

streamline treatments of the interpretability orders \leq_{κ}^* of Shelah, the key new notion being that of pseudosaturation.

In Chapter 4, we uniformize many ultrafilter constructions in Keisler’s order. As a particular application, we prove that for all $3 \leq k < k'$, $T_{k+1,k} \not\leq T_{k'+1,k'}$, where $T_{n,k}$ is the theory of the random k -ary n -clique free hypergraph. This improves the previous result of Malliaris and Shelah that $T_{k+1,k} \not\leq T_{k'+1,k'}$ for all $k < k' - 1$.

Borel complexity is a pre-order on sentences of $\mathcal{L}_{\omega_1\omega}$ measuring the complexity of countable models. In Chapter 5, we describe joint work with Richard Rast and Chris Laskowski on this order. Their key idea is the following: suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Define $\text{css}(\Phi)_{\text{ptl}}$ to be the set of all sentences $\phi \in \mathcal{L}_{\infty\omega}$, such that in some forcing extension $\mathbb{V}[G]$, ϕ becomes the canonical Scott sentence of some model of Φ . Define $\|\Phi\|$ to be the cardinality of $\text{css}(\Phi)_{\text{ptl}}$ (possibly ∞). We show that if $\Phi \leq_B \Psi$ then this induces an injection from $\text{css}(\Phi)_{\text{ptl}}$ to $\text{css}(\Psi)_{\text{ptl}}$, whence $\|\Phi\| \leq \|\Psi\|$. This is a potent new method for proving nonreducibilities in \leq_B , and we give several applications, including the first example of a complete first order theory T with non-Borel isomorphism relation, but which is not Borel complete.

In Chapter 6, we introduce the notion of thickness. The motivation is as follows: suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ with $\|\Phi\| = \infty$; we wish to still be able to apply counting arguments to $\text{css}(\Phi)_{\text{ptl}}$. The thickness spectrum $\tau(\Phi, \kappa)$ of Φ accomplishes this; roughly, $\tau(\Phi, \kappa) \approx |\text{CSS}(\Phi)_{\text{ptl}} \cap \mathbb{V}_{\kappa+}|$, although care must be taken to ensure that thickness is a \leq_B -reducibility invariant. We present several applications of the notion of thickness; in particular, we show that all the Friedman-Stanley jumps of torsion abelian groups are non-Borel complete. We also show that if Φ has the Schröder-Bernstein property (that is, whenever two countable models of Φ are biembeddable, then they are isomorphic), then under large cardinals, Φ is not Borel complete.

In Chapter 7, we describe joint work with Saharon Shelah on the complexity of countable torsion-free abelian groups. In particular, we show that if certain large cardinals fail, then torsion-free abelian groups are $a\Delta_2^1$ -complete, where $\leq_{a\Delta_2^1}$ is a well-known coarsening of \leq_B .

Abstract prepared by Douglas Ulrich.

E-mail: dsulrich@uci.edu

URL: <https://faculty.sites.uci.edu/douglasulrich/publications/>

VICTORIA NOQUEZ, *Vaught’s Two-Cardinal Theorem and Notions of Minimality in Continuous Logic*, University of Illinois at Chicago, USA, 2017. Supervised by David Marker. MSC: 03C65. Keywords: continuous logic, model theory.

Abstract

Much of the work in this thesis was motivated by an effort to prove a continuous analogue of the Baldwin-Lachlan characterization of uncountable categoricity: a theory T in a countable language is uncountably categorical (has one model up to isomorphism of size κ for some uncountable κ) if and only if T has no Vaughtian pairs and T is ω -stable.

As in the classical setting, we approach the forward direction by proving a continuous version of Vaught’s Two-Cardinal theorem. A continuous theory T has a (κ, λ) -model if there is $\mathcal{M} \models T$ with density character κ which has a definable subset with density character λ . We show that if T has a (κ, λ) -model for infinite cardinals $\kappa > \lambda$, then T has an (\aleph_1, \aleph_0) -model. We also show that with the additional assumption that T is ω -stable, if T has an (\aleph_1, \aleph_0) -model, then for any uncountable κ , T has a (κ, \aleph_0) -model. This provides us with the tools necessary to prove the forward direction of the Baldwin-Lachlan characterization of uncountable categoricity for continuous logic.

Towards the reverse direction, we introduce a continuous notion of strong minimality, saying that a set is minimal if every subset which is the zero set of a definable predicate is totally bounded, or its approximate complements are all totally bounded. We show that this characterization is equivalent to the classical definition of strong minimality, and that it allows us to use algebraic closure to define a notion of dimension which determines models up to isomorphism. However, the only known examples of strongly minimal theories in the

continuous setting are classical theories viewed as continuous with the discrete metric, and we see that this notion lacks the machinery necessary to prove the reverse direction of the Baldwin-Lachlan characterization of uncountable categoricity.

In an effort to better understand minimality in continuous logic, we also introduce a continuous notion of dp-minimality, and provide a few equivalent characterizations. We use these to show that the theory of infinite dimensional Hilbert spaces is a natural example of a dp-minimal continuous theory.

Abstract prepared by Victoria Noquez.

E-mail: vnoquez@gmail.com

PAUL K. GORBOW, *Self-similarity in the Foundations*, University of Gothenburg, Sweden, 2018. Supervised by Ali Enayat (primary), Peter LeFanu Lumsdaine, and Zachiri McKenzie (secondary). MSC: 03C62, 03E30, 03E35, 03E55, 03H99, 03G30. Keywords: set theory, ZF, KP, NF, NFU, nonstandard model, self-embedding, topos, algebraic set theory, endofunctor.

Abstract

This thesis concerns embeddings and self-embeddings of foundational structures in both set theory and category theory. In the first part, rank-initial self-embeddings of countable nonstandard models of set theory are comprehensively studied. In the second part, the theory of topoi is reformulated so as to accommodate a special self-embedding. This results in the formulation of first-order theories corresponding by equiconsistency to intuitionistic and classical variants of Quine's New Foundations. These research tracks are connected in that the techniques of the first part can be used to construct an array of classical models of the theory formulated in the second part.

The first part of the work on models of set theory consists in establishing a refined version of Friedman's theorem on the existence of embeddings between countable nonstandard models of a fragment of ZF, and an analogue of a theorem of Gaifman to the effect that certain countable models of set theory can be elementarily end-extended to a model with many automorphisms whose sets of fixed points equal the original model. The second part of the work on set theory consists in combining these two results into a technical machinery, yielding several results about nonstandard models of set theory relating such notions as self-embeddings, their sets of fixed points, strong rank-cuts, and set theories of different strengths.

In particular, back-and-forth constructions are carried out to establish various generalizations and refinements of Friedman's theorem on the existence of rank-initial embeddings between countable nonstandard models of the fragment $\text{KP}^{\mathcal{P}} + \Sigma_1^{\mathcal{P}}$ -Separation of ZF; and Gaifman's technique of iterated ultrapowers is employed to show that any countable model of $\text{GBC} +$ "the class of ordinals is weakly compact" can be elementarily rank-end-extended to models with well-behaved automorphisms whose sets of fixed points equal the original model. These theoretical developments are then utilized to prove various results relating self-embeddings, automorphisms, their sets of fixed points, standard systems, strong rank-cuts, and set theories of different strengths. Here is one example:

THEOREM. *Suppose that $\mathcal{M} \models \text{KP}^{\mathcal{P}} + \Sigma_1^{\mathcal{P}}$ -Separation + Choice is countable and nonstandard. The following are equivalent:*

- (a) *There is a strong rank-cut of \mathcal{M} that is isomorphic to \mathcal{M} .*
- (b) *\mathcal{M} expands to $(\mathcal{M}, \mathcal{A}) \models \text{GBC} +$ "the class of ordinals is weakly compact".*

The second part of the thesis consists in the formulation of a novel categorical set theory, ML_{cat} , which is proved to be equiconsistent to New Foundations (NF), and which can be modulated to correspond to intuitionistic or classical NF, with or without atoms. NF is a set theory that rescues the intuition behind naive set theory, by imposing a so called stratification constraint on the formulae featuring in the comprehension schema.

The axioms of the categorical theory developed here express that its structures have an endofunctor, with certain coherence properties. By means of this endofunctor, an appropriate