J. Austral. Math. Soc. (*Series A*) **50** (1991), 000–000

SYMMETRIC DUAL MULTIOBJECTIVE FRACTIONAL PROGRAMMING

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(Received 8 May 1989; Revised 16 October 1989)

Communicated by B. Mond

Abstract

A pair of symmetric dual multiobjective fractional programming problems is formulated and appropriate duality theorems are established.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 90 C 31, 90 C 32.

1. Introduction

Dorn [5] defined a program and its dual to be symmetric if the dual of the dual is the original problem. Dantzig, Eisenberg and Cottle [4] and Mond [7] gave symmetric dual theorems for programs involving a scalar functions $f(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ under the condition that $f(\cdot, y)$ is convex and $f(x, \cdot)$ is concave. More recently, Mond and Weir [8] have given a different pair of symmetric dual nonlinear programs which allows for a weakening of the convexity hypothesis for $f(x, y)$. Chandra, Craven and Mond [2] formulated a pair of symmetric dual fractional programs under suitable convexity hypothesis.

In [10] Weir and Mond discuss symmetric duality in multiobjective programming, generalizing [4] and [8]. The duals given there reduce to those known for scalar valued symmetric programming and also some more recent results in multiobjective programming duality.

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https://doi.org/10.1017/S1446788700032559 Published online by Cambridge University Press

The purpose of this paper is to formulate a pair of symmetric dual multiobjective fractional programs under suitable convexity assumptions. The relationship of the symmetric dual programs presented here to dual, nonsymmetric, fractional programming problems is also discussed.

2. Notation and preliminaries

The following conventions for vectors in $\mathbb{R}^n$ will be used:

$x > y$ if and only if $x_i > y_i$, $i = 1, 2, \dots, n$;
$x \geqq y$ if and only if $x_i \geqq y_i$, $i = 1, 2, \dots, n$;
$x \geq y$ if and only if $x_i \geqq y_i$, $i = 1, 2, \dots, n$, but $x \neq y$, $n \geq 2$;
$x \ngeq y$ is the negation of $x \geq y$.

If F is a twice differentiable functions from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$, then $\nabla_x F$ and $\nabla_y F$ denote gradient (column) vectors of F with respect to x and y respectively, and $\nabla_{yy} F$ and $\nabla_{yx} F$ denote respectively the $(m \times m)$ and $(n \times m)$ matrices of second partial derivatives.

If F is a twice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, then $\nabla_x F$ and $\nabla_y F$ denote respectively the $(n \times k)$ and $(m \times k)$ matrices of first partial derivatives. Consider the multiobjective programming problem:

$$\text{(P)} \qquad \text{minimize } f(x) \text{ subject to } x \in X.$$

Here $f: \mathbb{R}^n \to \mathbb{R}^k$ and $X \subset \mathbb{R}^n$. A feasible point z is said to be an efficient solution of (P) if $f_i(z) \geq f_i(x)$ for all $i = 1, 2, \dots, k$ implies $f_i(z) = f_i(x)$ for all $i = 1, 2, \dots, k$.

A feasible point z is said to be properly efficient [6] if it is efficient for (P) and if there exists a scalar $M > 0$ such that, for each i,

$$f_i(z) - f_i(x) \leq M(f_j(x) - f_j(z))$$

for some j such that $f_j(x) > f_j(z)$ wherever x is feasible for (P) and $f_i(x) < f_i(z)$.

A feasible point z is said to be a weak minimum [1] if there exists no other feasible point x for which $f(z) > f(x)$. If a feasible point z is efficient then it is also a weak minimum.

3. Duality

Consider the following pair of multiobjective symmetric fractional programs:

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Primal (FP)

minimize $\left(\frac{n_1(x, y)}{d_1(x, y)}, \ldots, \frac{n_k(x, y)}{d_k(x, y)}\right)^t$

subject to

$$\sum_{i=1}^{k} \omega_i(d_i \nabla_y n_i - n_i \nabla_y d_i) \leqq 0,$$

$$y^t \sum_{i=1}^{k} \omega_i(d_i \nabla_y n_i - n_i \nabla_y d_i) \geq 0, \qquad \omega > 0, \ \omega^t e = 1, \ x \geqq 0.$$

Dual (FD)

maximize $\left(\frac{n_1(u, v)}{d_1(u, v)}, \ldots, \frac{n_k(u, v)}{d_k(u, v)}\right)^t$

subject to

$$\sum_{i=1}^{k} \omega_i(d_i \nabla_x n_i - n_i \nabla_x d_i) \geqq 0,$$

$$u^t \sum_{i=1}^{k} \omega_i(d_i \nabla_x n_i - n_i \nabla_x d_i) \leq 0, \qquad \omega > 0, \ \omega^t e = 1, \ v \geqq 0.$$

Here $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^k$; n_i, $i = 1, 2, \ldots, k$, and d_i, $i = 1, 2, \ldots, k$, are twice differentiable functions from $\mathbb{R}^n \times \mathbb{R}^m$ to R, $n_1(\cdot, y)$ and $d_i(x, \cdot)$, $i = 1, 2, \ldots, k$, are convex $n_i(x, \cdot)$ and $d_i(\cdot, y)$, $i = 1, 2, \ldots, k$, are concave. It is assumed throughout that in the feasible regions $n_i > 0$, $i = 1, 2, \ldots, k$, and that each d_i is bounded.

In order to simplify notation we rewrite the primal and dual programs as follows.

Primal (FP$'$)

minimize $q = (q_1, q_2, \ldots, q_k)^t$

subject to

(1) $$q_i = n_i(x, y)/d_i(x, y), \qquad i = 1, 2, \ldots, k,$$

(2) $$\sum_{i=1}^{k} \omega_i(\nabla_y n_i - q_i \nabla_y d_i) \leqq 0,$$

(3) $$y^t \sum_{i=1}^{k} \omega_i(\nabla_y n_i - q_i \nabla_y d_i) \geq 0,$$

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$$\text{(4)} \qquad \omega > 0, \ \omega^t e = 1, \ x \geqq 0.$$

Dual (FD′)

maximize $\qquad p = (p_1, p_2, \dots, p_k)^t$

subject to

$$\text{(5)} \qquad p_i = n_i(u, v)/d_i(u, v), \qquad i = 1, 2, \dots, k,$$

$$\text{(6)} \qquad \sum_{i=1}^k \omega_i(\nabla_x n_i - p_i \nabla_x d_i) \geqq 0,$$

$$\text{(7)} \qquad u^t \sum_{i=1}^k \omega_i(\nabla_x n_i - p_i \nabla_x d_i) \leq 0, \qquad \omega > 0, \ \omega^t e = 1, \ v \geqq 0.$$

The following weak and strong duality theorems are stated in terms of (FP′) and (FP′) but apply equally to (FP) and (FD).

THEOREM 1 (weak duality). *Let* (x, y, w) *be feasible for* (FP′) *and let* (u, v, w) *be feasible for* (FP′). *Then* $q \nleq p$.

PROOF. From (4) and (7)

$$(x - u)^t \left[\sum_{i=1}^k \omega_i(n_{ik}(u, v) - p_i d_{ik}(u, v)) \right] \geq 0.$$

The convexity and concavity assumptions imply that $n_i(\cdot, v) - p_i d_i(\cdot, v)$, $i = 1, 2, \dots, k$, are convex; thus

$$\sum_{i=1}^k \omega_i(n_i(x, v) - p_i d_i(x, v)) \geq \sum_{i=1}^k \omega_i(n_i(u, v) - p_i d_i(u, v))$$

and from (5)

$$\text{(9)} \qquad \sum_{i=1}^k \omega_i(n_i(u, v) - p_i d_i(u, v)) \geq 0.$$

From (3) and (8)

$$(v - y)^t \left[\sum_{i=1}^k \omega_i(n_{iy}(x, y) - q_i d_{iy}(x, y)) \right] \leq 0.$$

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The convexity and concavity assumptions imply $n_i(x, \cdot) - q_i d_i(x, \cdot)$, $i = 1, 2, \ldots, k$, are concave; thus

$$\sum_{i=1}^{k} \omega_i(n_i(x, v) - q_i d_i(x, v)) \le \sum_{i=1}^{k} \omega_i(n_i(x, y) - q_i d_i(x, y))$$

and from (1)

$$\sum_{i=1}^{k} \omega_i(n_i(x, v) - q_i d_i(x, v)) \le 0. \tag{10}$$

Combining (9) and (10) gives

$$\sum_{i=1}^{k} \omega_i(q_i - p_i) d_i(x, v) \ge 0. \tag{11}$$

If, for some i, $q_i > p_i$ and for all $j \neq i$, $q_j \le p_j$, then since $d_i > 0$, $i = 1, 2, \ldots, k$, one would obtain a contradiction to (11); hence $q \nleq p$.

THEOREM 2 (strong duality). *Let (x_0, y_0, w_0) be a properly efficient solution for* (FP′)*; fix $w = w_0$ in* (FD)*; define q_0 by $q_{0i} = n_i(x_0, y_0)/d_i(x_0, y_0)$, $i = 1, 2, \ldots, k$. Assume that*

$$\sum_{i=1}^{k} \omega_{0i}(\nabla_{yy} n_i(x_0, y_0) - q_{0i} \nabla_{yy} d_i(x_0, y_0)) \tag{12}$$

is positive or negative definite and that the set

$$\{(\nabla_y n_1 - q_{01} \nabla_y d_1), (\nabla_y n_2 - q_{02} \nabla_y d_2), \ldots, (\nabla_y n_k - q_{0k} \nabla_y d_k)\} \tag{13}$$

is linearly independent. Then (x_0, y_0, ω_0) is a properly efficient solution of (FD′).

PROOF. Since (x_0, y_0, w_0) is a properly efficient solution of (FP′) then it is also a weak minimum. Hence there exist $a \in \mathbb{R}^m$, $b \in \mathbb{R}^k$ $r \in \mathbb{R}^m$, $s \in \mathbb{R}$, $t \in \mathbb{R}^k$, $z \in \mathbb{R}^n$, such that the following Fritz John conditions are

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satisfied at (x_0, y_0, ω_0) [3]:

(14) $$a_i + b_i d_i - \omega_{0i}(\nabla_y d_i)^t(r - sy_0) = 0, \qquad i = 1, 2, \ldots, k,$$

(15) $$\sum_{i=1}^{k}[b_i(\nabla_x n_i - q_{0i}\nabla_x d_i) + \omega_{0i}(\nabla_{yx} n_i - q_{0i}\nabla_{yx} d_i)(r - sy_0)] - z = 0,$$

(16) $$\sum_{i=1}^{k}[(b_i - s\omega_{0i})(\nabla_y n_i - q_{0i}\nabla_y d_i) + \omega_{0i}(\nabla_{yy} n_i - \omega_{0i}\nabla_{yy} d_i)(r - sy_0)] = 0,$$

(17) $$(r - sy_0)^t(\nabla_y n_k - q_{0i}\nabla_y d_i) - t_i = 0, \qquad i = 1, 2, \ldots, k,$$

(18) $$t^t\omega_0 = 0,$$

(19) $$z^t x_0 = 0,$$

(20) $$(a, r, s, t, z) \geqq 0,$$

(21) $$(a, b, r, s, t, z) \neq 0.$$

Since $\omega_0 > 0$ and $t \geqq 0$, then $t = 0$.

Multiplying (16) by $(r - sy_0)^t$ and applying (17) gives

$$(r - sy_0)^t\left[\sum_{i=1}^{k}\omega_{0i}(\nabla_{yy} n_i - q_{0i}\nabla_{yy} d_i)\right](r - sy_0) = 0.$$

Since (12) is assumed positive or negative definite then

(22) $$r = sy_0.$$

Thus, from (16),

$$\sum_{i=1}^{k}(b_i - s\omega_{0i})(\nabla_y n_i - q_{0i}\nabla_y d_i) = 0$$

and since, by assumption, the set (13) is linearly independent then

(23) $$b = s\omega_0.$$

If $s = 0$, then $b = 0$; from (14), $a = 0$; from (22) $r = 0$; from (15) $z = 0$; this combined with $t = 0$ contradicts (21); hence $s > 0$ and $b > 0$. From (22), $y_0 \geqq 0$ and from (15) and (23)

$$\sum_{i=1}^{k}\omega_{0i}(\nabla_x n_i - q_{0i}\nabla_x d_i) \geqq 0.$$

From (15), (23) and (19) it also follows that

$$x_0^t\sum_{i=1}^{k}\omega_{0i}(\nabla_x n_i - q_{0i}\nabla_x d_i) = 0.$$

Thus, (x_0, y_0, ω_0) is feasible for (FD) and the objective values of (FP′) and (FD′) are equal there. Clearly, (x_0, y_0, ω_0) is efficient for (FD′). If (x_0, y_0, ω_0) were improperly efficient, then for some feasible (u_i, v_i, ω_0) with $p_{1i} = n_i(u_i, v_i)/d_i(u_i, v_i)$, $i = 1, 2, \ldots, k$, and for some i, $p_{1i} - q_{0i} > M$ for any $M > 0$. Since d_i, $i = 1, 2, \ldots, k$, is bounded it follows that

$$\sum_{i=1}^{k} \omega_{0i}(q_{0i} - p_{1i})\, d_i(x_0, v_i) < 0,$$

which contradicts weak duality, equation (ii). Thus (x_0, y_0, ω_0) is properly efficient for (FD′).

4. Special cases

(i) If $n_i(x, y) = f_i(x) + y^t h(x)$, $i = 1, 2, \ldots, k$, and $d_i(x, y) = g_i(x)$, $i = 1, 2, \ldots, k$, where f_i, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, k$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ then programs (FP) and (FD) reduce to

(P1) minimize $((f_1(x) + y^t h(x))/g_1(x), \ldots, (f_k(x) + y^t h(x))/g_k(x))^t$

subject to

$$h(x) \leqq 0, \qquad y^t h(x) \geq 0, \qquad x \geqq 0,$$

and

(D1) maximize $((f_1(u) + v^t h(u))/g_1(u), \ldots, (f_k(u) + v^t h(u))/g_k(u))^t$

subject to

$$\sum_{i=1}^{k} \omega_i g_i(u) \nabla((f_i(u) + v^t h(u))/g_i(u)) \geqq 0,$$

$$u^t \sum_{i=1}^{k} \omega_i g_i(u) \nabla((f_i(u) + v^t h(u))/g_i(u)) \leq 0,$$

$$\omega > 0, \qquad \omega^t e = 1, \qquad v \geqq 0.$$

(Here $\nabla \equiv \nabla_x$.)

Since in (P1) $y^t h(x) \geq 0$, $g_i(x) > 0$, $i = 1, 2, \ldots, k$, we can take $y = 0$ and thus eliminate y from the problem. The problem (P1) is thus equivalent to

(P2) minimize $(f_1(x)/g_1(x), f_2(x)/g_2(x), \ldots, f_k(x)/g_k(x))^t$

subject to

$$h(x) \leqq 0, \qquad x \geqq 0.$$

This is a standard multiobjective fractional programming problem, with non-negativity constraints. Program (D1) is a Mond-Weir type dual for (P2).

(ii) If, in (FP) and (FD), $d_i(x, y) = 1$, we obtain symmetric dual problems of Weir and Mond [10]; there duality is proved under somewhat weaker convexity conditions.

(iii) If, in (FP) and (FD), $k = 1$, then we obtain pair of scalar symmetric dual fractional programs of Chandra, Craven and Mond [2].

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