J. Austral. Math. Soc. (Series A) 41 (1986), 1–12

UNIFORM ESTIMATES FOR FAMILIES OF SINGULAR INTEGRALS AND DOUBLE FOURIER SERIES

ELENA PRESTINI

(Received 12 January 1984)

Communicated by G. Brown

Abstract

It is an open problem to establish whether or not the partial sums operator $S_{NN^2}f(x, y)$ of the Fourier series of $f \in L_p$, $1 < p < 2$, converges to the function almost everywhere as $N \to \infty$. The purpose of this paper is to identify the operator that, in this problem of a.e. convergence of Fourier series, plays the central role that the maximal Hilbert transform plays in the one-dimensional case. This operator appears to be a singular integral with variable coefficients which is a variant of the maximal double Hilbert transform.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 42 B 05.

1. Introduction

Let f be in $L_p([-\pi, \pi] \times [-\pi, \pi])$, $p > 1$, and let $\sum_{n,m=-\infty}^{\infty} a_{nm} e^{i(nx+my)}$ be its Fourier series. It is well known that the square partial sums operator $S_{NN}f(x, y) = \sum_{|n|,|m| \leqslant N} a_{nm} e^{i(nx+my)}$ converges almost everywhere to $f(x, y)$ as N tends to infinity for every $f \in L_p$, $p > 1$ [3]. The analogous statement concerning the partial sums operator $S_{NM}f(x, y) = \sum_{|n| \leqslant N, |m| \leqslant M} a_{nm} e^{i(nx+my)}$ where N and M tend to infinity independently, is false for all $p \geqslant 1$ [4]. The partial sums operator $S_{NN^2}f(x, y) = \sum_{|n| \leqslant N, |m| \leqslant N^2} a_{nm} e^{i(nx+my)}$ (or $S_{NN^k}f(x, y)$, k any integer bigger than 1) can be thought as an intermeditate case between the two previous ones.

We are interested in the open problem of establishing whether or not $S_{NN^2}f(x, y)$ converges a.e. as $N \to \infty$ for any $f \in L_p$, $1 < p < 2$. (For $p \geqslant 2$ the answer is known to be positive [3]. The proof is simple. It uses the one-dimensional result [5] and the fact that the characteristic function of the parabola $y > x^2$ is a multiplier for L_2 (and L_2 only).) The operator to study is the following singular integral with variable coefficients

$$(1) \quad Tf(x, y) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(N(x,y)x' + N^2(x,y)y')} f(x - x', y - y')/x'y'\, dx'\, dy'$$

where $N(x, y)$ is any bounded integer-valued function [7]. One would like to prove that there exists a constant c_p, independent of $N(x, y)$ and f, such that

$$(2) \qquad \|Tf\|_p \leqslant c_p \|f\|_p, \qquad 1 < p < 2,$$

because this implies that $S_{NN^2} f(x, y) \to f(x, y)$ a.e. as $N \to \infty$ for every $f \in L_p$. We do not settle this question but we study special cases of operators as in (1). Our study leads us to identify certain singular integral operators that, in this problem of a.e. convergence of Fourier series, appear to play the same central role that the maximal Hilbert transform plays in the one-dimensional case [1], [2], [5]. We consider two families of functions $N(x, y)$ and we prove for the corresponding operator T defined in (1) the uniform estimate (2). In Section 1 we consider the case $N(x, y) = (\lambda N_0(x, y))^*$ for $\lambda > 10^{10}$ (here $(*)^*$ denotes the greatest integer function) under the assumption that $N_0(x, y)$ is differentiable and that $N_0(x, y)$, $\partial N_0/\partial x(x, y)$, $\partial N_0/\partial y(x, y)$ take on approximately the values C, A and B respectively with $0 < A, B, C \leqslant 1$ (i.e. $C/2 \leqslant N_0(x, y) \leqslant C$, etc.). This case suggests that $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (e^{iM(x)y'}/x'y') f(x - x', y - y')\, dx'\, dy'$, where $M(x)$ is any integer-valued bounded function, is the operator we are looking for. This is easily recognized as the double Hilbert transform. Let us observe that this case is just neater, but not very different from the case $N(x, y) = (\lambda x + \mu y)^*$, λ, $\mu > 10^{10}$ studied in [7], and it will be used in Section 2. We will present it briefly. In Section 2 we consider $N(x, y) = (\lambda xy)^*$, $\lambda > 10^{10}$. Its behavior sharply differs from that of $N(x, y)$ of Section 1 in the region close to the axes $x = 0$ and $y = 0$, so that new tools have to be introduced. In particular this case suggests a more complicated operator than the double Hilbert transform of Section 1, which is roughly speaking the following one

$$(3) \qquad \int\int_{(x'y') \in D} (1/x'y') f(x - x', y - y')\, dx'\, dy'$$

(together with its maximal operator), where D is a fixed region symmetrical with respect to the axes x' and y' (see Section 2 for the exact definition). Finally in Section 3 we take $N(x, y) = (\lambda x y^{\beta})^*$, $\lambda > 10^{10}$, $\beta \geqslant 1$. This case leads us to consider a more general singular integral with variable coefficients (and its maximal operator), which is roughly speaking the following one

$$(4) \qquad \int\int_{(x'y')\in D_y} 1/x'y' f(x-x', y-y')\,dx'\,dy',$$

where for every fixed y, D_y is a region symmetrical with respect to the axes x' and y' (see Section 3 for the exact definition).

In [8] the boundedness in L_p of the operators (3) and (4) has been proved and their maximal operators have been dominated pointwise by classical operators like the Hardy-Littlewood maximal function and the one-dimensional Hilbert transform. We shall state and make use of these results here. We hope that this work and [8] provide the insight and the necessary tools to handle the case, important for the complete solution of the problem, in which $N(x, y)$ is monotonically nondecreasing in x and y.

1.

We introduce the following operator, that we call the Carleson operator, $Cf(x) = \int_{-\pi}^{\pi}(e^{iM(x)x'}/x')f(x-x')\,dx'$ where $M(x)$ is any bounded, integer-valued function. From [5], there exists a constant c_p, independent of $M(x)$ and f, such that

$$\|Cf\|_p \leqslant c_p\|f\|_p, \qquad 1 < p < \infty.$$

For the rest of this section we also refer the reader to [7]. There exists a C^∞ function $\phi(t)$ supported on $\{|t| \leqslant 2\pi\}$ such that if we write $\phi_k(t) = 2^k\phi(2^k t)$ then $1/t = \sum_{k=0}^{\infty}\phi_k(t)$ for $|t| \leqslant \pi$. Now we can write the operator (1) as follows:

$$Tf(x, y) = \sum_{k,h=0}^{\infty} T_{kh}f(x, y),$$

where

$$T_{kh}f(x, y) = \int\int e^{i(N(x,y)x' + N^2(x,y)y')}\phi_k(x')\phi_h(y')f(x-x', y-y')\,dx'\,dy'.$$

Since T_{kh} acts independently on dyadic intervals $I \times J$, $|I| = 2\pi 2^{-k}$, $|J| = 2\pi 2^{-h}$ and $I, J \subseteq [0, 2\pi]$, we fix $I \times J$. Clearly $\|T_{kh}f\|_{L_p(I\times J)} \leqslant c_p\|f\|_{L_p(I^*\times J^*)}$ for $p > 1$, where I^* denotes the double of I. We shall consider pairs $\mathbf{p} = [I \times J, \omega_I]$, where $\omega_I \subset R$, $|\omega_I| = 2^k$ is any dyadic interval, and operators $T_{\mathbf{p}}f(x, y) = T_{kh}f(x, y)\chi_{E_{\mathbf{p}}}(x, y)$, where $E_{\mathbf{p}} = \{(x, y) \in I \times J\colon\ N(x, y) \in \omega_I\}$. Clearly $T_{kh}f(x, y) = \sum_{\omega_I \subset R} T_{\mathbf{p}}f(x, y)$ for $(x, y) \in I \times J$ and by Schwartz inequality $\|T_{\mathbf{p}}\|_2 \leqslant c(|E_{\mathbf{p}}|/|I \times J|)^{1/2} = cA(\mathbf{p})$, where $\|T_{\mathbf{p}}\|_2$ is the operator norm. The distinction between norms of operators and functions will be clear from the context.

https://doi.org/10.1017/S1446788700028020 Published online by Cambridge University Press

Similarly using $\mathbf{q} = [I \times J, \omega_J]$, $|\omega_J| = 2^h$, we can decompose $T_{kh}f(x, y) = \sum_{\omega_J \subset R} T_{\mathbf{q}} f(x, y)$ for $(x, y) \in I \times J$. We have $\|T_{\mathbf{q}}\|_2 \leqslant c(|E_{\mathbf{q}}|/|I \times J|)^{1/2} = cB(\mathbf{q})$ where $E_{\mathbf{q}} = \{(x, y) \in I \times J\colon N^2(x, y) \in \omega_J\}$. In [6] we proved the following

Lemma 1. *Let I be a fixed dyadic interval, $I \subseteq [0, 2\pi]$, $|I| = 2\pi 2^{-k}$ and $M(x)$ be a bounded integer-valued function. Let $\{\tilde{\mathbf{p}}\} = \{[I, \omega_I]\}$ be a collection of pairs such that $A(\tilde{\mathbf{p}}) = (|E_{\tilde{\mathbf{p}}}|/|I|)^{1/2} \leqslant \delta^{1/2}$, where $E_{\tilde{\mathbf{p}}} = \{x \in I\colon\ M(x) \in \omega_I\}$ and let $T_{\tilde{\mathbf{p}}} f(x, y) = (\int e^{iM(x)x'} \phi_k(x') f(x - x')\, dx')\chi_{E_{\tilde{p}}}(x)$. Then for $1 < p \leqslant 2$ and $1/p + 1/q = 1$*

$$\left\| \sum_{\tilde{p}} T_{\tilde{\mathbf{p}}} f \right\|_{L_p(I)} \leqslant c_p \delta^{1/2q} \| f \|_{L_p(I^*)}.$$

Now we consider for any $\lambda > 10^{10}$ and $x, y \in [0, 2\pi]$ the family of functions $N(x, y) = (\lambda N_0(x, y))^*$ with $N_0(x, y)$ differentiable. Furthermore we assume that there exist constants A, B, C such that $0 < A, B, C \leqslant 1$ and $C/2 \leqslant N_0(x, y) \leqslant C$, $A/2 \leqslant \partial N_0(x, y)/\partial x \leqslant A$, $B/2 \leqslant \partial N_0(x, y)/\partial y \leqslant B$ for every (x, y). We keep denoting by $Tf(x, y)$ the operator associated to this family by (1). We have the following

Theorem 1. *There exists a constant c_p independent of λ, A, B, C and f such that $\|Tf\|_p \leqslant c_p \|f\|_p$, $1 < p \leqslant 2$.*

Proof. We consider two cases: $\lambda > B/AC$ and $10^{10} < \lambda \leqslant B/AC$. First suppose $\lambda > B/AC$. We denote by $[a]$ the biggest dyadic number less than a, $a > 0$. We subdivide the proof into four steps.

Step 1. Let $2^{-k_1} = [1/2\pi^2(\lambda A)^{1/2}]$ and $2^{-h_1} = [1/2\pi^2\lambda(BC)^{1/2}]$. Consider the operator $G_1 f(x, y) = \int\int e^{i(N(x,y)x' + N^2(x,y)y')} \sum_{k \geqslant k_1, h \geqslant h_1} \phi_k(x')\phi_h(y') f(x - x', y - y')\, dx'\, dy'$. Then G_1 acts independently on dyadic intervals $I \times J$ such that $|I| = 2\pi 2^{-k_1}$, $|J| = 2\pi 2^{-h_1}$. Fix one of them and let x_I and y_J be the middle point of I and J. Then for $(x, y) \in I \times J$ we write

$$G_1 f(x, y) \simeq \int\int e^{i(N(x_I, y_J)x' + N^2(x, y)y')} \times \sum_{k \geqslant k_1, h \geqslant h_1} \phi_k(x')\phi_h(y') f(x - x', y - y')\, dx'\, dy' = M_1 f(x, y),$$

meaning that the error term which is equal to $G_1 f(x, y)$ minus the main term $M_1 f(x, y)$ satisfies the same estimate of M_1. This can be proved by the same method used in [7], which applies an expansion in Taylor series since

$$|N(x, y) - N(x_I, y_J)|\,|x'| < 1.$$

https://doi.org/10.1017/S1446788700028020 Published online by Cambridge University Press

We shall use this convention for the rest of this paper. It is immediate that $M_1 f(x, y) = \exp(iN(x_I, y_J)x) C_{y'} H_{x'}(\exp(-iN(x_I, y_J)x') f(x', y'))(x, y)$, where $H_{x'}$ denotes the Hilbert transform in x' (actually a variant of it since the kernel is smooth at $|x'| = 2\pi 2^{-k_1}$ just as it is for $C_{y'}$ [8]). So $\|G_1\|_p \leqslant c_p$, $1 < p < \infty$.

REMARK. Observe that for $(x, y) \in I \times J$ we also have

$$|N^2(x, y) - N^2(x, y_J)|\,|y'| < 1,$$

so that we can actually say that

$$G_1 f(x, y) \simeq M_1 f(x, y) \simeq \int\int \exp\big(i\big(N^2(x, y_J)y' + N(x_I, y_J)x'\big)\big) \times \sum_{k \geqslant k_1, h \geqslant h_1} \phi_k(x')\phi_h(y') f(x - x', y - y')\, dx'\, dy'.$$

The last operator (if we first integrate in x' and then in y'), to an exponential factor, is the double Hilbert transform.

STEP 2. Let $1/2\pi^2\lambda(BC)^{1/2} \leqslant 2^{-h} < A^{1/2}/(2\pi^2\lambda^{1/2}B)$ be fixed. For $(x, y) \in I \times J$, $|I| = 2\pi 2^{-k_1}$ and $|J| = 2\pi 2^{-h}$ we have that

$$G_h f(x, y) = \sum_{k \geqslant k_1} T_{kh} f(x, y) \simeq \int \exp\big(iN^2(x, y)y'\big)\phi_h(y') \times \int \exp\big(iN(x_I, y_J)x'\big) \sum_{k \geqslant k_1} \phi_k(x') f(x - x', y - y')\, dx'\, dy',$$

where (x_I, y_J) is the center of $I \times J$. Clearly $\|G_h\|_p \leqslant c_p \|f\|_p$, $1 < p < \infty$. We are going to improve this estimate in L_2. It is easy to check that the size of the sets $E^x_{[J,\omega_J]} = \{y \in J\colon N^2(x, y) \in \omega_J\}$ is not greater than $c2^h/(\lambda^2 BC)$ for every x. Hence $\|G_h\|_2 \leqslant c(2^{2h}/(\lambda^2 BC)^{1/2})$. Therefore, by interpolation, $\sum_h \|G_h\|_p \leqslant c_p \sum_h (2^{2h}/(\lambda^2 BC))^{1/2q} \leqslant c_p$ for every $1 < p \leqslant 2$, $1/p + 1/q = 1$. If instead $2^{-h} \geqslant A^{1/2}/2\pi^2\lambda^{1/2}B$ is fixed and $2^{-k_2} = [2^h/(2\pi^2\lambda B)]$ we let $\bar{G}_h f(x, y) = \sum_{k \geqslant k_2} T_{kh} f(x, y)$. Proceeding as above we have that $\sum_h \|\bar{G}_n\|_p \leqslant c_p$, $1 < p \leqslant 2$.

STEP 3. For every $2^{-k} \geqslant [1/(2\pi^2(\lambda A)^{1/2})]$ we let $2^{-h_2} = [2^k/(2\pi^2\lambda^2 AC)]$. Then as in Step 2, but switching the roles of the operators in x' and y', if $G_k = \sum_{h \geqslant h_2} T_{kh}$ we have that $\sum_k \|G_k\|_p \leqslant c_p \sum_k (2^{2k}/\lambda A)^{1/2q} \leqslant c_p$, $1 < p \leqslant 2$.

STEP 4. The remaining operators T_{kh} will be subdivided into three families. The first one is defined by $2^{-h} > A2^{-k}/B$ and so $2^h/(2\pi^2\lambda B) \leqslant 2^{-k} < \lambda C/2^h$. It is easy to check that $B(\mathbf{q}) \leqslant c(2^{2h}/(\lambda^2 BC))^{1/2}$, which implies

$$\sum_{k,h} \|T_{kh}\|_p \leqslant c_p \sum_h \big(2^{2h}/(\lambda^2 BC)\big)^{1/2q} \lg\big(\lambda^2 BC/2^{2h}\big) \leqslant c_p.$$

The second family is defined by $2^{-k}/\lambda C \leqslant 2^{-h} \leqslant A2^{-k}/B$. It is easy to check that $B(\mathbf{q}) \leqslant c(2^h 2^k/(\lambda^2 AC))^{1/2}$, which implies

$$\sum_{k,h} \|T_{kh}\|_p \leqslant c_p \sum_h \left(2^h/(\lambda^{3/2}A^{1/2}C)\right)^{1/2q} \leqslant c_p$$

since $\lambda > B/AC$. For the remaining T_{kh}'s we use the estimate $A(\mathbf{p}) \leqslant c(2^{2k}/(\lambda A))^{1/2}$ and so $\sum_{k,h}\|T_{kh}\|_p \leqslant c_p \sum_k (2^{2k}/(\lambda A))^{1/2q} \lg(\lambda A/2^{2k}) \leqslant c_p$.

This ends the case $\lambda > B/AC$. If instead $10^{10} < \lambda \leqslant B/AC$ then $y' = x'/\lambda C$ lies above $y' = Ax'/B$. This case be handled as the previous one and we leave it ot the interested reader. As we observed in the remark the main operator that controls the case is the double Hilbert transform.

2.

We shall need the following

LEMMA 2. (1) *The operators* $Hf(x) = \int \sum_{k=0}^{\infty} \phi_k(x') f(x - x')\,dx'$ *and* $\tilde{H}f(x) = \operatorname{Sup}_{k_0} |\int \sum_{k \leqslant k_0} \phi_k(x')\, f(x - x')\,dx'|$ *are bounded in* L_p, $1 < p < \infty$. *Moreover if* M *denotes the Hardy-Littlewood maximal function then*

$$\tilde{H}f(x) \leqslant c\{Mf(x) + M(Hf)(x)\}. \tag{5}$$

(2) *Suppose that* $B \subset N \times N$ *is a collection of pairs* (k, h) *of nonnegative integers such that for every* k *the section* $B_k = \{h \in N: (k, h) \in B\}$ *is a truncation of* N *possibly depending upon* k *and for every* h *the section* $B_h = \{k \in N: (k, h) \in B\}$ *is a truncation of* N *possibly depending upon* h. *Then the operators*

$$H_1 f(x, y) = \int\int \sum_{(k,h) \in B} \phi_k(x')\phi_h(y') f(x - x', y - y')\,dx'\,dy'$$

and

$$\tilde{H}_1 f(x, y) = \operatorname*{Sup}_{k_0, h_0} \left| \int\int \sum_{\substack{(k,h) \in B \\ k \leqslant k_0,\, h \leqslant h_0}} \phi_k(x')\phi_h(y') f(x - x', y - y')\,dx'\,dy' \right|$$

are bounded in L_p, $1 < p < \infty$, *with norm independent of* B. *Moreover the following inequality holds*:

$$\tilde{H}_1 f(x, y) \leqslant c\{M_{x'}M_{y'}f(x, y) + M_{x'}\tilde{H}_{y'}f(x, y) + M_{y'}\tilde{H}_{x'}f(x, y) + M_{x'}M_{y'}(H_1 f)(x, y)\}. \tag{6}$$

PROOF. This is Lemma 1 and Theorems 1 and 2 of [8].

Now we consider the operator $Tf(x, y)$ defined by (1) with $N(x, y) = (\lambda xy)^*$, $\lambda > 10^{10}$. We are going to prove the following

THEOREM 2. *There exists a constant* c_p, *independent of* λ *and* f, *such that* $\|Tf\|_p \leqslant c_p\|f\|_p$, $1 < p \leqslant 2$.

PROOF. As before $Tf(x, y) = \sum_{k,h=0}^{\infty} T_{kh}f(x, y)$. Now if we consider for all integers $m, n \geqslant 0$ the region $R_{mn} = \{(x, y)\colon 2\pi 2^{-m-1} < x \leqslant 2\pi 2^{-m}, 2\pi 2^{-n-1} < y \leqslant 2\pi 2^{-n}\}$ then by Theorem 1 we have that for $(x, y) \in R_{nm}$ the operator $G_{nm}f(x, y) = \sum_{k \geqslant m, h \geqslant n} T_{kh}f(x, y)$ is bounded on L_p, $1 < p \leqslant 2$. Clearly there is no problem in adding up any finite number of these estimates. In what follows we will see that if $2^{-n} \geqslant 2^{-2m}$ then we apply this argument only for $2^{-n} \geqslant \lambda^{-1/2}$ and $2^m 2^{3n/2}/\lambda \leqslant 1$; if $2^{-n} < 2^{-2m}$ only for $2^{-m} \geqslant \lambda^{-1/4}$ and $2^{2m}2^n/\lambda \leqslant 1$. So we are left, for any nonnegative integers n, m, with the operators

(1) $T_{kh}f(x, y)$ for $k \geqslant 0$, $h \geqslant n$ and $(x, y) \in I \times J$ where $I = [0, 2\pi 2^{-k})$, $|J| = 2\pi 2^{-h}$, $J \subseteq \{y\colon 2\pi 2^{-n-1} < y \leqslant 2\pi 2^{-n}\}$;

(2) $T_{kh}f(x, y)$ for $h \geqslant 0$, $k \geqslant m$ and $(x, y) \in I \times J$ where $|I| = 2\pi 2^{-k}$, $I \subseteq \{x\colon 2\pi 2^{-m-1} < x \leqslant 2\pi 2^{-m}\}$, $J = [0, 2\pi 2^{-h})$;

(3) $T_{kh}f(x, y)$ for $k \geqslant 0$, $h \geqslant 0$ and $(x, y) \in I \times J$ where $I = [0, 2\pi 2^{-k})$, $J = [0, 2\pi 2^{-h})$.

One can estimate the sum of the operators in (1) and (2) quite easily by the methods of Section 1. We leave this to the interested reader. Instead something new (Lemma 2) has to be introduced to handle the operators in (3). We are going to do it in this way. Let $1/p + 1/q = 1$. It is easy to check that the worst estimate for $A(\mathbf{p})$ and $B(\mathbf{q})$, namely the one corresponding to $\omega_I = [0, 2^k)$ and $\omega_J = [0, 2^h)$, is $A(\mathbf{p}) \leqslant c((2^{2k+h}/\lambda)\lg(\lambda/2^{2k+h}))^{1/2}$, $B(\mathbf{q}) \leqslant c((2^k 2^{3h/2}/\lambda)\lg(\lambda/2^{k+3h/2}))^{1/2}$. Clearly if $2^{-h} \geqslant 2^{-2k}$ it is more convenient to use $B(\mathbf{q})$, i.e. to write $T_{kh}f(x, y) = \sum_{\mathbf{q}} T_{\mathbf{q}}f(x, y)$, and to use $A(\mathbf{p})$ if $2^{-h} < 2^{-2k}$. The two cases being similar we are going to treat only the case $2^{-h} \geqslant 2^{-2k}$. First of all we shall add up all pairs $\mathbf{q} = [I \times J, \omega_J]$ such that $B(\mathbf{q}) \sim 1$ (i.e. $2^k 2^{3h/2}/8\pi^3\lambda \geqslant 1$, which makes $|N(x, y)x'|$, $|N^2(x, y)y'| < 1$). We show that for $1 < p \leqslant 2$

$$\left\| \sum_{B(\mathbf{q}) \sim 1} T_{\mathbf{q}} f \right\|_p \leqslant c_p \|f\|_p. \tag{7}$$

Secondly for $r \geqslant 0$ we show that if $B(\mathbf{q}) \sim 2^{-r-1}$ (i.e. $2^{-2r-2} \leqslant 2^k 2^{3h/2}/8\pi^3\lambda < 2^{-2r}$) then

$$\left\| \sum_{B(\mathbf{q}) \sim 2^{-r-1}} T_{\mathbf{q}} f \right\|_p \leqslant c_p 2^{-r/8q}(r + 1)\|f\|_p. \tag{8}$$

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Since it is no problem to add up to the estimates (8) over r this will end the proof. We start by proving (7). In our assumptions it has to be $\omega_J = [0, 2^h)$, since we are working close to the origin $x = y = 0$. We have that

$$F_{-1}f(x,y) = \sum_{B(\mathbf{q})\sim 1} T_{\mathbf{q}}f(x,y)$$

$$\simeq \int\int \sum_{\substack{(k,h)\in B_{-1}\\ k\geqslant k_0(x,y),\ h\geqslant h_0(x,y)}} \phi_k(x')\phi_h(y')f(x-x', y-y')\,dx'\,dy',$$

where $B_{-1} = \{(k,h)\colon 2^{-h} \geqslant 2^{-2k},\ 2^k 2^{3h/2}/8\pi^3\lambda \geqslant 1\}$, and $k_0(x,y)$, $h_0(x,y)$ are defined as follows. If $\pi \leqslant y \leqslant 2\pi$, $x \leqslant 2\pi[1/8\pi^3\lambda]$ then $2^{-k_0} = 2^{-k_0(x,y)} = [1/8\pi^3\lambda]$, $2^{-h_0} = 2^{-h_0(x,y)} = 1$; if $\pi/2 \leqslant y \leqslant \pi$ then, if $x \leqslant 2\pi[1/8\pi^3]$ we have that $2^{-k_0} = 2[1/8\pi^3\lambda]$, $2^{-h_0} = 1$, while if $2\pi[1/8\pi^3\lambda] \leqslant x \leqslant 4\pi[1/8\pi^3\lambda]$ we have the same 2^{-k_0} and $2^{-h_0} = 1/2$; and so on, until $y \leqslant 2\pi[1/(8\pi^3\lambda)^{1/2}]$; then, if $x \leqslant 2\pi[1/8\pi^3\lambda]$ we have $2^{-k_0} = [1/(8\pi^3\lambda)^{1/4}]$, $2^{-h_0} = 1, \ldots,$ while if $\pi[1/(8\pi^3\lambda)^{1/4}] \leqslant x \leqslant 2\pi[1/(8\pi^3\lambda)^{1/4}]$ then we have the same 2^{-k_0} and $2^{-h_0} = [1/(8\pi^3\lambda)^{1/2}]$. Therefore

$$|F_{-1}f(x,y)| \leqslant \left| \operatorname*{Sup}_{k_0,h_0} \int\int \sum_{\substack{(k,h)\in B_{-1}\\ k\geqslant k_0,\, h\geqslant k_0}} \phi_k(x')\phi_h(y')f(x-x', y-y')\,dx'\,dy' \right|.$$

Hence by Lemma 2 we have $\|F_{-1}\|_p \leqslant c_p$ and (7) is proved. Now let $B(\mathbf{q}) \sim 2^{-r-1}$. Since $2^{-h} \geqslant 2^{-2k}$ we have that $2^{2r}/8\pi^3\lambda \leqslant 2^{-k} \leqslant 2^{(r+1)/2}/(8\pi^3\lambda)^{1/4}$. If $T_{kh}f(x,y) = \Sigma_{\mathbf{q}} T_{\mathbf{q}}f(x,y)$ and $B(\mathbf{q}) \sim 2^{-r-1}$ then $\|T_{kh}\|_p \leqslant c_p 2^{-r/2q}$ by Lemma 1 of [6]. So, trivially, if $1/2^{r+1}(8\pi^3\lambda)^{1/4} \leqslant 2^{-k} \leqslant 2^{(r+1)/2}/(8\pi^3\lambda)^{1/4}$ we have that $\Sigma_{k,h}\|T_{kh}\|_p \leqslant c_p 2^{-r/2q}\lg(2^{3(r+1)/2})$. We are left to consider $2^{-k} < 2^{-r-1}(8\pi^3\lambda)^{-1/4}$ which makes $|N(x,y)x'| < 1$. In this case we keep the T_{kh}'s decomposed as sums of the $T_{\mathbf{q}}$'s, $\mathbf{q} = [I \times J, \omega_J]$. Depending upon ω_J we are going to subdivide the $T_{\mathbf{q}}$'s into $4(r+1)$ families $D_0, \ldots, D_{4(r+1)}$ and to show that $\|\Sigma_{\mathbf{q}\in D_s} T_{\mathbf{q}}f\|_p \leqslant c_p 2^{-r/2q}$ for every s. This proves (8). The first family D_0 is defined by $\omega_J = [0, 2^h)$. Since $(\lambda xy)^2 < 2^h$,, clearly

$$F_0 f(x,y) = \sum_{\mathbf{q}\in D_0} T_{\mathbf{q}}f(x,y)$$

$$\sim \int\int \sum_{\substack{(k,h)\in B_{-r-1}\\ 2^h > (\lambda xy)^2}} \phi_k(x')\phi_h(y')f(x-x', y-y')\,dx'\,dy',$$

where

$$B_{-r-1} = \left\{(k,h)\colon 2^{-h} \geqslant 2^{-2k},\ B(\mathbf{q}) \sim 2^{-r-1}, 2^{-k} < 1/2^{r+1}(8\pi^3\lambda)^{1/4}\right\}.$$

By Lemma 2 the last operator has L_p-norm smaller than c_p. This is not good enough. We are going to do better by splitting every $T_{\mathbf{q}}f(x,y) = T^1_{\mathbf{q}}f(x,y) + T^2_{\mathbf{q}}f(x,y)$ where $T^i_{\mathbf{q}}f(x,y) = T_{\mathbf{q}}f(x,y)\chi_{E^i_{\mathbf{q}}}(x,y)$ and

$$E^1_{\mathbf{q}} = E_{\mathbf{q}} \cap \{(x,y)\colon x \leqslant 2^{3h/2}2^r/\lambda\},\ E^2_{\mathbf{q}} = E_{\mathbf{q}} \setminus E^1_{\mathbf{q}}.$$

With obvious notations, $F_0 f = F^1_0 f + F^2_0 f$. If we put $E^y_{1\mathbf{q}} = \{x\colon (x,y) \in E^1_{\mathbf{q}}\}$ and $E^x_{2\mathbf{q}} = \{y\colon (x,y) \in E^2_{\mathbf{q}}\}$ it is easy to check that

$$|E^y_{1\mathbf{q}}|/|I|,\ |E^x_{2\mathbf{q}}|/|J| \leqslant c2^{-r}. \tag{9}$$

Now the estimates (5) and (6) of Lemma 2 can be made more precise [8], namely

$$\begin{aligned}
|F^1_0(x,y)| &= \Big|\sum_{\mathbf{q}\in D_0} T^1_{\mathbf{q}}f(x,y)\Big| \\
&\leqslant c\Bigg\{ \operatorname*{Sup}_{\substack{k_1\leqslant k\leqslant k_2\\ h_1\leqslant h\leqslant h_2}} (4kh)^{-1}\int_{-h}^{h}\int_{-k}^{k} |f(x-x',y-y')|\,dx'\,dy' \\
&\quad + \operatorname*{Sup}_{\substack{k_1\leqslant k\leqslant k_2\\ h_1\leqslant h\leqslant h_2}} (4kh)^{-1}\int_{-h}^{h}\int_{-k}^{k} |H_{x''}f(x-x',y-y')|\,dx'\,dy' \\
&\quad + \operatorname*{Sup}_{\substack{k_1\leqslant k\leqslant k_2\\ h_1\leqslant h\leqslant h_2}} (4kh)^{-1}\int_{-h}^{h}\int_{-k}^{k} |H_1 f(x-x',y-y')|\,dx'\,dy' \\
&\quad + \operatorname*{Sup}_{k_1\leqslant k\leqslant k_2} (2k)^{-1}\int_{-k}^{k} |\tilde{H}_{y'}f(x-x',y)|\,dx'\Bigg\}
\end{aligned}$$

with $k_i = k_i(x,y)$, $h_i = h_i(x,y)$.

The maximal function acting on the x' variable and (9) make the L_p-norm of each of these four operators less than or equal to $c2^{-r/p}$. Precisely we can always first integrate in y' and then (with y fixed) in x'. Then the support of $F^1_0 f(x,y)$ in the x variable (y is fixed) can be subdivided into disjoint intervals $\tilde{I}^y_1, \tilde{I}^y_2, \ldots$. These can be set in one-to-one correspondence with other disjoint intervals $I^{y'}_1, I^{y'}_2, \ldots$ such that $x \in \tilde{I}^y_i$, $y' = y - \pi$, $|I^{y'}_i| \sim 2^{-k_2(x,y)}$. For instance if $\pi < y \leqslant 2\pi$ then $F^1_0 f(x,y) = T^1_{\mathbf{q}_1}f(x,y)$ where $\mathbf{q}_1 = [[0,2\pi\alpha)\times[0,2\pi),\ [0,1)]$ and $\alpha = [2^r/\lambda]$. Then $\tilde{I}^y_1 = E^y_{1\mathbf{q}_1}$ and $I^{y'}_1 = [0,2\pi\alpha]$. If instead $\pi/2 \leqslant y < \pi$ then $F^1_0 f(x,y) = T^1_{\mathbf{q}_1}f(x,y)T^1_{\mathbf{q}_2}f(x,y)$, where $\mathbf{q}_2 = [[0,4\pi\alpha)\times[0,\pi),[0,2)]$. In this case $\tilde{I}^y_1$ and $I^{y'}_1$ are defined as above, while $\tilde{I}^y_2 = E^y_{1\mathbf{q}_2}\setminus E^y_{1\mathbf{q}_1}$ and $I^{y'}_2 = [2\pi\alpha, 4\pi\alpha]$. And so on. By (9) we have that

$$\frac{|\tilde{I}^y_i|}{|I^{y'}_i|} \leqslant c2^{-r}.$$

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So by the action of the maximal function (acting on the x' variable) that appears in each of the four terms that control $F_0^1 f(x, y)$ we have

$$\|F_0^1\|_p \leqslant c_p 2^{-r/p}. \tag{10}$$

Now we proceed to show that $\|F_0^2\|_p \leqslant c_p(r2^{-r})^{1/p}$. This is easier and it is proved by the following observation. There exists a family of sets $\{G_h\}$ and of two-by-two disjoint intervals $\{I_h \times I_h\}$ with the properties $G_h \subset I_h \times J_h$; $|G_h|/|I_h \times J_h| \leqslant c2^{-r}$; at any point (x, y) in G_h at most r among the $T_q^2 f(x, y)$'s are different from zero; if $(x, y) \in I_h \times J_h$ then

$$|F_0^2 f(x, y)| \leqslant r \operatorname*{Sup}_{I_h \times J_h \subseteq I \times J} \frac{1}{|I \times J|} \int_{I \times J} |f(x', y')| \, dx' dy'$$

if $(x, y) \in G_h$ and $F_0^2 f(x, y) = 0$ otherwise. Precisely $I_h = [2^h 2^{n/2}/\lambda, 2^{h+1} 2^{n/2}/\lambda]$, $J_h = [0, 2^{-h}]$ and $G_h = \{(x, y) \in I_h \times J_h\colon N^2(x, y) \in [0, 2^h)\}$. This proves the desired estimate. Now we are going to consider the family D_s, $1 \leqslant s \leqslant 4(r + 1)$, defined by $\mathbf{q} = [I \times J, \omega_J]$, $\omega_J \in V_s$ and

$$V_s = \bigcup_h \left\{ \left[2^{s-1} 2^h, (2^{s-1} + 1) 2^h\right), \ldots, \left[(2^s - 1) 2^h, 2^s 2^h\right) \right\}$$

(observe that $E_{\mathbf{q}} = \varnothing$ if $\omega_J \in V_s$, $s > 4(r + 1)$). The ω_J's in V_s are two-by-two disjoint so that the corresponding $T_{\mathbf{q}}$'s live on disjoint sets. This makes the matter easier and so it is left to the reader to prove that

$$\left\| \sum_{\mathbf{q} \in D_s} T_{\mathbf{q}} f \right\|_p \leqslant c_p 2^{-r/8p}$$

by considering a suitable maximal function.

The main operator we used is $\tilde{H}_1$. As we said it is easy to handle all pairs $\mathbf{q} = [I \times J, \omega_J]$, $|J| \geqslant |I|^2$, $\omega_J = [0, 2^h]$, $B(\mathbf{q}) = 1$ of the collection (1): it is enough to apply the double Hilbert transform. For instance if $J \subset \{y\colon 2^{-n}\pi \leqslant y \leqslant 2^{-n+1}\pi\}$, $2^{-n} > \lambda^{-1/4}$, then we observe that $N(x, y) = \lambda xy \sim \lambda 2^{-n} x$. So after applying the operator

$$\sum_{2^{-h} \leqslant 2^{-n}} e^{i(\lambda 2^{-n} x) y'} \phi_h(y') \sum_{[2^{3n/2}/\lambda] < 2^{-k} \leqslant [2^n/\lambda]^{1/2}} \phi_k(x') * f(x, y)$$

no pairs of norm 1 are left among the ones we started with.

The picture becomes considerably more complicated if we let $N(x, y) = (\lambda x y^\beta)^*$ where β is as big as we wish. To handle this case we are going to introduce an operator more powerful than H_1. This is the subject of the next section.

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3.

We shall make use of the following

LEMMA 3. *For every y fixed, let B_y be a subset of $N \times N$ with the property that for every k there exists an integer $r(y, k)$ such that $B_{yk} = \{h \in N: (k, h) \in B_y\} = \{h \geqslant r(y, k)\}$. Then there exists a constant c_p depending only upon p, $1 < p < \infty$, such that $\|H_2 f\|_p \leqslant c_p \|f\|_p$ where*

$$H_2 f(x, y) = \int\int \sum_{(k,h)\in B_y} \phi_h(y')\phi_k(x')f(x - x', y - y')\, dx'\, dy'.$$

Moreover if we define

$$\tilde{H}_2 f(x, y) = \operatorname*{Sup}_{k_0} \left| \int\int \sum_{\substack{(k,h)\in B_y \\ k \leqslant k_0}} \phi_h(y')\phi_k(x')f(x - x', y - y')\, dx'\, dy' \right|$$

then there exists an absolute constant c such that $\tilde{H}_2 f(x, y) \leqslant c\{M_{y'}H_{x'}f(x, y) + M_{\bar{y}}(H_2 f(x, \bar{y}))(y)\}$ and so $\tilde{H}_2$ is a bounded operator on L_p.

PROOF. This is Theorem 3 and Theorem 4 of [8].

Now we consider the operator $Tf(x, y)$ defined as in (1) with $N(x, y) = (\lambda x y^{\beta})^*$, $\lambda > 10^{10}$, $\beta \geqslant 1$. A new problem arises when β is big. Let us assume for instance that β is as big as we wish and $\lambda > \beta 2^{\beta}$. We are going to explain why $\tilde{H}_2$ is needed to control $\sum_{\mathbf{q}} T_{\mathbf{q}} f(x, y)$ where $\mathbf{q} = [I \times J, \omega_J]$, $\omega_J = [0, 2^h]$, $|J| \geqslant |I|^2$, $I = [0, 2^{-k}]$ and $B(\mathbf{q}) = 1$. Suppose for instance that $\pi \leqslant y \leqslant 2\pi$. The point is that y^{β} is approximately a constant (i.e. it takes on values lying in between two consecutive dyadic numbers) only on very small intervals, precisely over intervals of size not greater than $1/\beta$. So if among the pairs under consideration we single out those with $J \subseteq \{y: \pi \leqslant y \leqslant 2\pi\}$, then only the pairs with $|J| \leqslant 1/\beta$ can be dealt with by applying the double Hilbert transform. Similarly if $\pi 2^{-n} \leqslant y \leqslant \pi 2^{-n+1}$, and so on. Therefore, after applying the double Hilbert transform, we are left with many pairs having $J \subseteq \{y: \pi 2^{-n} \leqslant y \leqslant \pi 2^{-n+1}\}$ besides those having $I = [0, 2\pi 2^{-k}]$ and $J = [0, 2\pi 2^{-h}]$. It is easy to check that two pairs $[I_1 \times J_1, \omega_{J_1}]$ and $[I_2 \times J_2, \omega_{J_2}]$ with $|J_1| = |J_2| = 2^{-h}$ (and so $\omega_{J_1} = \omega_{J_2} = [0, 2^h]$) will have $I_1 \neq I_2$ unless $J_1 = J_2$. In other words, if we add up the $T_{\mathbf{q}}$'s corresponding to all pairs we are left with, the resulting operator will have a convolution kernel depending upon the height y at which the operator is evaluated. Such an operator, except for an error term, is $\tilde{H}_2$.

We will not go into more details, $\tilde{H}_2$ is essentially the only new tool needed to control $Tf(x, y)$.

We conclude by observing that our study suggests that $\tilde{H}_2$ and the double Hilbert transform play the same central role in the problem of a.e. convergence of double Fourier series we mentioned, that the Hilbert transform played in the one-dimensional problem.

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Istituto Matematico
Universita di Milano
Via C. Saldini, 50
20133 Milano
Italy

https://doi.org/10.1017/S1446788700028020 Published online by Cambridge University Press