-Frobenius ring. Using the annihilator characterization of quasi-Frobenius rings, instead of homological characterization, we may derive this theorem also from the following generalized formulation of Hall’s [15] theorem :

PROPOSITION 34. *Let B be a quasi-Frobenius ring and β be an automorphism of B . Let \mathfrak{Q} be a finitely generated projective B -left-module, \mathfrak{R} a B -right-module, and let there be given a regular 1 - β -scalar product \langle, \rangle of $\mathfrak{Q}, \mathfrak{R}$ in B ; then \mathfrak{R} is finitely generated B -right projective, by Prop. 15. For every B -left submodule \mathfrak{Q}_0 of \mathfrak{Q} (resp. B -right submodule \mathfrak{R}_0 of \mathfrak{R}) we have*

$$(23) \quad LR\mathfrak{Q}_0 = \mathfrak{Q}_0 \quad (\text{resp. } RL\mathfrak{R}_0 = \mathfrak{R}_0),$$

where L, R are the same as in Proposition 15, 2).

Proof. For the sake of completeness and convenience we briefly reproduce Hall’s proof in the form adapted to our present generalized formation. Expressing \mathfrak{R} as a direct sum of submodules isomorphic to right-ideals generated by an idempotent and applying Prop. 15 repeatedly, we may express $\mathfrak{R}, \mathfrak{Q}$ as direct sums

$$\begin{aligned} \mathfrak{R} &= v_1 B \oplus \cdots \oplus v_n B, \\ \mathfrak{Q} &= Bu_1 \oplus \cdots \oplus Bu_n \end{aligned}$$

with v_ν, u_ν such that $v_\nu B \approx e_\nu B$ (B -right), $Bu_\nu \approx B\beta e_\nu$ (B -left) by $v_\nu \leftrightarrow e_\nu, u_\nu \leftrightarrow \beta e_\nu, e_\nu$ being idempotents in B , and furthermore

$$\langle u_\mu, v_\nu \rangle = \delta_{\mu\nu} \beta e_\nu.$$

Denote by $\mathfrak{R}^*, \mathfrak{Q}^*$ the submodules $v_2 B \oplus \dots \oplus v_n B, Bu_2 \oplus \dots \oplus Bu_n$ of $\mathfrak{R}, \mathfrak{Q}$. The scalar product \langle, \rangle on $\mathfrak{R}, \mathfrak{Q}$ induces an (also $(1, \beta)$ -) scalar product on $\mathfrak{R}^*, \mathfrak{Q}^*$, which is also regular and which we denote also by \langle, \rangle . For a submodule \mathfrak{R}_0 of \mathfrak{R} we denote by \mathfrak{R}_0^* the intersection $\mathfrak{R}_0 \cap \mathfrak{R}^*$. Similarly we set $\mathfrak{Q}_0^* = \mathfrak{Q}_0 \cap \mathfrak{Q}^*$ for a submodule \mathfrak{Q}_0 of \mathfrak{Q} . Now, in order to prove our proposition by induction, with respect to n , we assume that the assertion for $\mathfrak{R}^*, \mathfrak{Q}^*$ (and their scalar product \langle, \rangle), in place of $\mathfrak{R}, \mathfrak{Q}$, (and \langle, \rangle), is valid. Then our induction argument is divided into several steps.

1) Let $u = b_1 u_1 + \dots + b_n u_n$ ($b_\nu \in B\beta e_\nu$) be an arbitrary element of \mathfrak{Q} . Then (23) holds for $\mathfrak{Q}_0 = Bu$, i.e.

$$LRBu = Bu.$$

To prove this, let $s_1 u_1 + \dots + s_n u_n \in LRBu$ ($s_\nu \in B\beta e_\nu$). Then $0 = \langle s_1 u_1 + \dots + s_n u_n, v_1 x_1 \rangle = s_1 \beta x_1$ for every $x_1 \in B$ with $b_1 \beta x_1 = 0$. Hence $s_1 \in lr b_1 = B b_1$, where l, r are left and right annihilator operations in the quasi-Frobenius ring B . So $s_1 = t b_1$ with $t \in B$. We may write

$$s_1 u_1 + \dots + s_n u_n - t(b_1 u_1 + \dots + b_n u_n) = s'_2 u_2 + \dots + s'_n u_n$$

($s'_\nu \in B$). Since this element belongs to $(LRBu)^*$, it belongs to $L^* R^* B(b_2 u_2 + \dots + b_n u_n)$, as we readily see, and we can write

$$s'_2 u_2 + \dots + s'_n u_n = t'(b_2 u_2 + \dots + b_n u_n) \quad (t' \in B)$$

in virtue of our induction assumption.

Now, for any element $w = b_1 z_1 = -(b_2 z_2 + \dots + b_n z_n)$ ($z_\nu \in B$) common to $b_1 B$ and the (not necessarily direct) sum $b_2 B + \dots + b_n B$, we have evidently $b_1 z_1 + \dots + b_n z_n = 0$ and thus $v_1 \beta^{-1} z_1 + \dots + v_n \beta^{-1} z_n \in RBu$. Since $t'(b_2 u_2 + \dots + b_n u_n) = s'_2 u_2 + \dots + s'_n u_n \in LRBu$, this entails $t'w = -t'(b_2 z_2 + \dots + b_n z_n) = \langle s'_2 u_2 + \dots + s'_n u_n, v_1 \beta^{-1} z_1 + \dots + v_n \beta^{-1} z_n \rangle = 0$. This shows

$$t' \in l(b_1 B \cap (b_2 B + \dots + b_n B)).$$

The right-hand side is equal to $l b_1 B + l(b_2 B + \dots + b_n B)$ and hence $t' = p + q$ with $p \in l b_1 B, q \in l(b_2 B + \dots + b_n B)$. Therefore

$$t'(b_2 u_2 + \cdots + b_n u_n) = \hat{p}(b_1 u_1 + b_2 u_2 + \cdots + b_n u_n) = \hat{p}u,$$

$$\begin{aligned} s_1 u_1 + \cdots + s_n u_n &= tu + (s'_2 u_2 + \cdots + s'_n u_n) \\ &= tu + t'(b_2 u_2 + \cdots + b_n u_n) = tu + \hat{p}u \\ &= (t + \hat{p})u \in Bu. \end{aligned}$$

This proves $LRBu(\subseteq \text{whence}) = Bu$.

2) Next we show that if u is an element of \mathfrak{Q} then $R^*(Bu)^* = (RBu + v_1 B)^*$. For, $L^*(RBu + v_1 B)^* \subseteq LRBu$ as we readily see. The right-hand side is Bu by the step 1). Thus $(Bu)^* \supseteq L^*(RBu + v_1 B)^*$, and therefore $R^*(Bu)^*$ is contained in $R^*L^*(RBu + v_1 B)^*$ which is in turn $(RBu + v_1 B)^*$ by our induction assumption. This proves a half of our contention, and the other half, i.e. the inclusion of the other direction, is evident.

3) Let \mathfrak{Q}_0 be a B -left submodule of \mathfrak{Q} . Then we wish to prove $R^*\mathfrak{Q}_0^* = (R\mathfrak{Q}_0 + v_1 B)^*$.

Again the inclusion \subseteq is evident, and so we have to prove \supseteq , i.e. that for any element v^* of $R^*\mathfrak{Q}_0^*$ there exists an element x_1 of $e_1 B$ such that

$$v_1 x_1 + v^* \in R\mathfrak{Q}_0.$$

For this purpose, let

$$(24) \quad u^{(\rho)} = b_1^{(\rho)} u_1 + \cdots + b_n^{(\rho)} u_n \quad (b_v^{(\rho)} \in B\beta e_v) \quad (\rho = 1, \dots, r)$$

be a (finite) system of B -left generators of \mathfrak{Q}_0 . Let $s \leq r$ and assume that there is an $x'_1 \in e_1 B$ satisfying

$$(25) \quad \langle u^{(\rho)}, v_1 x'_1 + v^* \rangle = 0 \quad \text{for } \rho = 1, \dots, s-1.$$

By the above step 1) there is $x''_1 \in e_1 B$ satisfying

$$(26) \quad \langle u^{(s)}, v_1 x''_1 + v^* \rangle = 0.$$

Now, let $w = y^{(1)} b_1^{(1)} + \cdots + y^{(s-1)} b_1^{(s-1)} = y^{(s)} b^{(s)} (y^{(\sigma)} \in B)$ be an element of $(Bb_1^{(1)} + \cdots + Bb_1^{(s-1)}) \cap Bb_1^{(s)}$. Then

$$y^{(1)} u^{(1)} + \cdots + y^{(s-1)} u^{(s-1)} - y^{(s)} u^{(s)} \in \mathfrak{Q}_0^*.$$

Hence

$$\langle y^{(1)} u^{(1)} + \cdots + y^{(s-1)} u^{(s-1)} - y^{(s)} u^{(s)}, v^* \rangle = 0.$$

In combination with (25), (26) this entails

$$\langle y^{(1)} u^{(1)} + \dots + y^{(s-1)} u^{(s-1)}, v_1 x' \rangle - \langle y^{(s)} u^{(s)}, v_1 x_1'' \rangle = 0, \text{ i.e.}$$

$$w\beta x_1' - w\beta x_1'' = w\beta(x_1' - x_1'') = 0.$$

We have thus seen

$$\beta(x_1' - x_1'') \in r((Bb_1^{(1)} + \dots + Bb_1^{(s-1)}) \cap Bb_1^{(1)}) = r(Bb_1^{(1)} + \dots + Bb_1^{(s-1)}) + rBb_1^{(1)}.$$

Hence $\beta(x_1' - x_1'') = p + q$ with

$$(27) \quad p \in r(Bb_1^{(1)} + \dots + Bb_1^{(s-1)}), \quad q \in rBb_1^{(1)}.$$

Put $x_1''' = x_1' - \beta^{-1}p (= x_1'' + \beta^{-1}q)$. Then by (25), (26) and (27)

$$(28) \quad \langle u^{(\rho)}, v_1 x_1''' + v^* \rangle = 0 \quad \text{for } \rho = 1, \dots, s.$$

(25) \rightarrow (28) makes $s - 1 \rightarrow s$ and we reach to our contension by recursion of this argument.

4) Let (24) be as in the preceding step, and $u = y_1 u_1 + \dots + y_n u_n$ ($y_\nu \in B\beta e_\nu$) be any element of $LR\mathcal{Q}_0$. If x_1 is an element of B such that $b_1^{(\rho)}\beta x_1 = 0$ for $\rho = 1, \dots, r$, then $v_1 x_1 \in R\mathcal{Q}_0$ and hence $y_1 \beta x_1 = 0$. Thus $y_1 \in l(rb_1^{(1)} \cap \dots \cap rb_1^{(r)}) = lr b_1^{(1)} + \dots + lr b_1^{(r)} = Bb_1^{(1)} + \dots + Bb_1^{(r)}$, whence

$$y_1 = p^{(1)} b_1^{(1)} + \dots + p^{(r)} b_1^{(r)} \quad (p^{(\rho)} \in B).$$

The element $u' = p^{(1)} u^{(1)} + \dots + p^{(r)} u^{(r)}$ of \mathcal{Q}_0 has a form

$$y_1 u_1 + y_2' u_2 + \dots + y_n' u_n \quad (y_\nu' \in B\beta e_\nu; \nu = 2, \dots, n).$$

We have $u - u' \in (LR\mathcal{Q}_0)^*$ and hence

$$\langle u - u', v^* \rangle = 0$$

for every $v^* \in (R\mathcal{Q}_0 + v_1 B)^*$, whence, by the preceding step 3), for every $v^* \in R^* \mathcal{Q}^*$. So $u - u' \in L^* R^* \mathcal{Q}_0^*$ and $\in \mathcal{Q}_0^*$ by our induction assumption. Hence

$$u = u' + (u - u') \in \mathcal{Q}_0 + \mathcal{Q}_0^* = \mathcal{Q}_0.$$

This completes our final step, proving thus Proposition 34. Now, as a second application of Prop 34, Hall's theorem (in generalized formulation), we combine it with Prop. 30 and Theorem 33 to obtain

THEOREM 35. *Let A be a β -Frobenius extension of a quasi-Frobenius ring B and \mathfrak{z} be a (two-sided) ideal of A which is a left d -ideal of A and satisfies*

$\mathfrak{J} \cap B = 0$. The residue-ring A/\mathfrak{J} is β -Frobenius over B if and only if there is an element a in A such that $r\mathfrak{J} = AaB = aA$ and $\pi(xya - xay) = 0$ for all $x \in A$, $y \in B$, where π is as in Prop. 4.

Proof. If $r\mathfrak{J} = AaB = aA$, then, since $r\mathfrak{J} = R_\pi \mathfrak{J}$, $\mathfrak{J} (= L_\pi R_\pi \mathfrak{J}) = L_\pi AaB (= L_\pi aA) = L_\pi aA$. Hence the "if" part of our theorem follows from Prop. 30. To prove the "only if", we have merely to note that the condition $\mathfrak{J} = L_\pi aA = L_\pi aA$ of Theorem 33 entails $R_\pi \mathfrak{J} (= r\mathfrak{J}) = R_\pi L_\pi aB = R_\pi L_\pi aA$ and to observe that here $R_\pi L_\pi aB = AaB$, $R_\pi L_\pi aA = aA$ in virtue of Prop. 34.

Remark. Our theorem fails to cover Satz 9 of Kasch [10]; particularly, our by-condition $\pi(xya - xay) = 0$ is very strong. However, the writers fail to convince themselves of the validity of the proof of [10], Satz 9.

REFERENCES (some repeating ones in Part I)

- [4] S. Eilenberg-T. Nakayama, On the dimension of modules and algebras II, Nagoya Math. J. **9** (1955), 1-16.
- [6] K. Hirata, On relative homological algebra of Frobenius extensions, Nagoya Math. J. **15** (1959), 17-28.
- [10] F. Kasch, Grundlagen einer Theorie der Frobeniusweiterungen, Math. Ann. **127** (1954), 453-474.
- [13] T. Nakayama-T. Tsuzuku, A remark on Frobenius extensions and endomorphism rings, Nagoya Math. J. **15** (1959), 9-16.
- [14] C. Curtis, Quasi-Frobenius rings and Galois theory, III. J. Math. **3** (1959), 134-144.
- [15] M. Hall, A type of algebraic closure, Ann. Math. **40** (1939), 360-369.
- [16] M. Ikeda, On a theorem of Gaschütz, Osaka Math. J. **5** (1953), 53-58.
- [17] T. Nakayama, On Frobeniusean algebras I, Ann. Math. **40** (1939), 611-633.
- [18] T. Nakayama, Galois theory of simple rings, Trans. Amer. Math. Soc. **73** (1952), 276-292.

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