THE DISTRIBUTION OF SEQUENCES AND SUMMABILITY

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1. We suppose that $0 < s_n \leqslant 1$ for every n, and denote by $n(\alpha, \beta)$ the number of $s_0, s_1, s_2, \ldots, s_n$ which fall in the interval $0 \leqslant \alpha < x \leqslant \beta \leqslant 1$.

If there exists a function $g(t)$, $0 \leqslant t \leqslant 1$, such that

$$\lim_{n\to\infty} \frac{n(\alpha, \beta)}{n} = g(\beta) - g(\alpha) \tag{1}$$

for every interval $(\alpha, \beta]$ with $0 \leqslant \beta - \alpha \leqslant 1$, the sequence (s_n) is said to have a distribution function $g(t)$, $0 \leqslant t \leqslant 1$, in the interval $[0, 1]$, (see **9**, p. 87). It follows from (1) that the function $g(t)$ is monotonic non-decreasing with $g(0) = 0$ and $g(1) = 1$.

In the special case when $g(t) \equiv t$ for every t in $[0, 1]$, the sequence (s_n) is said to be uniformly distributed in $[0, 1]$. H. Weyl (**8, 14**) proved that if a sequence (s_n) is uniformly distributed in $[0, 1]$ then for every Riemann integrable function $f(x)$ with $0 \leqslant x \leqslant 1$, the following relation holds:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(s_k) = \int_0^1 f(x)dx. \tag{2}$$

We now state:

THEOREM 1. *Let $A = (a_{mn})$ be a regular matrix, and let (s_n) be a real sequence such that $|s_n| < B$ for every n. Suppose that $\{x(n)\}$, $(n = 1, 2, \ldots)$ is the subsequence of positive integers such that $s_{x(n)} \leqslant x$ and let*

$$g_m(x) = \sum_{n=1}^{\infty} a_{m,x(n)}.$$

If $g_m(x)$ tends to a limit $g(x)$ as $m \to \infty$ for all x in $[-B, B]$, then

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n} f(s_n) = \int_{-B}^{B} f(x)dg(x) \tag{3}$$

for every continuous function $f(x)$ defined on $[-B, B]$, the integral being a Riemann-Stieltjes integral.

This theorem was proved by Henstock in the case $f(x) \equiv x$ (see **2, 5**). The first half of the proof is identical with Henstock's, but we include it for a later reference.

Proof. Since (a_{mn}) is a regular matrix, we have

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$$\sum_{n=1}^{\infty} |a_{m,n}| \leqslant M$$

for every $m \geqslant 1$; thus $g_m(x)$ exists for each m and x. Also $g_m(-B) = 0$, $g_m(B) = \sum a_{mn}$. Let $y > x$, then $\{x(n)\}$ is a subsequence of $\{y(n)\}$, so that

$$(4) \qquad |g_m(y) - g_m(x)| \leqslant \sum_{n=1}^{\infty} |a_{m,y(n)}| - \sum_{n=1}^{\infty} |a_{m,x(n)}|.$$

Hence, the variation of $g_m(x)$ in $[-B, B]$ is less than or equal to

$$\sum_{n=1}^{\infty} |a_{m,n}| \leqslant M$$

for all m; moreover, the Riemann-Stieltjes integral

$$\int_{-B}^{B} f(x) dg_m(x)$$

exists for all continuous $f(x)$.

Now let us select an arbitrary $\epsilon > 0$ and subdivide $[-B, B]$ by means of the points $\{x_i\}$ $(i = 0, 1, \ldots, k)$ into subintervals $[x_i, x_{i+1}]$ so small that the oscillation of $f(x)$ is less than ϵ/M on every interval $[x_i, x_{i+1}]$. Then if $\sum_{p(i)}$ denotes summation over all integers $p(i)$ in $\{x_i(n)\}$ that are not in $\{x_{i-1}(n)\}$, we have

$$\left| \sum_{n=1}^{\infty} a_{m,n} f(s_n) - \sum_{i=1}^{k} f(x_i)\{g_m(x_i) - g_m(x_{i-1})\} \right|$$
$$= \left| \sum_{i=1}^{k} \sum_{p(i)} a_{m,p(i)} \{f(s_{p(i)}) - f(x_i)\} \right|$$
$$\leqslant \sum_{i=1}^{k} \sum_{p(i)} |a_{m,p(i)}| \frac{\epsilon}{M} \leqslant \epsilon.$$

Hence we have:

$$(5) \qquad \sum_{n=1}^{\infty} a_{m,n} f(s_n) = \int_{-B}^{B} f(x) dg_m(x).$$

From (4) we have that the variation of $g_m(x)$ is less than M for all m, and so using a theorem of Helly (see **10**, p. 232),

$$(6) \qquad \lim_{m \to \infty} \int_{-B}^{B} f(x) dg_m(x) = \int_{-B}^{B} f(x) dg(x)$$

and our theorem follows directly from (5) and (6).

A submatrix of a regular matrix is formed by removing rows from the original matrix. It is clear that if (s_n) is bounded and not A-summable there will be many different submatrices of A which sum (s_n). We now prove:

PROPOSITION 1. *If $|s_n| \leqslant B$, there is a submatrix of $A = (a_{mn})$ which sums (s_n) and $\{f(s_n)\}$, where $f(x)$ is any continuous function on the interval $[-B, B]$.*

Proof. From (4) we have that the functions $g_m(x)$ are of uniformly bounded variation and uniformly bounded. Hence, we can apply a second theorem of Helly's (see **10**, p. 222) to find a subsequence $\{g_{m_k}(x)\}$ that converges everywhere in $[-B, B]$ to a function of bounded variation $g(x)$. If we apply Theorem 1 to this submatrix, we have proved our assertion.

Suppose it is known that (3) is true for every continuous function $f(x)$ and some fixed function $g(x)$. What can be said about the nature of $g(x)$? We now prove:

PROPOSITION 2. *If* (3) *is true for some* $g(x)$ *and every continuous function* $f(x)$ *in* $[-B, B]$ *then there is a subsequence of the integers* $\{m_k\}$ *such that*

$$g(x) = \lim_{k\to\infty} g_{m_k}(x) \tag{7}$$

for all values of x *with the exception of a countable set.*

Proof. By Proposition 1, we can select a subsequence $\{m_k\}$ in such a way that

$$g'(x) = \lim_{k\to\infty} g_{m_k}(x)$$

for all x. Also, from Proposition 1,

$$\lim_{k\to\infty} \sum a_{m_k n} f(s_n) = \int_{-B}^{B} f(x)dg'(x).$$

Since submatrices of a regular matrix A sum all A-summable sequences to the same value,

$$\int_{-B}^{B} f(x)d[g(x) - g'(x)] = 0$$

for every continuous $f(x)$ on $[-B, B]$. From this relation and a theorem of F. Riesz, (**13**, p. 243), our conclusion follows.

If a bounded sequence (s_n) has a finite set of limit points, Cooke and Barnett (**2**, **1**) showed that (s_n) is summed by the regular matrix $A = (a_{mn})$ if a certain finite set of sequences of 0's and 1's is A-summable. The particular members of this finite set depend on the limit points of the sequence. For the case $f(x) \equiv x$, Theorem 1 is an extension of this idea to the case of the general bounded sequence. Henstock (**5**) showed that if $\lim g_m(x)$ exists for a countable everywhere-dense set of x, then (s_n) is A-summable; this is an improvement on the non-countable set implied in Theorem 1. We now prove a theorem which complements another of Henstock's (see **5**, p. 31).

THEOREM 2. *If* $a_{n,k} \geqslant 0$ *and* $\lim g_m(x)$ *exists for an everywhere-dense set in* $[-B, B]$, *then a* $g(x)$ *exists such that*

$$\lim_{m\to\infty} \sum a_{mn} f(s_n) = \int_{-B}^{B} f(x)dg(x)$$

for every continuous function $f(x)$ *in* $[-B, B]$.

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Proof. From the properties of a regular matrix it is clear that

$$\lim_{m\to\infty} g_m(x)$$

will exist at $x = -B$ and $x = B$. Now if $a_{mn} \geqslant 0$ and $\lim g_m(x)$ exists on an everywhere-dense set, then if $x_1 \geqslant x_2$ we have $\lim g_m(x_1) \geqslant \lim g_m(x_2)$. We may complete the definition of $g(x)$ by writing

$$g(x) = \lim_{u\to x} g(u)$$

when u approaches x over points of the everywhere-dense set which are less than x. The function $g(x)$ is now defined for all values of x in $[-B, B]$; it is clearly non-decreasing. Our theorem now follows from a theorem due to Hilbert (**13**, p. 245), and a reference to (4).

2. Let (s_n) be a sequence of numbers satisfying $0 < s_n \leqslant 1$ for every $n = 1, 2, 3, \ldots$, then (s_n) is said to be well distributed if and only if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=n+1}^{n+p} I_{[a,b]}(s_k) = b - a$$

holds uniformly in n for every interval $[a, b]$, $I_{[a,b]}(x)$ being the characteristic function of the interval $[a, b]$ (see **12**). Clearly all well-distributed sequences are also uniformly distributed, where uniform distribution is defined in § 1 (see also **14**).

In this paragraph and in the remainder of the paper, we shall use the notation $\{\theta\}$ for $\theta - [\theta]$, where $[\theta]$ is the largest integer less than or equal to θ. Where there is no possibility of confusion we shall write $\{s_n\}$ for $(\{s_n\})$. From the sequence (s_n) we define another one (a_n/n) as follows:

$$(8) \qquad a_n = r \quad \text{where} \quad \frac{r}{n} \leqslant \{s_n\} < \frac{r+1}{n},$$

for $n = 1, 2, 3, \ldots$ and $0 \leqslant r \leqslant n - 1$. We also write

$$(9) \qquad \alpha = \sum_{n=1}^{\infty} \frac{a_n}{n!}, \qquad 0 < \alpha \leqslant 1.$$

From (8) it is clear that

$$\lim_{n\to\infty} \left| \{s_n\} - \frac{a_n}{n} \right| = 0.$$

However, it is shown in **(6)** that if $\lim |\{s_n\} - \{t_n\}| = 0$, then both of the sequences $\{s_n\}$ and $\{t_n\}$ are either well distributed or not well distributed. Hence the sequence $\{s_n\}$ is well distributed if and only if (a_n/n) is well distributed. On the other hand it is shown in **(6)** that the sequence (a_n/n) is well distributed if and only if the sequence $\{n!\,\alpha\}$ is well distributed. This means that corresponding to every sequence $\{s_n\}$ there is a number α, $0 < \alpha \leqslant 1$ such that $\{s_n\}$ is well distributed if and only if the sequence $\{n!\,\alpha\}$ is well distributed. A similar situation holds with respect to uniform distribution as may be seen from the theorem proved in **(6)**.

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The following theorem is due to Weyl (see **8** or **14**).

THEOREM 3. *If* $g(x)$ *has the positive integers for its domain and* $g(x) \neq g(y)$ *for* $x \neq y$, *then* $\{g(n)\alpha\}$ *is a uniformly distributed sequence for almost all* α.

From this theorem it is clear that the sequence $\{n!\,\alpha\}$ is uniformly distributed for almost all α. In a certain sense, almost all sequences $\{s_n\}$ are uniformly distributed. Here we shall show that the sequence $\{n!\,\alpha\}$ is well distributed for almost no α.

First, however, we shall make a few remarks on homogeneous sets. Let E be a subset of $(0, 1)$ and let the density $\Delta(a, b)$ of E in the interval (a, b), $0 \leqslant a < b \leqslant 1$, be defined by the following relation

$$\Delta(a, b) = \frac{\text{outer measure } (E \cap (a, b))}{|b - a|}.$$

If E is of measure one, it is clear that $\Delta(a, b) = 1$ for every interval (a, b); likewise if E is of measure 0, it is clear that $\Delta(a, b) = 0$ for every interval (a, b). Sets having the same density for every interval in $(0, 1)$ are called homogeneous. If E is measurable of measure r, $0 < r < 1$, then the complement of E can be covered by a set of open intervals whose total length is $(1 - r) + \epsilon$ where ϵ is an arbitrarily small positive quantity. Denote this set of open intervals by S; then the outer measure of $E \cap S$ is less than ϵ. For at least one of the intervals (α, β) belonging to S, it follows that

$$\Delta(\alpha, \beta) \leqslant \frac{\epsilon}{(1 - r) + \epsilon}.$$

If, on the other hand, we examined the open set covering E, we could show the existence of intervals such that

$$\Delta(\alpha, \beta) \geqslant \frac{r}{r + \epsilon}.$$

It is now clear that a necessary and sufficient condition for a measurable set to be homogeneous is that its measure be either zero or one. Moreover, if E is measurable, $\Delta(a, b) \geqslant \delta > 0$ for all intervals (a, b), then E is homogeneous and of measure one; see also Knopp (**7**, p. 413, Satz 4).

We now prove the following:

THEOREM 4. *Let* $(n(k))$ *be a subsequence of the integers,*

$$\frac{n(k)}{n(k-1)} = r(k), \qquad r(k) \nearrow \infty,$$

then for almost all α, $0 < \alpha \leqslant 1$, *the sequence* $\{n(k)\alpha\}$ *is not well distributed.*

Proof. If the sequence $\{n(k)\alpha\}$ is well distributed, then we cannot have, for instance,

$$\{n(k)\alpha\} \leqslant \tfrac{1}{2} \quad \text{for} \quad k_\nu \leqslant k \leqslant k_\nu + [\log_2 \nu] \tag{10}$$

for infinitely many ν. For if $I(x)$ is the characteristic function of $[0, \frac{1}{2}]$ we would then have

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$$(11) \qquad \frac{1}{[\log_2 \nu]} \sum_{n=k_\nu+1}^{k_\nu+[\log_2 \nu]} I(s_n') = 1 \qquad \text{with} \qquad s_n' = \{n(k)\alpha\}$$

for infinitely many k and the sequence $\{n(k)\alpha\}$ is not well distributed.

We first prove that the set of α for which $\{n(k)\alpha\}$ is not well distributed contains a set of positive measure. Let E_k be the set of α for which $\{n(k)\alpha\} \leqslant \frac{1}{2}$. This set consists of the first half of each of the intervals

$$\left(0, \frac{1}{n(k)}\right), \left(\frac{1}{n(k)}, \frac{2}{n(k)}\right), \ldots, \left(\frac{n(k)-1}{n(k)}, 1\right)$$

and $\mu(E_k) = \frac{1}{2}$. We first remark that contained in the interval

$$J'(r, n(k)) = \left(\frac{r}{n(k)}, \frac{2r+1}{2n(k)}\right)$$

there will be at least $[\frac{1}{2}r(k+1)] - 1$ intervals of the form

$$J(r, n(k+1)) = \left(\frac{r}{n(k+1)}, \frac{r+1}{n(k+1)}\right)$$

for there can be at most two intervals of the form $J(r, n(k+1))$ which intersect $J'(r, n(k))$ but do not lie completely in $J'(r, n(k))$ and their combined length may be either

$$\frac{r(k+1)}{2} \frac{1}{n(k+1)} - \left[\frac{r(k+1)}{2}\right] \frac{1}{n(k+1)}$$

or

$$\frac{r(k+1)}{2} \frac{1}{n(k+1)} - \left(\left[\frac{r(k+1)}{2}\right] - 1\right) \frac{1}{n(k+1)}.$$

Hence the number of intervals of the form $J'(r, n(k+1))$ completely contained in E_k is at least

$$([\tfrac{1}{2}r(k+1)] - 1)n(k) \geqslant (\tfrac{1}{2}r(k+1) - 2)n(k).$$

Each of the intervals $J'(r, n(k+1))$ in turn contains at least $[\frac{1}{2}r(k+2)] - 1$ intervals of the form $J'(r, n(k+2))$. It follows that

$$\bigcap_{n=k}^{k+p} E_n$$

contains at least

$$n(k)(\tfrac{1}{2}r(k+1) - 2) \ldots (\tfrac{1}{2}r(k+p) - 2)$$

intervals of the form $J'(r, n(k+p))$. This means that

$$\mu\left(\bigcap_{n=k}^{k+p} E_n\right) \geqslant \frac{1}{2} \frac{n(k)}{n(k+p)} (\tfrac{1}{2}r(k+1) - 2) \ldots (\tfrac{1}{2}r(k+p) - 2)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{r(k+1)}\right) \ldots \left(\frac{1}{2} - \frac{2}{r(k+p)}\right) = \left(\frac{1}{2}\right)^{p+1} - \lambda(k, p) = L(k, p),$$

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where

(12) $$\lim_{k\to\infty} \lambda(k, p) = 0 \qquad (p = 1, 2, \ldots).$$

On the other hand, there can be no more than $[\frac{1}{2}r(k+1)]$ intervals of the form $J(r, n(k+1))$ wholly contained in $J'(r, n(k))$, and indeed $J'(r, n(k))$ is wholly contained by no more than $[\frac{1}{2}r(k+1)] + 2 \leqslant \frac{1}{2}r(k+1) + 2$ intervals of the form $J(r, n(k+1))$. This is also an upper bound for the number of intervals of the form $J'(r, n(k+1))$ which intersect $J'(r, n(k))$. It is now clear that

$$\mu\left(\bigcap_{n=k}^{k+p} E_n\right) \leqslant \frac{1}{2}\left(\frac{1}{2} + \frac{2}{r(k+1)}\right) \ldots \left(\frac{1}{2} + \frac{2}{r(k+p)}\right) = \left(\frac{1}{2}\right)^{p+1} + \zeta(k,p) = U(k,p)$$

where

(13) $$\lim_{k\to\infty} \zeta(k, p) = 0 \qquad (p = 1, 2, \ldots),$$

and that

(14) $$\lim_{k\to\infty} \frac{U(k, p)}{L(k, p)} = 1 \qquad (p = 1, 2, \ldots).$$

For $\nu = 1, 2$, let $\xi_\nu = E_{k_\nu}$ where $k_2 > k_1$ and $r(k_1) \geqslant 16$. For $\nu \geqslant 3$ let

(15) $$\xi_\nu = \bigcap_{r=k_\nu}^{k_\nu + p(\nu)} E_r$$

and choose first $p(\nu) = [\log_2 \nu]$, $\nu \neq 2^k$; then by virtue of (12) and the fact that $2^{-[\log_2 \nu]} > 1/\nu$,

(16) $$L(k, p(\nu)) \geqslant 1/\nu$$

for $k_\nu \geqslant K$. For $\nu = 2^k$, let $p(\nu) = k - 1$ and again by virtue of (12)

$$L(k, p(\nu)) \geqslant 2^{1-k} - \epsilon \geqslant 2/\nu - \epsilon > 1/\nu$$

for $k_\nu \geqslant K'$. Since

$$(\nu/2^{[\log_2 \nu]}) \leqslant (2\nu/2^{\log_2 \nu}) \leqslant 2,$$

it is evident that for $\nu \neq 2^k$

$$L(k, p(\nu)) \leqslant 2/\nu,$$

and this will also be true for $\nu = 2^k$. In either case, because of (14) it is possible to choose k_ν so large that

(17) $$U(k_\nu, p(\nu)) \leqslant 5/2\nu$$

and $k_\nu \geqslant k_{\nu-1} + p(\nu - 1)$. With this choice of $p(\nu)$ and then of k_ν, we have

(18) $$5/2\nu \geqslant \mu(\xi_\nu) \geqslant 1/\nu.$$

Moreover, from (13) and (15) and the argument in the sentences preceding (13), it is clear that ξ_ν is covered by intervals of the form $J'(r, n(k_\nu + p(\nu)))$

whose number does not exceed $n(k_\nu)U(k_\nu, p(\nu))$. Each of the intervals of the form $J'(r, n(k_\nu + p(\nu)))$ is in turn intersected by no more than

$$\left[\frac{n(k_{\nu+h})}{2n(k_\nu + p(\nu))}\right] + 2$$

intervals of the form $J'(r, n(k_{\nu+h}))$. Since $r(k_1) \geqslant 16$, we also have

$$\left[\frac{n(k_{\nu+h})}{2n(k_\nu + p(\nu))}\right] + 2 \leqslant \frac{5n(k_{\nu+h})}{8n(k_\nu + p(\nu))}.$$

It is also clear that each of the intervals of the form $J'(r, n(k_{\nu+h}))$ is intersected by no more than $U(k_{\nu+h}, p(\nu + h))$ intervals of the form $J'(r, n(k_{\nu+h} + p(\nu + h)))$ which belongs to $\xi_{\nu+h}$. Hence we have

$$\mu(\xi_\nu\xi_{\nu+h}) \leqslant \tfrac{1}{2}n(k_\nu)U(k_\nu, p(\nu)) \cdot \frac{5n(k_{\nu+h})}{8n(k_\nu + p(\nu))} \cdot U(k_{\nu+h}, p(\nu + h)) \cdot \frac{1}{n(k_{\nu+h} + p(\nu + h))} \leqslant \frac{125}{64} \cdot \frac{1}{\nu} \cdot \frac{1}{\nu + h}.$$

Now for any n_0, $M \geqslant N \geqslant n_0$ so that

$$\sum_{k=N}^{M} \frac{1}{k} = a$$

is arbitrarily close to $\frac{1}{2}$. For $N \leqslant n \leqslant M$, we have

$$\mu\left(\xi_n \cap \bigcup_{k=N}^{n-1} \xi_k\right) = \mu\left(\bigcup_{k=N}^{n-1} \cap \xi_n\xi_k\right) \leqslant \frac{125}{64} \cdot \frac{1}{n} \sum_{k=N}^{n-1} \frac{1}{k} \leqslant a \cdot \frac{125}{64} \cdot \frac{1}{n}, \qquad \text{for } n > N.$$

Therefore, it follows that

$$\mu\left(\xi_n - \bigcup_{k=N}^{n-1} \xi_k\right) \geqslant \frac{1}{n} - a \cdot \frac{125}{64} \cdot \frac{1}{n},$$

$$\mu\left(\bigcup_{n=n_0}^{\infty} \xi_n\right) \geqslant \mu\left(\bigcup_{n=N}^{M} \xi_n\right) = \sum_{n=N}^{M} \mu\left(\xi_n - \bigcup_{k=N}^{n-1} \xi_k\right) \geqslant \left(1 - \frac{125a}{64}\right) \sum_{n=N}^{M} \frac{1}{n} = \left(1 - \frac{125}{64} a\right)a.$$

This last expression can be made arbitrarily close to 3/256, so that $\mu(\overline{\lim}\, \xi_n) \geqslant 3/256$. We have seen that if $\alpha \in \overline{\lim}\, \xi_n$, $\{n(k)\alpha\}$ is not well distributed. In a different context, this argument is the same as one used by Lorentz **(9**, p. 134; see also Halmos **3)**.

We shall now show that the set F of α for which $\{n(k)\alpha\}$ is not well distributed is of measure one. Let (a, b) be an interval such that $0 \leqslant a < b \leqslant 1$; then if k_0 is sufficiently large, (a, b) will contain intervals of the form $J(r, n(k_0))$. In fact the number of such intervals wholly contained in (a, b) will be

$$[|b - a|n(k_0)] \qquad \text{or} \qquad [|b - a|n(k_0)] - 1.$$

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Now it is evident that the sets $\xi_\nu(k_0)$ $(\nu = 1, 2, \ldots)$, $k_1 = k_0$, could be constructed in much the same way as before. Moreover, it is clear that the upper and lower bounds obtained in (12) and (13) and the arguments preceding them are of such a type that they could be adapted to proving

$$\mu(\overline{\lim}\,\xi_\nu \cap J(r, n(k_0))) \geqslant \frac{3}{256n(k_0)}$$

in much the same way as for the interval (0, 1). This means

$$\mu(\overline{\lim}\,\xi_\nu(k_0) \cap (a, b)) \geqslant ([b - a|n(k_0)] - 1)\frac{3}{256n(k_0)}$$
$$\geqslant \frac{3}{256}|b - a| - \frac{6}{256n(k_0)}.$$

The set $\overline{\lim}\,\xi_\nu(k_0)$ $(k_0 = 1, 2, \ldots)$ is a Borel set and hence measurable. Let the set G consist of those

$$\alpha \in \bigcup_{k_0=1}^{\infty} \overline{\lim}\,\xi_\nu(k_0).$$

It is clear that G, being an enumerable union of measurable sets, is also measurable. Moreover,

$$\mu(G \cap (a, b)) \geqslant \frac{3}{256}|b - a|$$

and the density of G in any interval is greater than 3/256. From our preliminary remarks this evidently means that the set G is homogeneous and of measure one. The set F of α for which $\{n(k)\alpha\}$ is not well distributed includes G and so is measurable, with measure one.

A normal number is one in whose decimal expansion all digits occur with equal frequency and in fact all blocks of digits of the same length occur with equal frequency. Let $x = .x_1x_2\ldots$ be an infinite decimal to base r and let X_n denote the block of digits $x_1x_2\ldots x_n$. Let $N(B_k, X_n)$ denote the number of occurrences of the block $B_k = b_1\ldots b_k$ in X_n. Then x is normal to the base r if

$$\lim_{n\to\infty}\frac{1}{n}N(B_k, X_n) = r^{-k}$$

for all B_k, $k = 1, 2, 3, \ldots$. A complete discussion of normal numbers can be found in **(11)**. The following has been proved by Wall **(11**, p. 110**)**:

THEOREM 5. *The number x is normal to base r if and only if the sequence $\{r^kx\}$ is uniformly distributed.*

In **(6)** it is stated that:

THEOREM 6. *The sequence $\{r^k\theta\}$ is not well distributed for any θ when r is any rational number.*

The proof given for Theorem 6 is not correct and the complete truth of the matter remains open. We note that the argument could be modified to give the result, when r is integral. However, in this special case, the following approach is simpler.

THEOREM 6′. *The sequence $\{r^k\theta\}$ is not well distributed for any θ when r is an integer.*

Proof. In the first place the sequence $\{r^k\theta\}$ must be uniformly distributed or it cannot be well distributed. This means, according to Theorem 5, that it must be normal to the base r. However, from the definition of a normal number, it is clear that for every $p = 1, 2, 3, \ldots$ there is an n_p such that if $\theta = .x_1x_2\ldots$ to base r, then

$$x_{n_p} = x_{n_p+1} = \ldots = x_{n_p+p} = 0.$$

It follows that the terms of $\{r^k\theta\}$ are less than or equal to $1/r$ for

$$n_p + 1 \leqslant k \leqslant n_p + p.$$

Hence if $I(x)$ is the interval function for $(0, 1/r)$, then

$$\frac{1}{p}\sum_{n_p+1}^{n_p+p} I(\{r^k\theta\}) = 1$$

and $\{r^k\theta\}$ cannot be well distributed. This proves the theorem.

REFERENCES

1. R. G. Cooke and A. M. Barnett, *The "right" value for the generalized limit of a bounded divergent sequence*, J. London Math. Soc., *23* (1948), 211–221.
2. R. G. Cooke, *Infinite matrices and sequence spaces* (London, 1950).
3. P. R. Halmos, *Measure theory* (Princeton, 1950).
4. G. H. Hardy, *Divergent series* (Oxford, 1949).
5. R. Henstock, *The efficiency of matrices for bounded sequences*, J. London Math. Soc., *25* (1950), 27–33.
6. F. R. Keogh, B. Lawton, and G. M. Petersen, *Well distributed sequences*, Can. J. Math. *10* (1958), 572–576.
7. K. Knopp, *Mengentheoretische Behandlung einiger Probleme der diophantischen Approximationen und der transfiniten Wahrscheinlichkeiten*, Math. Ann., *95* (1926), 409–426.
8. J. F. Koksma, *Diophantische Approximationen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. IV (Berlin, 1936).
9. G. G. Lorentz, *Borel and Banach properties of methods of summation*, Duke Math. J. *22* (1955), 129–141.
10. I. P. Natanson, *Theory of functions of a real variable*, Translated by Boron and Hewitt from the Russian (New York, 1955).
11. I. Niven, *Irrational numbers*, Carus Monographs, no. *11* (1956).
12. G. M. Petersen, *Almost convergence and uniformly distributed sequences*, Quart. J. Math. (Oxford), *7* (1956), 188–191.
13. H. S. Wall, *Analytic theory of continued fractions* (New York, 1948).
14. H. Weyl, *Ueber die Gleichverteilung von Zahlen mod. Eins*, Math. Ann., *77* (1916), 313–352.
15. A. Zygmund, *Trigonometric series*, Vol. I (Cambridge, 1959), see p. 140.

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