ON A Ramanujan’s Identity

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1. Introduction. In [4], Ramanujan stated the following beautiful identity which was later proved by Bailey [2] and [3]:

\[ q \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^5}{(1 - q^n)} = \sum_{n=0}^{\infty} \frac{q^{5n+1}}{(1 - q^{5n+1})^2} - \frac{q^{5n+2}}{(1 - q^{5n+2})^2} - \frac{q^{5n+3}}{(1 - q^{5n+3})^2} + \frac{q^{5n+4}}{(1 - q^{5n+4})^2}. \] (1)

We denote by \( S(n) \) the coefficient of \( q^n \) on the right hand side of (1). The object of this paper is to prove the following two theorems.

**Theorem 1.** (a) \( S(n) \) is multiplicative. (b) Let \( p \) be a prime. We have the following:

(i) if \( p = 5 \), then \( S(5^n) = 5^n \);

(ii) if \( p \equiv \pm 1 \) (mod 5), then \( S(p^n) = (p^{n+1} - 1)/(p - 1) \);

(iii) if \( p \equiv \pm 2 \) (mod 5), then \( S(p^n) = \begin{cases} (p^{n+1} - 1)/(p + 1) & \text{if } n \text{ is odd} \\ (p^{n+1} + 1)/(p + 1) & \text{if } n \text{ is even} \end{cases} \)

**Theorem 2.** Let \( R(n) \) denote the number of integral solutions of the Diophantine equation

\[ 5(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_2x_3 - x_3x_4) + x_1 + x_2 + x_3 + x_4 = n. \]

Then

\[ R(n) = S(n + 1). \]

2. Proof of Theorem 1. For \( 1 \leq k \leq 4 \), define

\[ \sigma_k(n) = \sum_{d \mid n, d \equiv k \text{ (mod 5)}} n/d. \]

and note that

\[ \sum_{n=0}^{\infty} \frac{q^{5n+k}}{(1 - q^{5n+k})^2} = \sum_{n=1}^{\infty} \sigma_k(n)q^n. \]

As \( S(n) \) denotes the coefficients of \( q^n \) on the right hand side of (1), we have

\[ S(n) = \sigma_1(n) - \sigma_2(n) - \sigma_3(n) + \sigma_4(n). \] (2)

Now, the part (a) of the Theorem 1 easily follows by observing that

\[ S(n) = \sum_{d \mid n} (n/d)\chi(d), \]

where \( \chi \) is the Dirichlet character defined by \( \chi(2) = -1 \). Part (b) of the Theorem 1 follows by using (2).

3. Proof of Theorem 2. In this section we shall prove the following lemma from which Theorem 2 follows immediately.

**Lemma.** For $|q| < 1$,
\[ \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^5}{(1 - q^n)} = \sum_{n=0}^{\infty} R(n)q^n. \]

**Proof.** First we note that
\[ \sum_{n \in \mathbb{Z}} z^{An} q^{Bn} + Cn = \prod_{n=1}^{\infty} (1 - q^{2Bn})(1 + z^A q^{2Bn+C-B})(1 + z^{-A} q^{2Bn-C-B}), \]
by Jacobi's triple product identity [1, p. 21]. Hence $\prod_{n=1}^{\infty} (1 - q^n)^{-1}$ is the constant term (i.e. the term independent of $z$) in
\[ \prod_{n=1}^{\infty} (1 + zq^{n-2})(1 + z^{-1}q^{n+1}). \]

Now consider
\[
\sum_{y_1, y_2, y_3, y_4, y_5 \in \mathbb{Z}} \left\{ q^{S/2(\Sigma_{i=1}^5 y_i^2)} \right\} \left\{ (y_1 + y_2 + 3y_3 + 5y_4 + 7y_5) z^{y_1 - y_2 - y_3 - y_4 - y_5} \right\}
\]
\[
= \left\{ \prod_{n=1}^{\infty} (1 - q^{5n})(1 + zq^{5n-2})(1 + z^{-1}q^{5n-3}) \right\} \left\{ \prod_{n=1}^{\infty} (1 - q^{5n})(1 + z^{-1}q^{5n-2})(1 + zq^{5n-3}) \right\}
\times \left\{ \prod_{n=1}^{\infty} (1 - q^{5n})(1 + z^{-1}q^{5n-1})(1 + zq^{5n-4}) \right\} \left\{ \prod_{n=1}^{\infty} (1 - q^{5n})(1 + z^{-1}q^{5n})(1 + zq^{5n-5}) \right\}
\times \left\{ \prod_{n=1}^{\infty} (1 - q^{5n})(1 + z^{-1}q^{5n+1})(1 + zq^{5n-6}) \right\}
\]
\[
= \prod_{n=1}^{\infty} (1 - q^{5n})^5(1 + zq^{n-2})(1 + z^{-1}q^{n+1}). \quad (4)
\]

By comparing the constant terms on both sides of (3) we have $y_1 = y_2 + y_3 + y_4 + y_5$. Hence by making the change of variables
\[ x_1 = y_2 + y_3 + y_4 + y_5, \quad x_2 = y_3 + y_4 + y_5, \quad x_3 = y_4 + y_5, \quad x_4 = y_5 \]
to get the polynomial in the form given in Theorem 2, the lemma follows.

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**References**
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