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MAXIMUM PRINCIPLE FOR NON-LINEAR DEGENERATE EQUATIONS OF THE PARABOLIC TYPE

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This paper establishes a weak maximum principle for the difference $u - v$ of solutions to nonlinear degenerate parabolic differential inequality

$$\alpha(t, x)u_t - f\left(t, x, u(t, x), Du(t, x), D^2u(t, x)\right) \le \alpha(t, x)v_t - f\left(t, x, v(t, x), Dv(t, x), D^2v(t, x)\right) .$$

The function α is non-negative and f is assumed to be parabolic with respect to u in the sense that there exists a non-negative function κ such that

$$f\left(t, x, u(t, x), Du(t, x), r_1\right) - f\left(t, x, Du(t, x), r_2\right) \ge \kappa(t, x)\mathrm{Tr}\left(r_1 - r_2\right) ,$$

whenever r_1 and r_2 are symmetric matrices and $r_1 \ge r_2$. The crucial assumption is that $\alpha + \kappa$ is bounded away from zero.

The results are then applied to the uniqueness of the Cauchy problem for the degenerate parabolic equation

$$\alpha(t, x)u_t = f\left(t, x, u, Du, D^2u\right)$$

under various growth conditions similar to those used in uniqueness theorems for parabolic (non-degenerate) equations.

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The main purpose of this note is to prove a weak maximum principle for non-linear degenerate equations of the parabolic type. There is an extensive literature on the maximum principle for parabolic and elliptic equations (see [8] and [10]). In recent years the maximum principle was extended to degenerate elliptic parabolic equations and has been studied by several authors [1], [4], [5], [6], [9], [12], [13], [14]. We begin in Section 1 by considering a maximum principle for non-linear degenerate equations of parabolic type in a bounded domain. Theorem 1 is an extension to non-linear equations of the weak maximum principle proved by Ippolito [5]. In Section 2 we extend these results to an infinite strip by using the method of growth damping factors (see [2] and [7]). As an application we obtain the uniqueness of the first initial-boundary value problem in a bounded cylinder and that of the Cauchy problem on a half space.

1.

We consider a differential inequality of the form

$$(1)\quad \alpha(t, x)u_t - f\big(t, x, u(t, x), Du(t, x), D^2u(t, x)\big) \le \alpha(t, x)v_t - f\big(t, x, v(t, x), Dv(t, x), D^2v(t, x)\big)$$

in $(0, T] \times \Omega$, where Ω is an open set in R_n. Du denotes the gradient of u with respect to x, D^2u is the Hessian matrix of the second order derivatives (also with respect to the variable x), D_i denotes the derivative with respect to x_i, $\alpha(t, x)$ is a non-negative function on $(0, T] \times R_n$. Let $Q = (0, T] \times \Omega$. We denote by $\partial_p Q$ the parabolic boundary of Q; that is, $\partial_p Q = \overline{Q} - Q$.

We assume that $f(t, x, u, p, r)$ is defined for $(t, x) \in Q$, $u \in R$, $p \in R_n$ and $r \in R_{n^2}$.

A function $u(t, x)$ is said to be regular on Q if it is continuous on $\overline{Q}$ and Du, D^2u and u_t are continuous on Q (at $t = T$ the derivative u_t is understood as the left-hand derivative).

Given a regular function u, the function f is said to be weakly parabolic with respect to u if there exists a non-negative function $\kappa = \kappa(t, x)$ such that

$$f\big(t, x, u(t, x), Du(t, x), r_1\big) - f\big(t, x, Du(t, x), r_2\big) \geq \kappa(t, x)\operatorname{Tr}\big(r_1-r_2\big)$$

holds for $(t, x) \in Q$, whenever r_1 and r_2 are symmetric matrices and $r_1 \geq r_2$ (that is, the quadratic form $\sum_{j,k=1}^{n} \big(r_{1jk}-r_{2jk}\big)\lambda_j\lambda_k$ is positive semidefinite).

This definition has been introduced by Besala [2] (see also Szarski [10] and [11]).

THEOREM 1. *Suppose that*

(i) Ω *is bounded,*

(ii) f *is decreasing with respect to* u *,*

(iii) there exists a positive constant L *such that*

$$|f(t, x, u, p, r)-f(t, x, u, \bar{p}, \bar{r})| \leq L(|p-\bar{p}|+|r-\bar{r}|)$$

holds for all $(t, x) \in Q$ *and arbitrary* $u, p, \bar{p}, r$ *and* $\bar{r}$ *,*

(iv) there exists a positive constant k *such that*

$$\alpha(t, x) + \kappa(t, x) \geq k$$

for $(t, x) \in Q$ *,*

(v) u *and* v *are solutions to* (1) *regular in* Q *,*

(vi) f *is weakly parabolic with respect to* u *or* v *.*

Then, if $u - v$ *has a positive maximum on* $\bar{Q}$ *, this maximum is attained at some point of* $\partial_p Q$ *.*

Proof. We prove the theorem with the assumption that f is weakly parabolic with respect to u. If f is weakly parabolic with respect to v the proof is similar. We set

$$M = \max\{u(t, x)-v(t, x);\ (t, x) \in \bar{Q}\} ,$$

$$m = \sup\{0, u(t, x)-v(t, x);\ (t, x) \in \partial_p Q\} .$$

Assume, contrary to what we want to prove, that $M > m$. Choose

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$x^0 \in R_n - \overline{\Omega}$, we may assume that $1 \le |x-x^0| \le C$ for all $x \in \overline{\Omega}$ and some positive constant C . Let $t_0 < 0$ and γ a positive number satisfying

$$\gamma > 2L(n+C)/k \ .$$

We define an auxiliary function ϕ by

$$\phi(t, x) = \exp\left[-\gamma|x-x^0|^2-\gamma^2(t-t_0)\right] ,$$

and choose $\varepsilon > 0$ in such a way that

$$\varepsilon\phi(t, x) < M - m , \tag{2}$$

for all $(t, x) \in \overline{Q}$. Let

$$w(t, x) = u(t, x) - v(t, x) + \varepsilon\phi(t, x) .$$

It follows from (2) that $w < M$ on $\partial_p Q$. Hence w attains its maximum (which is greater than M) over $\overline{Q}$ at a point $(\bar{t}, \bar{x}) \in Q$. Obviously

$$w_t \ge 0 , \quad D_i u - D_i v = 2\varepsilon\gamma\left(\bar{x}_i - x_i^0\right)\phi \quad (i = 1, \ldots, n) ,$$

$$D^2 w \le 0 ,$$

at $(\bar{t}, \bar{x})$. By assumption *(vi)* we have

$$\begin{aligned} 0 &\le \alpha(\bar{t}, \bar{x})w_t(\bar{t}, \bar{x}) \\ &= \alpha(\bar{t}, \bar{x})\left(u_t(\bar{t}, \bar{x})-v_t(\bar{t}, \bar{x})\right) - \varepsilon\gamma^2\alpha(\bar{t}, \bar{x})\phi(\bar{t}, \bar{x}) \\ &\le f\left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), Du(\bar{t}, \bar{x}), D^2u(\bar{t}, \bar{x})\right) \\ &\qquad - f\left(\bar{t}, \bar{x}, v(\bar{t}, \bar{x}), Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})\right) - \varepsilon\gamma^2\alpha(\bar{t}, \bar{x})\phi(\bar{t}, \bar{x}) \\ &= J - \varepsilon\gamma^2\alpha(\bar{t}, \bar{x})\phi(\bar{t}, \bar{x}) . \end{aligned} \tag{3}$$

Observe that

$$\begin{aligned} D^2u(\bar{t}, \bar{x}) &= \left[D^2w(\bar{t}, \bar{x})-4\varepsilon\gamma^2 r\phi(\bar{t}, \bar{x})\right] + \left[2\gamma\varepsilon I\phi(\bar{t}, \bar{x})+D^2v(\bar{t}, \bar{x})\right] \\ &= A + B , \end{aligned}$$

where r is the matrix with the entries

$$\left(\bar{x}_i - x_i^0\right)\left(\bar{x}_j - x_j^0\right)$$

and I is the identity matrix. Now

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(4)

$$J = [f(\bar t,\ \bar x,\ u(\bar t,\ \bar x),\ Du(\bar t,\ \bar x),\ A+B)-f(\bar t,\ \bar x,\ u(\bar t,\ \bar x),\ Du(\bar t,\ \bar x),\ B)]$$
$$+ [f(\bar t,\ \bar x,\ u(\bar t,\ \bar x),\ Du(\bar t,\ \bar x),\ B)-f(\bar t,\ \bar x,\ v(\bar t,\ \bar x),\ Du(\bar t,\ \bar x),\ B)]$$
$$+ [f(\bar t,\ \bar x,\ v(\bar t,\ \bar x),\ Du(\bar t,\ \bar x),\ B)-f(\bar t,\ \bar x,\ v(\bar t,\ \bar x),\ Dv(\bar t,\ \bar x),\ D^2v(\bar t,\ \bar x))]$$
$$= J_1 + J_2 + J_3 .$$

Using the weak parabolicity we estimate

(5)
$$J_1 \le \kappa(\bar t,\ \bar x)\left[\sum_{i=1}^{n} D_i^2 w(\bar t,\ \bar x)-4\varepsilon\gamma^2|\bar x-x_0|^2\phi(\bar t,\ \bar x)\right]$$
$$\le -4\varepsilon\gamma^2\kappa(\bar t,\ \bar x)\phi(\bar t,\ \bar x) .$$

The inequality $w(\bar t,\ \bar x) \ge M$ implies

$$u(\bar t,\ \bar x) - v(\bar t,\ \bar x) \ge M - \varepsilon\phi(\bar t,\ \bar x) \ge m \ge 0$$

and since f is decreasing

(6)
$$J_2 \le 0 .$$

Finally, in view of assumption *(iii)*, we have

(7)
$$J_3 \le L\{|Du(\bar t,\ \bar x)-Dv(\bar t,\ \bar x)|+2\gamma\varepsilon n\phi\}$$
$$= 2L\gamma\varepsilon\left(|\bar x-\bar x_0|+n\right)\phi(\bar t,\ \bar x)$$
$$\le 2\gamma\varepsilon L(C+n)\phi(\bar t,\ \bar x) .$$

Combining (3), (4), (5), (6) and (7) we obtain

$$0 \le \alpha(\bar t,\ \bar x)w_t(\bar t,\ \bar x)$$
$$\le \gamma\varepsilon\phi(\bar t,\ \bar x)[-\gamma\alpha(\bar t,\ \bar x)-4\gamma H(\bar t,\ \bar x)+2L(C+n)]$$
$$\le \gamma\varepsilon\phi(\bar t,\ \bar x)[-\gamma k+2L(C+n)] < 0 .$$

This contradiction completes the proof.

As an immediate consequence of Theorem 1 we obtain the uniqueness of the first initial-boundary value problem

(8)
$$\alpha(t,\ x)u_t = f(t,\ x,\ u,\ Du,\ D^2u) \quad \text{for} \quad (t,\ x) \in Q ,$$

(9)
$$u(t,\ x) = g(t,\ x) \quad \text{for} \quad (t,\ x) \in \partial_p Q ,$$

while g is a given continuous function on $\partial_p Q$.

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COROLLARY 1. *Assume that conditions (i), (ii), (iii) and (iv) of Theorem* 1 *hold. Let* u *and* v *be regular solutions to the problems* (8) *and* (9). *If* f *is weakly parabolic with respect to* u *or* v *then* $u \equiv v$ *on* $\bar{Q}$.

2.

In this section we shall investigate the uniqueness of the Cauchy problem for non-linear equation (8). The method of Section 1 can be used to prove the following extension of Corollary 1 to the infinite strip $(0, T] \times R_n$. Put $S = (0, T] \times R_n$.

THEOREM 2. *Assume that conditions (i), (ii), (iii) and (iv) of Theorem* 1 *hold for* $(t, x) \in S$. *Let* u *and* v *be regular solutions of* (8) *such that* $u(x, 0) = v(x, 0)$ *for* $x \in R_n$ *and* $\lim_{|x| \to \infty} [u(t, x) - v(t, x)] = 0$ *uniformly in* $[0, T]$. *If* f *is weakly parabolic with respect to* u *or* v , *then* $u \equiv v$ *on* $\bar{S}$.

In general, we cannot expect uniqueness even in the class of bounded functions. Indeed, the function

$$u(t, x) \equiv t$$

satisfies the equation

$$\Delta u - 0.u_t = 0$$

and vanishes at $t = 0$. In the above example $\alpha \equiv 0$; however, if there exist positive constants β and R such that $\alpha(t, x) \geq \beta$ for $|x| \geq R$ and $t \in [0, T]$, then we can formulate a uniqueness theorem in the class of functions growing not faster than $e^{k|x|^2}$, where $k > 0$.

We shall need the following condition on f :

(A) *there exist positive constants* L_0, L_1 *and* L_2 *such that*

$$|f(t, x, u, p, r) - f(t, x, \bar{u}, \bar{p}, \bar{r})| \leq L_0 |r - \bar{r}| + L_1 (1 + |x|) |p - \bar{p}| + L_2 (1 + |x|^2) |u - \bar{u}|$$

for all $(t, x) \in S$ *and arbitrary* $u, \bar{u}, p, \bar{p}, r$ *and* $\bar{r}$. *Moreover, there exist positive constants* R *and* L_3 *such that*

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$$f(t, x, u, p, r) - f(t, x, \bar{u}, p, r) \le -L_3(u-\bar{u})$$

for all $|x| < R$, $t \in [0, T]$, $u \ge \bar{u}$ *and arbitrary* p *and* r .

Here $|r-\bar{r}|$ is defined by

$$|r-\bar{r}| = \sum_{i,j=1}^{n} |r_{ij}-\bar{r}_{ij}| .$$

Let β be a fixed positive number and define

$$F_\beta(x, k, \rho) = \begin{cases} L_2 + 4L_0kn + 4nL_1|x| + \left(16L_0k^2n^2+4nkL_1+L_2\right)|x|^2 \\ \qquad - 2\beta k\rho\left(1+|x|^2\right) \quad \text{for } |x| \ge R , \\ 4L_0kn + 4nkL_1|x| + \left(16k^2n^2L_0+4nkL_1\right)|x|^2 \\ \qquad - L_3 - 2\alpha k\rho\left(1+|x|^2\right) \quad \text{for } |x| < R . \end{cases}$$

There exists a k such that $F_\beta(x, k, 0) < 0$ for all $|x| < R$, to every such k there exists a $\rho(k)$ such that $F_\beta(x, k, \rho) < 0$ for all $x \in R_n$ and all $\rho > \rho(k)$. Put

$$k_0 = \sup\{k > 0;\ \text{there exists } \rho(k) > 0 \text{ such that } F_\beta(x, k, \rho) < 0 \text{ for all } \rho \ge \rho(k) \text{ and } x \in R_n\} .$$

We are now in a position to define a class of functions in which we shall investigate the uniqueness of the Cauchy problem.

We shall say that a function $u(t, x)$ defined on $\bar{S}$ belongs to $E_+(k_0)$ $\left(E_-(k_0)\right)$ if there exist positive constants M and $k < k_0$ such that

$$u(t, x) \le Me^{k|x|^2} \quad \left(u(t, x) \ge -Me^{k|x|^2}\right)$$

for all $(t, x) \in \bar{S}$.

Put

$$E(k_0) = E_+(k_0) \cap E_-(k_0) .$$

THEOREM 3. *Suppose that* $f(t, x, u, p, r)$ *satisfies condition* (A)

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and moreover

(i) *there exists a positive constant* β *such that*

$$\alpha(t, x) \ge \beta \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad |x| \ge R$$

and

$$\alpha(t, x) \ge 0 \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad |x| < R ,$$

(ii) u *and* v *are regular solutions to* (1) *in* S *such that* $u \in E_+(k_0)$ *and* $v \in E_-(k_0)$,

(iii) f *is weakly parabolic with respect to* u *or* v *with parabolicity function* $H \equiv 0$.

If $u(0, x) \le v(0, x)$ *for* $x \in R_n$, *then*

$$u(t, x) \le v(t, x) \quad \text{for} \quad \bar{S} .$$

Proof. Since $u \in E_+(k_0)$ and $v \in E_-(k_0)$ there exist positive constants M and $k_1 < k_0$ such that

$$u(t, x) - v(t, x) \le Me^{k_1|x|^2} \tag{10}$$

for all $(t, x) \in \bar{S}$. Let $k_1 < k < k_0$ and

$$H(t, x) = \exp\left[\frac{2k\left(1+|x|^2\right)}{1-\rho t}\right]$$

for $(t, x) \in [0, 1/2\rho] \times R_n$, where $\rho = \rho(k)$ (see the definition of $E_+(k_0)$ and $E_-(k_0)$).

Define

$$\tilde{u} = u/H \quad \text{and} \quad \tilde{v} = v/H .$$

It follows from (10) that given $\varepsilon > 0$ there is a $R_\varepsilon > 0$ such that

$$\tilde{u}(t, x) - \tilde{v}(t, x) \le \varepsilon \tag{11}$$

for $t \in [0, 1/2\rho]$ and $|x| \ge R_\varepsilon$. We may suppose that $R \le R_\varepsilon$. In order to prove that $u \le v$ in $[0, 1/2\rho] \times R_n$, it suffices to prove that this inequality is true in the cylinder $[0, 1/2\rho] \times \left(|x| \le R_\varepsilon\right)$. Assuming that there is a point $(\bar{t}, \bar{x}) \in (0, 1/2\rho] \times \left(|x| < R_\varepsilon\right)$ such that

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$$\tilde{u}(\bar{t}, \bar{x}) - \tilde{v}(\bar{t}, \bar{x}) = \mathrm{Max}\{\tilde{u}(t, x)-\tilde{v}(t, x);\ (t, x) \in [0, 1/2\rho] \times \{|x| < R_\varepsilon\}\} > \varepsilon ,$$

we shall derive a contradiction. Note that

$$D[\tilde{u}(\bar{t}, \bar{x})-\tilde{v}(\bar{t}, \bar{x})] = 0 , \quad \tilde{u}_t(\bar{t}, \bar{x}) - \tilde{v}_t(\bar{t}, \bar{x}) \geq 0$$

and

$$D^2[\tilde{u}(\bar{t}, \bar{x})-\tilde{v}(\bar{t}, \bar{x})] \leq 0 .$$

Consequently by assumptions (A) and *(iii)* and by using an argument similar to [*10*], pp. 205-208, we obtain

$$\alpha(\bar{t}, \bar{x})H(\bar{t}, \bar{x})[\tilde{u}_t(\bar{t}, \bar{x})-\tilde{v}_t(\bar{t}, \bar{x})] \leq H(\bar{t}, \bar{x})(1-\rho\bar{t})^{-2}[\tilde{u}(\bar{t}, \bar{x})-\tilde{v}(\bar{t}, \bar{x})]F_\beta(\bar{x}, k, \rho) < 0 .$$

This contradiction completes the proof if $1/2\rho = T$; otherwise the proof can be completed by a finite number of applications of the above argument on $[1/2\rho, 1/\rho] \times R_n$, $[1/\rho, 3/2\rho] \times R_n$, and so on.

COROLLARY 2. *Suppose that assumptions* (A), *(i) and (ii) of Theorem* 3 *hold. Let* u *and* v *be regular solutions of* (8) *in* S *belonging to* $E(k_0)$. *If* f *is weakly parabolic with respect to* u *or* v *and* $u(x, 0) = v(x, 0)$ *for* $x \in R_n$ *then* $u \equiv v$ *on* S .

We conclude by giving some variants of Theorem 3. The first variant is:

THEOREM 3'. *Theorem* 3 *remains true if we replace assumptions* (A) *and (i) by*

(i') $\alpha(t, x) \geq 0$ *for* $(t, x) \in \bar{S}$,

(A') *there exist positive constants* L_0, L_1 *and* L_2 *such that*

$$[f(t, x, u, p, r)-f(t, x, \bar{u}, \bar{p}, \bar{r})] \leq L_0|r-\bar{r}| + L(1+|x|)|p-\bar{p}| - L_2(1+|x|^2)(u-\bar{u})$$

for all $(t, x) \in \bar{S}$, $u \geq \bar{u}$ *and arbitrary* $p, \bar{p}, r$ *and* $\bar{r}$.

In this case we define the corresponding functions F and H and a constant k_0 as follows:

$$F(x,\ k) = 4L_0kn + 4nkL_1|x| + \left(16k^2n^2L_0+4nkL_1\right)|x|^2 - L_2\left(1+|x|^2\right)$$

for all $x \in R_n$,

$$H(x) = \exp|2k\left(1+|x|^2\right)| ,$$

$$k_0 = \sup\{k;\ F(x,\ k) < 0 \text{ for all } x \in R_n\} .$$

Note that in this variant the function H is independent of t .

In variant 2 of Theorem 3 we replace assumptions (A) *and (i) by*

(i") $\alpha(t,\ x) \geq 0$ *for* $(t,\ x) \in [0,\ \infty) \times R_n$,

(A") *There exist positive constants* L_0, L_1 *and* L_2 *such that*

$$[f(t,\ x,\ u,\ p,\ r)-f(t,\ x,\ \bar{u},\ \bar{p},\ \bar{r})] \leq L_0|r-\bar{r}| + L_1|p-\bar{p}| - L_2(u-\bar{u})$$

for $(t,\ x) \in [0,\ \infty) \times R_n$, $u \geq \bar{u}$ *and arbitrary* p, $\bar{p}$, r *and* $\bar{r}$.

We define

$$F(t,\ x,\ \delta) = L_0\delta^2 \sum_{i,j=1}^{n} tgh\delta x_i tgh\delta x_j + L_1\delta \sum_{i=1}^{n} tgh\delta x_i - L_2 - \delta\alpha(t,\ x)tgh\delta t$$

for $(t,\ x) \in (0,\ \infty) \times R_n$,

$$\delta_0 = \sup\{\delta > 0;\ F(t,\ x,\ \delta) < 0 \text{ for all } (t,\ x) \in (0,\ \infty) \times R_n\} .$$

With δ_0 we associate the following class of functions.

A function $u(t,\ x)$ defined on $[0,\ \infty) \times R_n$ belongs to $A_+(\delta_0)$ $\left(A_-(\delta_0)\right)$ if there exist positive constants M and $\delta < \delta_0$ such that

$$u(t,\ x) \leq M \exp\left[\delta\left(\sum_{i=1}^{n} |x_i|+t\right)\right] \quad \left(u(t,\ x) \geq -M \exp\left[\delta\left(\sum_{i=1}^{n} |x_i|+t\right)\right]\right)$$

for all $(t,\ x) \in [0,\ \infty) \times R_n$.

Put

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$$A(\delta_0) = A_+(\delta_0) \cap A_-(\delta_0) .$$

We now state the result as

THEOREM 4. *Suppose that the functions* f *and* α *satisfy assumptions (i") and* (A") *and moreover*

(ii) u *and* v *are regular solutions to* (1) *in* $(0, \infty) \times R_n$ *such that* $u \in A_+(\delta_0)$ *and* $v \in A_-(\delta_0)$ *,*

(iii) f *is weakly parabolic with respect to* u *or* v *with parabolicity function* $H \equiv 0$ *.*

If $u(0, x) \leq v(0, x)$ *for* $x \in R_n$ *, then*

$$u(t, x) \leq v(t, x) \quad \textit{for} \quad (t, x) \in [0, \infty) \times R_n .$$

The proof is similar to that of Theorem 3. We only mention that the function H is defined by

$$H(t, x) = \cosh \delta t \prod_{i=1}^{n} \cosh \delta x_i$$

for $(t, x) \in [0, \infty) \times R_n$.

From the above variant of Theorem 3 and Theorem 4 one can derive the uniqueness criteria for the Cauchy problem for equation (8) in pretty much the same way as we derived Corollary 2 from Theorem 3.

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