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# FUNCTIONAL VERSIONS OF SOME REFINED AND REVERSED OPERATOR MEAN INEQUALITIES

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**Abstract**

We present refined and reversed inequalities for the weighted arithmetic mean–harmonic mean functional inequality. Our approach immediately yields the related operator versions in a simple and fast way. We also give some operator and functional inequalities for three or more arguments. As an application, we obtain a refined upper bound for the relative entropy involving functional arguments.

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## 1. Introduction

Let $H$ be a complex Hilbert space and $\mathcal{B}(H)$ be the $\mathbb{C}^*$-algebra of bounded linear operators acting on $H$. We denote by $\mathcal{B}^{+*}(H)$ the open cone of all (self-adjoint) positive invertible operators in $\mathcal{B}(H)$. Throughout this section, let $T, S \in \mathcal{B}^{+*}(H)$ and $\lambda \in (0, 1)$ be a real number. The means

$$T\nabla_\lambda S = (1-\lambda)T + \lambda S,$$
$$T!_\lambda S = ((1-\lambda)T^{-1} + \lambda S^{-1})^{-1}$$

are known as the weighted operator arithmetic and harmonic means of $T$ and $S$, respectively. For $\lambda = 1/2$, we denote them by $T\nabla S$ and $T!S$. These operator means satisfy the inequality

$$T!_\lambda S \le T\nabla_\lambda S, \tag{1.1}$$

where the notation $T \le S$ means that $T$ and $S$ are self-adjoint and $S - T$ is positive.

The arithmetic and harmonic operator means have been extended from positive operators to convex functionals (see [10]). Precisely, let $f, g : H \longrightarrow \mathbb{R} \cup \{\infty\}$ be two given functionals (convex or not). Then

$$\mathcal{A}_\lambda(f, g) = (1-\lambda)f + \lambda g,$$
$$\mathcal{H}_\lambda(f, g) = ((1-\lambda)f^* + \lambda g^*)^*$$

are called, by analogy, the weighted functional arithmetic and harmonic means of $f$ and $g$, respectively. For $\lambda = 1/2$, they are simply denoted by $\mathcal{A}(f, g)$ and $\mathcal{H}(f, g)$. Here, the notation $f^*$ refers to the Fenchel conjugate of $f$ defined for $x^* \in H$ by

$$f^*(x^*) = \sup_{x \in H}\{\mathrm{Re}\,\langle x^*, x\rangle - f(x)\}. \tag{1.2}$$

We also have the functional inequality

$$\mathcal{H}_\lambda(f, g) \le \mathcal{A}_\lambda(f, g), \tag{1.3}$$

where $\le$ denotes here the point-wise order, that is, $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in H$. The functional means are extensions of their related operator means in the sense that

$$\mathcal{A}_\lambda(f_T, g_S) = f_{T\nabla_\lambda S}, \quad \mathcal{H}_\lambda(f_T, g_S) = f_{T!_\lambda S},$$

where the notation $f_T$ refers to the convex quadratic form generated by the positive operator $T$, that is, $f_T(x) = \frac{1}{2}\langle Tx, x\rangle$ for all $x \in H$. Thus, $f_T^*(x^*) = \frac{1}{2}\langle T^{-1}x^*, x^*\rangle$ or, in short, $f_T^* = f_{T^{-1}}$, whenever $T$ is a positive invertible operator. With this, (1.1) is an immediate consequence of (1.3) (see, for example, [10]).

Recently, refinements of the inequalities (1.1) as well as their reverses have drawn attention (see [7, 8] and the references therein). For example,

$$2r_\lambda(T\nabla S - T!S) \le T\nabla_\lambda S - T!_\lambda S \le 2R_\lambda(T\nabla S - T!S) \tag{1.4}$$

for all $T, S \in \mathcal{B}^{+*}(H)$, where $r_\lambda = \min(1-\lambda, \lambda)$ and $R_\lambda = \max(1-\lambda, \lambda)$. See also [1, 2] for more detail and information. The usual approach to such operator inequalities is to start from their scalar versions and use the functional calculus (see, for example, [2, 12]). In general, this procedure is lengthy and becomes more difficult for three or more operator variables, because the $\mathbb{C}^*$-algebra $\mathcal{B}(\mathrm{H})$ is neither commutative nor totally ordered. As far as we know, an analogue of (1.4) for $n$ operator arguments has not yet been proposed.

The fundamental goal of this paper is to give some refinements and reverses of the functional inequalities (1.3). In particular, we obtain refinements and analogues of the inequalities (1.4) when the operator variables $T$ and $S$ are replaced by (convex) functionals $f$ and $g$. Our present approach brings many advantages.

- The proofs of the functional inequalities are direct, simple and short.
- The related operator inequalities, like (1.1) and (1.4), are immediately deduced from the functional results without any more analysis.
- Our approach gives generalisations of the functional inequalities and their related operator versions for $n$ functional or operator arguments.
- The functional results can be applied to give inequalities for the relative functional entropy and the relative operator entropy.
- The results can be extended from Hilbert space to a locally convex space, as explained in the following remark.

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REMARK 1.1. The approach presented in this paper is based on Fenchel duality and is developed here for Hilbert spaces. As is well known, Fenchel duality is in fact defined for a locally convex space $E$ where the inner product in (1.2) is replaced by the bracket duality between $E$ and $E^*$, the topological dual of $E$, that is, $\langle x^*, x\rangle = x^*(x)$ for all $x \in E$ and $x^* \in E^*$. So, with some precautions, the approach developed here and the results can be extended from the case where the underlying space is a Hilbert space to the case where the underlying space is a locally convex space. In this setting, the bounded operators should be considered acting from $E$ into $E^*$. In order not to lengthen the present paper, we omit the details about such an extension here and hope to present it in a forthcoming paper soon.

## 2. Basic notions and tools

Throughout, we denote by $\widetilde{\mathbb{R}}^H$ the (extended) space of all functionals defined from $H$ into $\mathbb{R} \cup \{+\infty\}$. For $f \in \widetilde{\mathbb{R}}^H$, let $f^*$ be its conjugate defined through (1.2). As already remarked, if $f, g \in \widetilde{\mathbb{R}}^H$ we write $f \le g$ if and only if $f(x) \le g(x)$, that is, $g(x) - f(x) \ge 0$, for all $x \in H$, with the usual convention $(+\infty) - (+\infty) = +\infty$. Because of this convention, we must be careful with proofs of functional equalities and inequalities. For example, the equality $f - g = -(g - f)$ is not always true, since the functionals can take the value $+\infty$. For the same reason, the two inequalities $f \le g$ and $f - g \le 0$ are not always equivalent, although $f \le g$ and $g - f \ge 0$ are equivalent.

The following lemma will be needed later.

LEMMA 2.1. *Let $f_i \in \widetilde{\mathbb{R}}^H$ and $p_i \ge 0$ for $i = 1, 2, \ldots, n$, with $\sum_{i=1}^n p_i = 1$. Then*

$$\Big(\sum_{i=1}^n p_i f_i\Big)^* \le \sum_{i=1}^n p_i f_i^*. \tag{2.1}$$

*In particular, if $T_i \in \mathcal{B}^{+*}(H)$ for $i = 1, 2, \ldots, n$, then*

$$\Big(\sum_{i=1}^n p_i T_i\Big)^{-1} \le \sum_{i=1}^n p_i T_i^{-1}. \tag{2.2}$$

PROOF. The lemma is well known for $n = 2$ and, for $n \ge 2$, it can be proved by mathematical induction. We can also proceed in the following simple way. By (1.2), for all $x^* \in H$,

$$\begin{aligned}\Big(\sum_{i=1}^n p_i f_i\Big)^*(x^*) &= \sup_{x\in H}\Big\{\mathrm{Re}\,\langle x^*, x\rangle - \sum_{i=1}^n p_i f_i(x)\Big\} = \sup_{x\in H}\Big\{\sum_{i=1}^n p_i(\mathrm{Re}\,\langle x^*, x\rangle - f_i(x))\Big\}\\ &\le \sum_{i=1}^n p_i \sup_{x\in H}\{\mathrm{Re}\,\langle x^*, x\rangle - f_i(x)\} = \sum_{i=1}^n p_i f_i^*(x^*),\end{aligned}$$

giving (2.1). Taking $f_i = f_{T_i}$ in (2.1) and observing that $f_i^* = f_{T_i^{-1}}$ yields (2.2) after a simple manipulation. The proof of the lemma is complete. □

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The functional inequality (2.1) is a discrete Jensen's inequality, meaning that the duality map $f \longmapsto f^*$ is point-wise convex, whereas (2.2) means that the inverse map $T \longmapsto T^{-1}$ is operator convex.

REMARK 2.2. The operator inequality (2.2) is, of course, well known in the literature. We mention it here because it will be needed later and it also illustrates how the functional approach and functional inequalities such as (2.1) yield related operator inequalities in a simple and short manner.

For the sake of simplicity, for $f \in \widetilde{\mathbb{R}}^H$ and all $x, x^* \in H$, we set

$$\mathcal{F}_f(x, x^*) = f(x) + f^*(x^*) - \operatorname{Re}\langle x^*, x\rangle \tag{2.3}$$

and we recall Fenchel's inequality

$$\mathcal{F}_f(x, x^*) \geq 0 \quad \text{for all } x^*, x \in H. \tag{2.4}$$

In what follows, $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in (\widetilde{\mathbb{R}}^H)^n$ and $\mathbf{T} = (T_1, T_2, \ldots, T_n) \in (\mathcal{B}^{+*}(H))^n$. We also set

$$\mathbf{P}_n = \Big\{\mathbf{p} := (p_1, p_2, \ldots, p_n) : p_i \geq 0,\ i = 1, 2, \ldots, n;\ \sum_{i=1}^n p_i = 1\Big\}.$$

The weighted arithmetic and harmonic operator and functional means, previously defined for two variables, can be immediately extended to $n$ arguments as follows:

$$\mathcal{A}_n^F(\mathbf{f}; \mathbf{p}) = \sum_{i=1}^n p_i f_i, \quad \mathcal{H}_n^F(\mathbf{f}; \mathbf{p}) = \Big(\sum_{i=1}^n p_i f_i^*\Big)^*$$

and

$$\mathcal{A}_n^O(\mathbf{T}; \mathbf{p}) = \sum_{i=1}^n p_i T_i, \quad \mathcal{H}_n^O(\mathbf{T}; \mathbf{p}) = \Big(\sum_{i=1}^n p_i T_i^{-1}\Big)^{-1}.$$

If $p = (1/n, 1/n, \ldots, 1/n) \in \mathbf{P}_n$, then we simply write these means as $\mathcal{A}_n^F(\mathbf{f})$, $\mathcal{H}_n^F(\mathbf{f})$, $\mathcal{A}_n^O(\mathbf{T})$ and $\mathcal{H}_n^O(\mathbf{T})$, respectively. Since $f^{**} \leq f$ for each $f \in \widetilde{\mathbb{R}}^H$, Lemma 2.1 gives $\mathcal{A}_n^F(\mathbf{f}; \mathbf{p}) \geq \mathcal{H}_n^F(\mathbf{f}; \mathbf{p})$ and $\mathcal{A}_n^O(\mathbf{T}; \mathbf{p}) \geq \mathcal{H}_n^O(\mathbf{T}; \mathbf{p})$.

We will also use the following lemma.

LEMMA 2.3. *Let* $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in (\widetilde{\mathbb{R}}^H)^n$ *and* $p = (p_1, p_2, \ldots, p_n) \in \mathbf{P}_n$. *For all* $x \in H$,

$$\mathcal{A}_n^F(\mathbf{f}; \mathbf{p})(x) - \mathcal{H}_n^F(\mathbf{f}; \mathbf{p})(x) = \inf_{x^* \in H} \sum_{i=1}^n p_i \mathcal{F}_{f_i}(x, x^*). \tag{2.5}$$

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Proof. Suppose that $x \in H$. By (1.2) and simple manipulations,

$$\begin{aligned}
\mathcal{A}_n^F(\mathbf{f};\mathbf{p})(x) - \mathcal{H}_n^F(\mathbf{f};\mathbf{p})(x) &:= \sum_{i=1}^n p_i f_i(x) - \Big(\sum_{i=1}^n p_i f_i^*\Big)^*(x)\\
&= \sum_{i=1}^n p_i f_i(x) - \sup_{x^*\in H}\Big\{\mathrm{Re}\,\langle x^*, x\rangle - \sum_{i=1}^n p_i f_i^*(x^*)\Big\}\\
&= \inf_{x^*\in H}\Big\{\sum_{i=1}^n p_i f_i(x) + \sum_{i=1}^n p_i f_i^*(x^*) - \mathrm{Re}\,\langle x^*, x\rangle\Big\}\\
&= \inf_{x^*\in H}\Big\{\sum_{i=1}^n p_i(f_i(x) + f_i^*(x^*) - \mathrm{Re}\,\langle x^*, x\rangle)\Big\}.
\end{aligned}$$

When combined with (2.3), this yields the desired result. □

## 3. Main results

We preserve the same notations as previously. The following theorem is our first main result.

Theorem 3.1. *Let $\mathbf{p}, \mathbf{q} \in \mathbf{P}_n$ with $\mathbf{q} = (q_1, q_2, \dots, q_n)$ and $q_i > 0$ for $i = 1, 2, \dots, n$. Then*

$$m(\mathcal{A}_n^F(\mathbf{f};\mathbf{q}) - \mathcal{H}_n^F(\mathbf{f};\mathbf{q})) \le \mathcal{A}_n^F(\mathbf{f};\mathbf{p}) - \mathcal{H}_n^F(\mathbf{f};\mathbf{p}) \le M(\mathcal{A}_n^F(\mathbf{f};\mathbf{q}) - \mathcal{H}_n^F(\mathbf{f};\mathbf{q})) \qquad (3.1)$$

*and*

$$m(\mathcal{A}_n^O(\mathbf{T};\mathbf{q}) - \mathcal{H}_n^O(\mathbf{T};\mathbf{q})) \le \mathcal{A}_n^O(\mathbf{T};\mathbf{p}) - \mathcal{H}_n^O(\mathbf{T};\mathbf{p}) \le M(\mathcal{A}_n^O(\mathbf{T};\mathbf{q}) - \mathcal{H}_n^O(\mathbf{T};\mathbf{q})), \quad (3.2)$$

*where we set*

$$m := \min_{1\le i\le n}\Big(\frac{p_i}{q_i}\Big) \quad \text{and} \quad M := \max_{1\le i\le n}\Big(\frac{p_i}{q_i}\Big).$$

Proof. First, (3.2) is an immediate consequence from (3.1) when we take $f_i = f_{T_i}$ for $i = 1, 2, \dots, n$. We now prove (3.1). From (2.4),

$$\begin{aligned}
\min_{1\le i\le n}\Big(\frac{p_i}{q_i}\Big)\sum_{i=1}^n q_i\mathcal{F}_{f_i}(x, x^*) \le \sum_{i=1}^n p_i\mathcal{F}_{f_i}(x, x^*) &= \sum_{i=1}^n \Big(\frac{p_i}{q_i}\Big) q_i\mathcal{F}_{f_i}(x, x^*)\\
&\le \max_{1\le i\le n}\Big(\frac{p_i}{q_i}\Big)\sum_{i=1}^n q_i\mathcal{F}_{f_i}(x, x^*).
\end{aligned}$$

Taking the infimum over $x^* \in H$ and using (2.5) gives the desired inequalities. □

Remark 3.2. In [1], Dragomir proved inequalities similar to those of Theorem 3.1 for real-valued convex functions (see [1, Theorem 1, page 472]). The proof presented there is long and seems to be nonstandard, although it corresponds to scalar convex functions. By comparison, the proof of the previous functional inequalities is simple and short. Furthermore, it immediately yields the inequalities related to the operator version. This, with other results and applications discussed later, shows the interest of the present approach.

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Choosing $q = (1/n, 1/n, \ldots, 1/n) \in \mathbf{P}_n$ in the above theorem, we immediately obtain the following corollary.

Corollary 3.3. *If* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{P}_n$, *then*

$$n \min_{1\le i\le n}(p_i)(\mathcal{A}_n^F(\mathbf{f}) - \mathcal{H}_n^F(\mathbf{f})) \le \mathcal{A}_n^F(\mathbf{f};\mathbf{p}) - \mathcal{H}_n^F(\mathbf{f};\mathbf{p}) \le n \max_{1\le i\le n}(p_i)(\mathcal{A}_n^F(\mathbf{f}) - \mathcal{H}_n^F(\mathbf{f}))$$

*and*

$$n \min_{1\le i\le n}(p_i)(\mathcal{A}_n^O(\mathbf{T}) - \mathcal{H}_n^O(\mathbf{T})) \le \mathcal{A}_n^O(\mathbf{T};\mathbf{p}) - \mathcal{H}_n^O(\mathbf{T};\mathbf{p}) \le n \max_{1\le i\le n}(p_i)(\mathcal{A}_n^O(\mathbf{T}) - \mathcal{H}_n^O(\mathbf{T})).$$

The next corollary is just the case $n = 2$ of Corollary 3.3.

Corollary 3.4. *Let* $f, g \in \widetilde{\mathbb{R}}^H$ *and* $T, S \in \mathcal{B}^{+*}(H)$. *For* $\lambda \in (0, 1)$,

$$2r_\lambda(\mathcal{A}(f,g) - \mathcal{H}(f,g)) \le \mathcal{A}_\lambda(f,g) - \mathcal{H}_\lambda(f,g) \le 2R_\lambda(\mathcal{A}(f,g) - \mathcal{H}(f,g)) \tag{3.3}$$

*and*

$$2r_\lambda(T\nabla S - T!S) \le T\nabla_\lambda S - T!_\lambda S \le 2R_\lambda(T\nabla S - T!S), \tag{3.4}$$

*where* $r_\lambda := \min(1-\lambda, \lambda)$ *and* $R_\lambda := \max(1-\lambda, \lambda)$.

It is clear that, if $\lambda = 1/2$, the inequalities (3.3) and (3.4) become equalities. Note that (3.4) is just (1.4). As already pointed out in the introduction, (3.4) has already been proved by the standard method (see [2, 12]). In the present approach, (3.4) is immediately deduced from (3.3), which, in turn, is proved in a simple and short way. Further, by Corollary 3.3, analogues of (3.3) and (3.4) for several functional or operator variables are also valid. More discussion of this point as well as some applications will be presented in the following section.

## 4. Application to functional entropy

In this section we apply our previous results to obtain a refined upper bound for the relative functional entropy. The operator version is immediately obtained as well. We first need to recall some basic notions.

For $T, S \in \mathcal{B}^{+*}(H)$ and $\lambda \in (0, 1)$, the relative operator entropy $\mathcal{S}(T|S)$ is defined by

$$\mathcal{S}(T|S) = T^{1/2}\log(T^{-1/2}ST^{-1/2})T^{1/2}$$

(see, for example, [3]). Various extensions of $\mathcal{S}(T|S)$ have been introduced and related operator inequalities have been investigated (see [4–6, 9]).

After [10], the relative entropy can be extended from the case where the variables are positive operators to the case where the variables are (convex) functionals, as follows:

$$\mathcal{E}(f|g) = \int_0^1 \frac{\mathcal{H}_t(f,g) - f}{t}\,dt. \tag{4.1}$$

Some inequalities giving lower and upper bounds of $\mathcal{E}(f|g)$ can be found in [11]. For example, for all $x \in H$ such that $f(x) < +\infty$,

$$\mathcal{E}(f|g)(x) \le g(x) - f(x).$$

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Our aim here is to apply our results to obtain a refinement of this functional inequality, as stated in the following theorem.

THEOREM 4.1. *Let $f, g \in \widetilde{\mathbb{R}}^H$ and $\lambda \in (0,1)$. For all $x \in H$ such that $f(x) < +\infty$, we have the functional inequalities*

$$\mathcal{E}(f|g)(x) \leq g(x) - f(x) - (2\ln 2)(\mathcal{A}(f,g)(x) - \mathcal{H}(f,g)(x)) \leq g(x) - f(x).$$

PROOF. The right-hand inequality is obvious, since $\mathcal{A}(f,g)(x) - \mathcal{H}(f,g)(x) \geq 0$ for all $x \in H$. Now let $x \in H$ be such that $f(x) < +\infty$. We can write

$$\begin{aligned}\mathcal{H}_t(f,g)(x) - f(x) &= (\mathcal{H}_t(f,g)(x) - \mathcal{A}_t(f,g)(x)) + (\mathcal{A}_t(f,g)(x) - f(x))\\ &= (\mathcal{H}_t(f,g)(x) - \mathcal{A}_t(f,g)(x)) + t(g(x) - f(x)).\end{aligned}$$

According to the left-hand side of (3.3),

$$\mathcal{H}_t(f,g)(x) - f(x) \leq -2r_t(\mathcal{A}(f,g)(x) - \mathcal{H}(f,g)(x)) + t(g(x) - f(x)).$$

By dividing this latter inequality by $t > 0$, integrating the two sides over $t \in (0,1)$ and using (4.1),

$$\mathcal{E}(f|g)(x) \leq g(x) - f(x) - 2\int_0^1 \frac{r_t}{t}\,dt(\mathcal{A}(f,g)(x) - \mathcal{H}(f,g)(x)).$$

Since $r_t := \min(1-t, t)$, a simple computation gives

$$\int_0^1 \frac{r_t}{t}\,dt = \int_0^{1/2} \frac{r_t}{t}\,dt + \int_{1/2}^1 \frac{r_t}{t}\,dt = \int_0^{1/2} \frac{t}{t}\,dt + \int_{1/2}^1 \frac{1-t}{t}\,dt = \ln 2.$$

This yields the desired result. □

In the same way as before, the operator version of the above theorem follows immediately.

COROLLARY 4.2. *Let $T, S \in \mathcal{H}^{+*}(H)$ and $\lambda \in (0,1)$. Then*

$$\mathcal{S}(T|S) \leq S - T - (2\ln 2)(T\nabla S - T!S) \leq S - T.$$

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