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AN ALGORITHM FOR CONSTRUCTING LOCALLY OPTIMAL MIN-MAX TRIANGULATION

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Several interesting criteria for constructing triangulations associated with a given set of points in a plane have been introduced. In order to obtain optimal triangulation with respect to the min-max-angle criterion, it is essential to study the nature of neutral cases with respect to the criterion. Our aim in this paper is to establish precise equations for neutral set curves with respect to the min-max-angle criterion and to develop an algorithm to obtain a locally optimal triangulation with respect to the criterion.

1. INTRODUCTION

Triangulations play a very significant role in problems of interpolation and approximation of bivariate functions. Bivariate splines which are certain piecewise polynomials defined over triangulations of domains in $\mathbb{R}^2$ have proved to be useful tools in approximation processes, finite element-methods, computer aided geometric designs and other such fields. Lawson [4] and Schumaker [5] have designed several interesting criteria for constructing triangulations of the convex hull of a given set of points in a plane. Most popular choices for triangulation criteria are the max-min-angle criterion and the min-max angle criterion. The max-min-angle criterion is quite well examined and efficient algorithms have been developed for finding optimal triangulations with respect to this criterion (see Correc and Chapius [1], Field [2], Lawson [4] and others). Considering the min-max triangulation criterion, Hansford [3] has demonstrated the graphic construction of neutral sets of given three points. In the present paper we aim to obtain precise equations for the neutral set curves with respect to the min-max angle criterion. Moreover we design a practical algorithm for finding a locally optimal triangulation with respect to this criterion.

2. PREFATORY CONDITIONS

Let $D = \{(x_i, y_i, f_i) : x_i, y_i \in \mathbb{R}\}_{i=1}^{n}$ be a set of scattered data where $V = \{v_i = (x_i, y_i) : x_i, y_i \in \mathbb{R}\}_{i=1}^{n}$ is a set of n distinct points in the plane. Let Ω be the convex

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hull of V. In order to obtain a surface model to the data D in Ω, we triangulate the region Ω. Given a set V of vertices, Ω can be triangulated in several ways. Different triangulation criteria enable us to choose a triangulation suitable for the practical needs of our problem. A triangulation criterion chooses the preferred triangulation among several alternatives, thus defining an ordering on the set of all triangulations. We use the notation $T' < T$ to denote that the criterion prefers T' to T.

An optimal triangulation of Ω with respect to given criterion is a triangulation T' such that $T' < T$ for every triangulation T of Ω. An optimal triangulation of Ω always exists since there are only a finitely many triangulations of Ω, yet it might be difficult to attain this optimal triangulation. Thus, in general, we try to get a *locally* optimal triangulation. Let T be a triangulation of Ω, e an internal edge of T, and Q a quadrilateral formed from the two triangles having e as a common edge. Then edge e is called locally optimal if one of the following conditions holds:

1. The quadrilateral Q is not strictly convex:
2. The quadrilateral Q is strictly convex and $T < T'$ where T' is obtained from T by of replacing e by the other diagonal of Q.

A locally optimal triangulation of Q is a triangulation in which all edges are locally optimal.

DEFINITION 2.1: Min-max-angle criterion. Given a triangle T, let $\alpha(T)$ = (maximum angle in T). Corresponding to a triangulation Γ, let $\alpha(\Gamma)$ = max $\{\alpha(T)\colon T \in \Gamma\}$. Then Γ is said to be better than triangulation Γ' with respect to the min-max-angle cirterion provided that $\alpha(\Gamma) \leqslant \alpha(\Gamma')$.

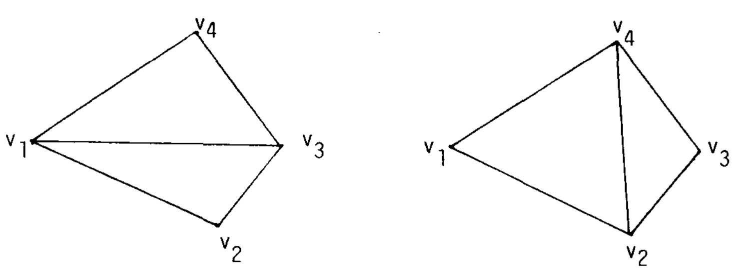

Figure 2.1

Let v_1, v_2, v_3, v_4 be four given points. Then there are two diagonals and correspondingly two triangulations as shown in Figure 2.1.

A triangulation criterion determines which diagonal should be used to obtain a better triangulation. A neutral case with respect to a triangulation-criterion occurs in a convex quadrilateral when either of the two diagonals may be chosen. Given three points, if a fourth point is determined so that a convex quadrilateral with the neutral case is formed then the locus of such a fourth point is called the neutral set for the given three points. In the case of the max-min-angle criterion, the neutral set is the circum circle of the given three points. However, in the case of the min-max-angle criterion the determination of the neutral set is slightly more involved. We shall obtain the precise equations to the curve/pieces of curves of the neutral sets corresponding to all possible relative positions of the given three points.

3. The neutral set for the min-max-angle criterion

Let v_1, v_2 and v_3 be three given points in anticlockwise order. In triangle $v_1v_2v_3$ let the measure of angles at vertices v_1, v_2 and v_3 be A, B and C respectively and let the sides opposite these angles be l_1, l_2 and l_3 respectively. Let x be the position of a fourth point such that in the convex quadrilateral $v_1v_2v_3x$ the neutral case occurs with respect to the min-max-angle criterion. With v_1 as origin and v_1v_2 as initial line let x be the point (r, θ) so that $v_1x = r$ and $\angle v_2v_1x = \theta$. The locus of point x determines the neutral set for the given three points. The following lemma (see Hansford [**3**]) gives an important assertion about the neutral case for the min-max-angle criterion:

Lemma 3.1. *In order that the neutral case with respect to the min-max-angle criterion may occur in a convex quadrilateral, two adjacent angles in it must be equal and they must be the maximum angles in the quadrilateral.*

We proceed to obtain the neutral sets in different situations of the given three points $v_1v_2v_3$.

Case 1. $B \leqslant 90°$:

Here we consider the case when $\angle v_1v_2v_3 = B \leqslant 90°$ as shown in Figure 3.1.

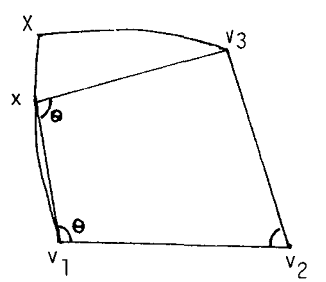

Figure 3.1

According to Lemma 3.1, in order that a neutral case may occur in the quadrilateral $v_1v_2v_3x$, two adjacent angles must be equal and must be the maximum angles in the

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quadrilateral. We first suppose the equal and maximum angles to be at vertices v_1 and x. Thus $\angle v_2 v_1 x = \theta = \angle v_1 x v_3$. Obviously $\theta \geqslant 90°$. Using the sine formula in triangle $x v_1 v_3$ we easily find that

$$r = l_2 \frac{\sin(2\theta - A)}{\sin\theta}. \tag{3.1}$$

This is a curve which starts from v_1 where θ is $90° + A/2$. When x moves along the curve, the vectorial angle θ for x decreases whereas $\angle v_2 v_3 x$ goes on increasing. Thus at a point X where $\theta = 120° - B/3$, we find that $\angle v_2 v_1 X = \angle v_1 X v_3 = \angle v_2 v_3 X$. At this point X, radius vector r_X is given by

$$r_X = l_2 \frac{\sin(60° + C + B/3)}{\sin(60° + B/3)}. \tag{3.2}$$

It is interesting to observe that if θ is decreased further, $\angle v_2 v_3 x$ exceeds $\angle v_2 v_1 x$ and we may have a neutral case for the quadrilateral only if the angle at v_3 is one of the maximum, adjacent and equal angles for the quadrilateral. Therefore now we suppose that the maximum and equal angles are at v_3 and x. Let $\overline{x} = (\overline{r}, \overline{\theta})$ be a position of x in this situation. Therefore if $\angle v_1 \overline{x} v_3 = \angle v_2 v_3 \overline{x} \equiv \alpha$, then $2\alpha = 360° - \overline{\theta} - B$. Hence from $\triangle v_1 \overline{x} v_3$ we have

$$\overline{r} = l_2 \frac{\sin(\alpha - C)}{\sin\alpha}.$$

Therefore the equation to this second part of the locus of point x is given by

$$r = l_2 \frac{\sin(\theta/2 + B/2 + C)}{\sin(\theta/2 + B/2)}. \tag{3.3}$$

Obviously at $\theta = A$, we have $r = l_2$. Further we see that when $B = 90°$, $r_X = l_1$.

It is easy to see that at $\theta = 120° - B/3$ the common values r_X of the radius vector r for the two parts of the locus of x is given by (3.2). Thus we have proved the following:

THEOREM 3.1. *Given three points v_1, v_2 and v_3 such that $\angle v_1 v_2 v_3 \leqslant 90°$, the neutral set with respect to the min-max angle criterion opposite to vertex v_2 is a continuous piecewise curve from v_1 to v_3 whose two parts $v_1 X$ and $v_3 X$ are given by equations (3.1) and (3.3) respectively.*

REMARK 3.1. It is interesting to note that the curve constituting the neutral set has two pieces whose domains are given by $90° + A/2 \geqslant \theta \geqslant 120° - B/3$ and $120° - B/3 \geqslant \theta \geqslant A$ respectively. Therefore first part of the neutral set curve will disappear if $3A + 2B \leqslant 180$ and the neutral set curve from v_1 to v_3 will be given by equation (3.3). Similarly neutral-set-curve will consist of a single curve given by (3.1), when $A + B/3 \geqslant 120°$.

4. Case 2: The neutral cases when $B > 90°$

Before we prove our main result for this case, we shall prove one proposition for the situation $90° < B < 120°$.

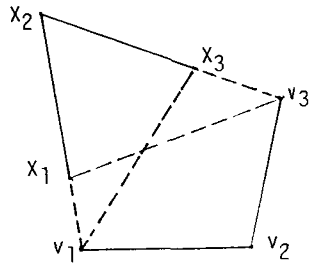

Figure 4.1

Let us suppose that $90° < B < 120°$ and adjacent angles $\angle v_1v_2v_3$ and $\angle v_2v_1x$ are the maximum and equal angles of the convex quadrilateral $v_1v_2v_3x$. We observe that the neutral case for the quadrilateral occurs till $\angle v_1xv_3 \leqslant B$ and $\angle v_2v_3x \leqslant B$. Clearly when $\angle v_2v_3x < B$, and the equal maximum angles of the quadrateral are at v_1 and v_2 then the locus of the fourth point x giving the neutral case in the quadrilateral is along the line v_1x. In fact, we find that the segment of line v_1x from $X_1 = (r_1, B)$ to $X_2 = (r_2, B)$ forms a part of the locus of x where

$$\text{(4.1a)} \qquad r_1 v_1 X_1 = -l_2 \frac{\sin(3B + C)}{\sin B},$$

$$\text{(4.1b)} \qquad r_2 = v_1 X_2 = -1_2 \frac{\sin(B - C)}{\sin 3B}.$$

Further we observe that the neutral case for the quadrilateral $v_1v_2v_3x$ occurs also when the equal and maximum angles are at v_2 and v_3. The locus of such a point x is the line segment from $X_2 = (r_2, \theta_2)$ to $X_3 = (r_3, \theta_3)$, where

$$\text{(4.2a)} \qquad r_2 = -1_2 \frac{\sin(B - C)}{\sin 3B}, \qquad \theta_2 = B;$$

$$\text{(4.2b)} \qquad r_3 = 1_2 \frac{\sin(B - C)}{\sin B}, \qquad \theta_3 = 360° - 3B.$$

We find that the two line segments meet at the point X_2. Therefore the locus of x is the continuous broken line from X_1 to X_3.

Thus we have proved the following:

Proposition 4.1. *Given three points v_1, v_2, v_3 with $\angle v_1v_2v_3 = B$ such that $90° < B < 120°$, if one of the adjacent, maximum and equal angles for the convex quadrilateral $v_1v_2v_3x$ is at v_2 then the locus of the fourth vertex x for which the neutral case occurs in the quadrilateral, consists of two line segments along v_1X and v_3X which meet at a point X, and are given by (4.1) and (4.2) respectively.*

Now we are set to discuss the neutral set for points v_1, v_2, v_3 so located that $90° < \angle B < 120°$.

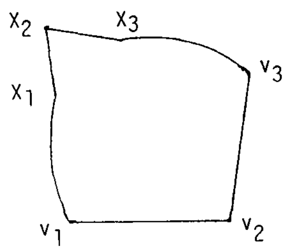

Figure 4.2

To start with the process of obtaining the neutral set in this situation, let us first suppose that $x_1 = (r, \theta)$ is a point such that $\angle v_2v_1x_1$ and $\angle v_1x_1v_3$ are the maximum, equal and adjacent angles so that the neutral case occurs in the convex quadrilateral $v_1v_2v_3x_1$. By considering the sine formula in $\Delta v_1v_2v_3$, we get (3.1) as the relation between r and θ. Thus when θ decreases from $\theta = 90° + A/2$ to $\theta = B$ the locus of a point x_1 is the curve v_1X_1 whose equation is given by (3.1).

When θ approaches value B, we have $\angle v_3v_2v_1 = \angle v_2v_1x_1 = \angle v_1x_1v_3$ in the quadrilateral and now we can have the maximum, equal and adjacent angles at v_2 and v_1. As discussed in Proposition 4.1 we can have a point x_2 such that $\angle x_2v_1v_2$ and $\angle v_1v_2v_3$ are the equal and maximum angles of the convex quadrilateral $v_1v_2v_3x_2$. Thus the locus of point x_2 is the line segment X_1X_2 given by (4.1). Further, we observe that when x_2 reaches X_2, $\angle x_2v_3v_2 = B$, and now we can have the neutral case with a point x_3 such that $\angle x_3v_3v_2 = \angle v_3v_2v_1$. Thus the locus of the neutral point x_3 is the line segment X_2X_3 given by (4.2).

At point X_3 we have $\angle v_1X_3v_3 = \angle X_3v_3v_2 = \angle v_3v_2v_1 = B$ and the vectorial angle for X_3 is $360° - 3B$. If we decrease θ still further we can have points x_4 such that $\angle v_1x_4v_3 = \angle x_4v_3v_2$, and they are the maximum angles in the convex quadrilateral $v_1v_2v_3x_4$. Therefore from $\Delta v_1x_4v_3$, we have

$$v_1x_4 = r = 1_2\frac{\sin(\alpha - C)}{\sin\alpha}, \quad \text{where } \alpha = \angle v_1x_4v_3.$$

Hence the locus of the point x_4 is the curve X_3v_3 whose equation is given by (3.3).

We have thus proved the following:

THEOREM 4.1. *Given three points v_1, v_2 and v_3 with $90° < \angle v_1v_2v_3 < 120°$, the locus of the neutral point with respect to the min-max-angle criterion, consists of a continuous piecewise curve from v_1 to v_3 whose four pieces are given by equations (3.1), (4.1), (4.2) and (3.3) respectively.*

Next we consider the case when $120° \leqslant B < 180°$.

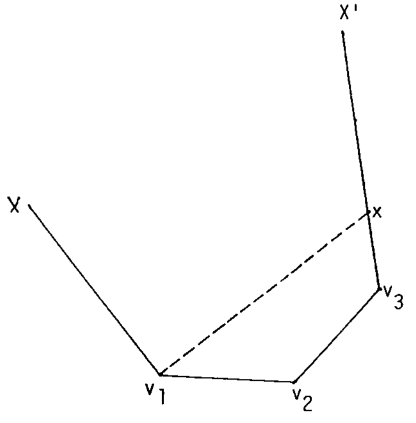

Figure 4.3

In view of Lemma 2.1, a little consideration shows that in this case, to have a neutral case in the convex quadrilateral $v_1v_2v_2x$ we must choose $\angle v_1v_2v_3$ as one of the maximum, adjacent and equal angles of the quadrilateral. Therefore, the only possible pairs of adjacent, equal and maximum angles are either $\angle v_1v_2v_3$ and $\angle v_2v_1x$, or the pair $\angle v_1v_2v_3$ and $\angle v_2v_3x$. Accordingly the locus of the neutral point is the whole line v_1X where $\angle Xv_1v_2 = \angle v_1v_2v_3$ or the whole line v_3X' where $\angle v_2v_3X' = \angle v_1v_2v_3$.

Clearly these two parts of the locus of point x do not meet to form a closed curve.

This proves the following:

THEOREM 4.2. *Given three points v_1, v_2, v_3 in that order, with $\angle v_1v_2v_3 \geqslant 120°$, the locus of the fourth point x such that the neutral case with respect to the min-max angle criterion occurs in the convex quadrilateral $v_1v_2v_3x$, consists of two infinite lines v_1X and v_3X', originating from v_1 and v_3, such that $\angle Xv_1v_2 = \angle v_1v_2v_3 = \angle v_2v_3X'$.*

Thus given any three non-collinear points v_1, v_2, v_3, the locus of a fourth point x

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such that the neutral case occurs in the quadrilateral $v_1v_2v_3x$, is completely determined by Theorems 3.1, 4.1 and 4.2.

5. Observations

5.1. The neutral sets as determined by Theorems 3.1 and 4.1 are closed curves. Thus, given three points in a plane, if no angle in the triangle is larger than $120°$, the neutral sets for them will be closed curves.

5.2. Given three points v_1, v_2, v_3, in that order such that $\angle v_1v_2v_3 < 120°$, so that the neutral set is a closed curve and if the fourth point v_4 is outside this closed curve, the triangulation with v_1v_3 as one edge would be better than that for choice v_2v_4 as an edge. This observation gives a procedure for obtaining a better triangulation with respect to the min-max-angle criterion.

5.3. In the case when $B \geqslant 120°$, the neutral set is not a closed curve, but consists of two infinite lines. Then for a point v_4 lying inside this region, edge v_2v_4 should be preferred to have a better triangulation.

6. A practical algorighm

A few preliminaries are needed before we define the algorithm:

Definition 6.1: *Triangle Number.* Triangle numbers are given from 1 onwards. When a new triangle is formed we give it the next number.

Definition 6.2: *Status of a triangle.*

Alive: Every newly formed triangle is alive.

Dead: 1. If it looses its existence,

or 2. All its sides are dead.

Definition 6.3: *Status of a side.*

Alive: A side of a triangle is alive by default,

Dead: 1. If it is a common edge of two triangles,

2. If it loses its existence,

3. If it is on the boundary of the convex hull containing $v_1, v_2, \ldots, v_n$.

Remark. When a triangle dies all its sides also die. A side which has died may become alive in a new triangle.

Algorithm.

Step 1 Choose any three points amongst v_i $i = 1, 2, \ldots, n$, such that they form a triangle and there is no point inside or on the boundary of the triangle.

Mark the triangle as 'alive'. Also mark the status of sides as alive or dead (according to Definition 6.3). Set 'number of triangles considered' to 0.

Step 2 Consider the alive triangle whose number is (number of triangle considered +1).

If this is dead then consider the next triangle whose number is 1 more and so on until we get an alive triangle. If no alive triangle is left, go to Step 6.

Set 'number of sides considered' to 0.

Step 3 Consider an alive side of the triangle under consideration.

If all its sides are dead go to Step 2.

Step 4 Suppose the triangle under consideration is ABC and the side under consideration is AC.

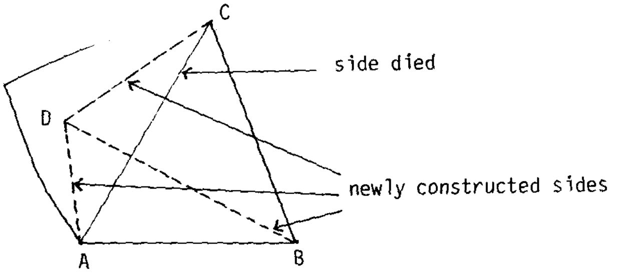

Figure 6.1

Consider the neutral set curve of $\triangle ABC$ opposite to the side AC. Choose a proper point D. (Definition of Proper point: A point D is a proper point if

1. It is not on AC.
2. AD, BD, BC do not cross any of the sides of existing triangles.
3. $\triangle ADC$ contains no point from the v_i's.)

If we are unable to choose D inside a neutral set go to Step 5.

If there are more than one proper points then choose any one of them (and call it D).

Form $\triangle ABD$ and give it the next number.

Form $\triangle BDC$ and give it the next number.

Mark sides AD and CD as alive and AC as dead.

The status of AB and BC remain the same.

Mark $\triangle ABC$ as dead.

Go to Step 4 (again).

Step 5 If there is no proper point inside the neutral set we consider a proper point outside the neutral set curve. Preferably the point chosen should be near AC. (If no such point exists, the triangulation is complete. Go to step 6). Let the point be D.

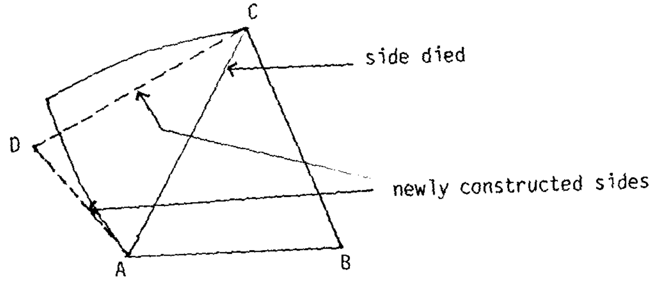

Figure 6.2

Then form the triangle ACD and assign it the next triangle number. Mark AC as dead in $\triangle ABC$ and $\triangle ACD$. Go to Step 3.

Step 6 The triangulation is complete.

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