ON INTEGERS n RELATIVELY PRIME TO $f(n)$

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1. Introduction. If m and n are two integers chosen at random, the probability that they are relatively prime **(2**, p. 267**)** is $6\pi^{-2}$. This result may still hold when m and n are functionally related. Thus, Watson **(3)** recently proved that for α irrational, the positive integers n for which $(n, [\alpha n]) = 1$, have density $6\pi^{-2}$. A different proof of a slightly more general result was given by Estermann **(1)**. The present authors found that the number of positive integers not exceeding x, with $(n, [n^{\frac{1}{2}}]) = 1$, is $6\pi^{-2}x + O(x^{\frac{1}{2}} \log x)$. In this paper we generalize the latter result. Roughly speaking, *if $f(1), f(2), \ldots$ is a non-decreasing sequence of non-negative integers, tending slowly to infinity, and if the intervals over which $f(m) = n$ increase slowly with n, then the probability that n be relatively prime to $f(n)$ is $6\pi^{-2}$.*

2. Notation. As usual, let $[\alpha]$ denote the largest integer not exceeding α, and let (m, n) be the greatest common divisor of m and n. Small Roman letters usually denote positive integers. Let $f = \{f(1), f(2), \ldots\}$ be any sequence of non-negative integers; then we define as follows:

$Q_f(x)$ *is the number of $n \leqslant x$ for which* $(n, f(n)) = 1$.

If

$$\lim_{x\to\infty} x^{-1} Q_f(x)$$

exists, we denote it by P_f, and call it *the probability that n and $f(n)$ are relatively prime.*

$R_f\,(x; a, b)$ *is the number of multiples of a, not exceeding x, which are mapped onto b by f.*

$S_f\,(x; a, b)$ *is the number of multiples of a, not exceeding x, which are mapped onto multiples of b by f.* Usually, the suffix f in the functions defined above will be omitted.

$f^*(n)$ denotes *the number of m such that $f(m) = n$.*

3. Preliminaries. We assume at the outset that

(i) *f is non-decreasing.*

The following relations between the functions defined are then immediate:

$$S(x; a, b) = \sum_{k=0}^{[b^{-1}f(x)]} R(x; a, kb), \tag{3.1}$$

$$S(x; 1, 1) = x, \tag{3.2}$$

$$f^*(b) = R(\infty; 1, b). \tag{3.3}$$

We now prove several lemmas.

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LEMMA 1.

$$Q(x) = \sum_{d=1}^{f(x)} \mu(d)\ S(x; d, d).$$

Proof. The Möbius function μ has the following property (**2**, p. 234):

$$\sum_{d|n} \mu(d) = 1 \quad (n = 1), \quad \sum_{d|n} \mu(d) = 0 \quad (n > 1). \tag{3.4}$$

Hence,

$$Q(x) = \sum_{\substack{n \leqslant x \\ (n, f(n))=1}} 1 = \sum_{n \leqslant x} \sum_{d|(n, f(n))} \mu(d) = \sum_{d=1}^{f(x)} \mu(d)\ S(x; d, d),$$

since $S(x; d, d)$ is the number of $n \leqslant x$ such that $d|n$ and $d|f(n)$. The fact that f is non-decreasing ensures that the last summation need not be carried beyond $f(x)$.

LEMMA 2. $|R(x; a, b) - a^{-1} R(x; 1, b)| < 1$.

Proof. By (i), the set of $R(x; 1, b)$ numbers mapped on b by f consists of consecutive integers. The $R(x; a, b)$ multiples of a mapped on b form a subset whose neighboring elements differ by a. The required result now follows from the fact that every set of a consecutive numbers contains exactly one multiple of a, while every set of fewer than a consecutive numbers contains at most one multiple of a.

LEMMA 3. $|S(x; a, b) - a^{-1} S(x; 1, b)| < b^{-1} f(x)$.

Proof. Replace b by kb in Lemma 2 and use (3.1).

We now assume

(ii) *f^* is finite and non-decreasing.*

LEMMA 4.

$$0 \leqslant \sum_{k=0}^{s} f^*(bk) - b^{-1} \sum_{k=0}^{sb} f^*(k) \leqslant f^*(sb).$$

Proof. The result follows by summing over k the obvious inequalities

$$f^*(bk) \geqslant b^{-1}(f^*(bk) + f^*(bk - 1) + \ldots + f^*(bk - b + 1))$$

and

$$f^*(bk) \leqslant b^{-1}(f^*(bk) + f^*(bk + 1) + \ldots + f^*(bk + b - 1)).$$

LEMMA 5. $S(x; 1, b) - b^{-1} S(x; 1, 1) = O(f^*(f(x)))$.

Proof. We observe that in (3.1) all terms of the sum, with the possible exception of the last, are unaltered by replacing x by ∞. The last term is $O(f^*(f(x)))$. Hence, using (3.3), we have

$$\begin{aligned} S(x; 1, b) &= \sum_{k=0}^{[b^{-1} f(x)]} R(\infty; 1, kb) + O(f^*(f(x))) \\ &= \sum_{k=0}^{[b^{-1} f(x)]} f^*(kb) + O(f^*(f(x))). \end{aligned} \tag{3.5}$$

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Also, putting $b = 1$ in (3.5) and absorbing the last $f(x) - b[b^{-1}f(x)]$ terms in the error term, we obtain

$$b^{-1}S(x;1,1) = b^{-1}\sum_{k=0}^{b[b^{-1}f(x)]} f^*(k) + O(f^*(f(x))). \tag{3.6}$$

The required result now follows from (3.5), (3.6), Lemma 4 with $s = [b^{-1}f(x)]$, and the fact that $f^*(b[b^{-1}f(x)]) \leqslant f^*(f(x))$.

4. Results. We proceed to estimate $S(x;a,b)$ and $Q(x)$. Consider the identity

$$S(x;a,b) = (ab)^{-1}S(x;1,1) + (S(x;a,b) - a^{-1}S(x;1,b)) + a^{-1}(S(x;1,b) - b^{-1}S(x;1,1)). \tag{4.1}$$

Using (3.2) and Lemmas 3 and 5, we obtain from (4.1) that

$$S(x;a,b) = (ab)^{-1}x + O(b^{-1}f(x)) + O(a^{-1}f^*(f(x))). \tag{4.2}$$

It is well known that

$$\sum_{d=1}^{r} d^{-1} = \log r + O(1) \tag{4.3}$$

and

$$\sum_{d=1}^{r} \mu(d)\, d^{-2} = 6\pi^{-2} + O(r^{-1}). \tag{4.4}$$

Using Lemma 1, (4.2) with $a = b = d$, (4.3) and (4.4), we obtain

THEOREM 1. *If* (i) f *is non-decreasing and* (ii) f^* *is finite and non-decreasing, then*

$$Q(x) = 6\pi^{-2}x + O(f(x)\log f(x)) + O(f^*(f(x))\log f(x)) + O(xf(x)^{-1}).$$

Example 1. $f(x) = [x^{\frac{1}{2}}], f^*(x) = 2x + 1, Q(x) = 6\pi^{-2}x + O(x^{\frac{1}{2}}\log x)$.

More generally, if k is an integer >1, and $f(x) = [x^{1/k}]$, then

$$Q(x) = 6\pi^{-2}x + O(x^{1-1/k}\log x).$$

An easy consequence of Theorem 1 is

THEOREM 2. *If* (i) *and* (ii) *hold, as well as* (iii) $f(x)\log f(x) = o(x)$ *and* (iv) $f^*(f(x))\log f(x) = o(x)$, *then* $P_f = 6\pi^{-2}$.

Proof. By Theorem 1 and the definition of P_f it suffices to check that $xf(x)^{-1} = o(x)$, that is, $f(x) \to \infty$, and this is a consequence of (i) and (ii).

5. Discussion. Clearly P_f is unaffected by changing the value of $f(x)$ on any set of zero density. Thus one can easily construct functions for which $P_f = 6\pi^{-2}$ but none of (i) to (iv) hold. On the other hand, none of the conditions (i) to (iv) are superfluous for Theorem 2, and they are therefore independent, as may be seen from Examples 2 to 5.

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Example 2. $f(2x) = 2[(x/2)^{\frac{1}{2}}]$, $f(2x+1) = f(2x) + 1$. Here $f^*(2x) = f^*(2x + 1) = 2x + 1$. Only (i) is violated. However, since $(n, f(n)) \geqslant 2$ for n even, $Q(x) \leqslant [\frac{1}{2}x]$, and if P exists then $P \leqslant \frac{1}{2} < 6\pi^{-2}$.

Example 3. $f(x) = 2[x^{\frac{1}{2}}]$. Only (ii) is violated, but again $(n, f(n)) \geqslant 2$ for n even, so that $P \neq 6\pi^{-2}$.

Example 4. $f(x) = x$. Only (iii) is violated, but clearly $Q(x) = 1$ and $P = 0$.

Example 5. $f(x) = [\log_{10} x]$. Only (iv) is violated. Let $x = 10^{2r+1}$ and consider all $n = 10^{2r} + 2s \leqslant x$, $s \geqslant 1$, so that $(n, f(n)) \geqslant 2$. Their number is $\frac{1}{2}(10^{2r+1} - 10^{2r}) = 0.45x$. Hence $Q(x) \leqslant 0.55x$, and $P \neq 6\pi^{-2}$. Actually it is not difficult to see that for this f, P_f does not exist.

References

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