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NONEXISTENCE RESULTS OF POSITIVE ENTIRE SOLUTIONS FOR QUASILINEAR ELLIPTIC INEQUALITIES

Dedicated to Professor Junji Kato on his 60th birthday

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ABSTRACT. This paper treats the quasilinear elliptic inequality

$$\operatorname{div}(|Du|^{m-2}Du) \geq p(x)u^{\sigma}, \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, $m > 1$, $\sigma > m-1$, and $p\colon \mathbb{R}^N \longrightarrow (0,\infty)$ is continuous. Sufficient conditions are given for this inequality to have no positive entire solutions. When p has radial symmetry, the existence of positive entire solutions can be characterized by our results and some known results.

1. **Introduction and the statement of results.** This paper is concerned with the quasilinear elliptic inequalities of the form

$$L_m u \equiv \operatorname{div}(|Du|^{m-2}Du) \geq p(x)u^{\sigma}, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $m > 1$, $\sigma > m-1$, and $p\colon \mathbb{R}^N \longrightarrow (0,\infty)$ is continuous. When $m = 2$, L_m reduces to the usual Laplacian; when $m \neq 2$, L_m is referred to as the degenerate Laplacian. A *positive entire solution* of (1.1) is defined to be a positive function $u \in C^1(\mathbb{R}^N)$ such that $|Du|^{m-2}Du \in C^1(\mathbb{R}^N)$ and satisfies (1.1) at each $x \in \mathbb{R}^N$.

The importance of such inequalities in mathematical analysis has been widely recognized in recent years. Interesting existence theorems and asymptotic theory for positive entire solutions of such inequalities have been obtained by many authors; see, *e.g.*, [4, 5, 7, 8, 9, 12, 14]. Among such results, we recall those obtained by [4]. The main existence theorem in [4] may be described roughly as follows:

THEOREM A [4, THEOREMS 2.1, 3.1, AND 3.2]. *Let p be radially symmetric. Then (1.1) has a positive radial entire solution if, for some $\varepsilon > 0$*

$$\begin{cases} \limsup_{|x|\to\infty} |x|^{m+\varepsilon} p(x) < \infty & \text{in the case } m < N; \\ \limsup_{|x|\to\infty} |x|^{m} (\log|x|)^{\sigma+1+\varepsilon} p(x) < \infty & \text{in the case } m = N; \\ \limsup_{|x|\to\infty} |x|^{N+\frac{\sigma(m-N)}{m-1}+\varepsilon} p(x) < \infty & \text{in the case } m > N. \end{cases} \tag{1.2}$$

Actually, in [4] we can find more than mentioned above. It is therefore natural to ask whether or not (1.1) does possess any positive entire solutions if (1.2) is violated. Our

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main objective is to give partial answers to this question. In fact, we can show that the decaying order imposed on $p(x)$ in Theorem A is optimal in some sense; that is, if $p(x)$ decays more slowly than indicated in Theorem A, (1.1) does not possess positive entire solutions.

The first result is as follows:

THEOREM 1. *Let $m > 1$ be arbitrary. If*

$$\liminf_{|x|\to\infty} |x|^m p(x) > 0, \tag{1.3}$$

then, inequality (1.1) has no positive entire solutions.

In the case $m < N$, Theorem A shows the sharpness of Theorem 1. On the other hand, in case $m \geq N$, one can improve Theorem 1 considerably as seen below:

THEOREM 2. *Let $m > N$. If*

$$\liminf_{|x|\to\infty} |x|^{N+\frac{\sigma(m-N)}{m-1}} p(x) > 0, \tag{1.4}$$

then, inequality (1.1) has no positive entire solutions.

THEOREM 3. *Let $m = N$. If*

$$\liminf_{|x|\to\infty} |x|^m (\log|x|)^{\sigma+1} p(x) > 0, \tag{1.5}$$

then, inequality (1.1) has no positive entire solutions.

The paper is organized as follows. In Section 2 an important lemma (Lemma 2.1) is stated and proved. By means of this lemma we can reduce the multi-dimensional problem under study to a one-dimensional problem. The proofs of Theorems 1, 2, and 3 are given in Sections 3, 4, and 5, respectively.

The problem of nonexistence of positive entire solutions has been studied in various situations. For the case $m = 2$, we refer to [2, 3, 6, 10, 13, 16] and, for the case $m \neq 2$, we refer to [1, 11, 12].

2. A comparison lemma.

Consider the ordinary differential equation

$$(r^{N-1}|v'|^{m-2}v')' = r^{N-1}q(r)v^{\sigma}, \quad r > 0, \tag{2.1}$$

where $q\colon [0,\infty) \longrightarrow (0,\infty)$ is a continuous function satisfying

$$p(x) \geq q(|x|), \quad x \in \mathbb{R}^N. \tag{2.2}$$

We define an entire solution of (2.1) by a solution v of (2.1) with $v'(0) = 0$ which exists on the interval $[0,\infty)$. It should be noted that the leading term of (2.1) is the so-called polar form of L_m.

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Let v be a solution of (2.1) with $v'(0) = 0$. Suppose that $[0, R)$ $(R \le \infty)$ be the maximal interval on which v is defined and remains positive. Then we have $v'(r) > 0$ for $0 < r < R$. In fact, an integration of (2.1) over $[0, r], r < R$, yields

$$|v'(r)|^{m-2}v'(r) = r^{1-N}\int_0^r s^{N-1}q(s)v^\sigma(s)\,ds, \quad 0 < r < R.$$

Hence $v'(r) > 0$ for $0 < r < R$. Moreover we know from this fact that, if $R < \infty$, then v must blow up at R: $v(R-0) = \infty$.

It is worthwhile to note that a positive entire solution v of (2.1) satisfies

$$v'(r) = \left(r^{1-N}\int_0^r s^{N-1}q(s)v^\sigma(s)\,ds\right)^{\frac{1}{m-1}}, \quad r \ge 0, \tag{2.3}$$

and

$$v(r) = v(0) + \int_0^r \left(s^{1-N}\int_0^s t^{N-1}q(t)v^\sigma(t)\,dt\right)^{\frac{1}{m-1}} ds, \quad r \ge 0. \tag{2.4}$$

The following lemma plays an important role in proving our results.

LEMMA 2.1. *If inequality (1.1) has a positive entire solution u, then there exists a positive entire solution v of (2.1).*

To prove Lemma 2.1, we prepare the following lemma.

LEMMA 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let u be a positive entire solution of (1.1) and let $v \in C(\overline{\Omega}) \cap C^1(\Omega)$ be a positive function satisfying $|Dv|^{m-2}Dv \in C^1(\Omega)$. If $L_m v \le p(x)v^\sigma$ in Ω and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

REMARK. Lemma 2.2 was also obtained in [1, 15], although we give a proof for the sake of completeness.

PROOF. Let $\phi: \mathbb{R} \longrightarrow [0, \infty)$ be a C^1-function which vanishes on $(-\infty, 0]$ and is strictly increasing on $(0, \infty)$. For example, $\phi(u) = 0$ for $u \le 0$ and $\phi(u) = u^2$ for $u > 0$. We have

$$(L_m u - L_m v)\phi(u-v) \ge p(x)(u^\sigma - v^\sigma)\phi(u-v) \quad \text{in } \Omega.$$

As a consequence of the divergence theorem, it follows that

$$-\int_\Omega (|Du|^{m-2}Du - |Dv|^{m-2}Dv)\cdot(Du - Dv)\phi'(u-v)\,dx \ge \int_\Omega p(x)(u^\sigma - v^\sigma)\phi(u-v)\,dx.$$

Since $(|Du|^{m-2}Du - |Dv|^{m-2}Dv)\cdot(Du - Dv) \ge 0$ in Ω, we have

$$\int_\Omega p(x)(u^\sigma - v^\sigma)\phi(u-v)\,dx \le 0.$$

Thus, we conclude that $u \le v$ in Ω. ∎

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PROOF OF LEMMA 2.1. Assume to the contrary that no such function v exists. Take $a > 0$ such that $a < u(0)$. Let v be a solution of (2.1) with initial values $v(0) = a$ and $v'(0) = 0$. Since v can not be continued to ∞, the maximal interval of existence of v is of the form $[0, R)$, $R < \infty$, and we have $v'(r) > 0$ for $0 < r < R$ and v blows up at R: $v(R-0) = \infty$. We therefore can find an $R_1 \in (0, R)$ so that

$$v(R_1) \geq \max\{u(x) : |x| = R_1\}.$$

Define $\Omega = \{x \in \mathbb{R}^N : |x| < R_1\}$. Then $L_m v \leq p(x)v^\sigma$ in Ω and $v \geq u$ on $\partial\Omega$. By Lemma 2.2, $u \leq v$ in Ω, which contradicts $v(0) = a < u(0)$. Thus, the proof is complete. ■

3. **Proof of Theorem 1.** In this section Theorem 1 is proved. Assume that (1.3) holds. Then there is a constant $c > 0$ such that

$$p(x) \geq \frac{c}{1+|x|^m}, \quad x \in \mathbb{R}^N.$$

Putting

$$q(r) = \frac{c}{1+r^m}, \quad r \geq 0,$$

we find that q satisfies (2.2), and

$$q(r) \geq C_0 r^{-m}, \quad r \geq R_0, \tag{3.1}$$

for some $C_0, R_0 > 0$.

PROOF OF THEOREM 1. Suppose to the contrary that (1.1) admits a positive entire solution. Then, (2.1) has a positive entire solution $v(r)$ by Lemma 2.1. First we show that

$$\lim_{r\to\infty} v(r) = \infty. \tag{3.2}$$

Since (2.4) holds and v is increasing, it follows that

$$v(r) \geq v(0) + v(0)^{\frac{\sigma}{m-1}} \int_0^r \left(s^{1-N}\int_0^s t^{N-1}q(t)\,dt\right)^{\frac{1}{m-1}} ds.$$

From (3.1), we observe that

$$\lim_{r\to\infty}\int_0^r \left(s^{1-N}\int_0^s t^{N-1}q(t)\,dt\right)^{\frac{1}{m-1}} ds = \infty.$$

Thus we obtain (3.2).

Integrating (2.1) twice over $[R, r]$, $R \geq R_0$, we see that

$$v(r) \geq v(R) + \int_R^r \left(\int_R^s \left(\frac{t}{s}\right)^{N-1} q(t)v^\sigma(t)\,dt\right)^{\frac{1}{m-1}} ds, \quad r \geq R \geq R_0. \tag{3.3}$$

Using (3.1) and the inequality

$$\left(\frac{t}{s}\right)^{N-1} \geq \frac{1}{2^{N-1}} \quad \text{for } R \leq t \leq s \leq 2R$$

in (3.3), we have

$$v(r) \geq v(R) + C_1 \int_R^r \left(\int_R^s t^{-m} v^\sigma(t)\,dt\right)^{\frac{1}{m-1}} ds, \quad R \leq r \leq 2R, \tag{3.4}$$

where $C_1 = (C_0/2^{(N-1)})^{1/(m-1)}$. Now let us define the function $w(r)$ on $R \leq r \leq 2R$, by the right hand side of (3.4). Then w satisfies $w(R) = v(R)$ and, for $R \leq r \leq 2R$, $w(r) \leq v(r)$,

$$w'(r) = C_1 \left(\int_R^r s^{-m} v^\sigma(s)\,ds\right)^{\frac{1}{m-1}} \geq 0,$$

and

$$(|w'|^{m-2} w')' = C_1^{m-1} r^{-m} v^\sigma \geq C_1^{m-1} r^{-m} w^\sigma. \tag{3.5}$$

Multiplying (3.5) by $w' \geq 0$ and integrating the resulting inequality on $[R, r]$ ($R \leq r \leq 2R$), we see that

$$\frac{m-1}{m} |w'(r)|^m \geq C_1^{m-1} r^{-m} \int_R^r w^\sigma(s) w'(s)\,ds = \frac{C_1^{m-1}}{\sigma+1} r^{-m} [w^{\sigma+1}(r) - w^{\sigma+1}(R)],$$

which implies

$$[w^{\sigma+1}(r) - w^{\sigma+1}(R)]^{-\frac{1}{m}} w'(r) \geq C_2 r^{-1}, \quad R < r < 2R,$$

where $C_2 = (\frac{mC_1^{m-1}}{(\sigma+1)(m-1)})^{1/m} > 0$. Integrating over $[R, 2R]$, we have

$$\int_{v(R)}^\infty [s^{\sigma+1} - w^{\sigma+1}(R)]^{-\frac{1}{m}} ds \geq \int_{w(R)}^{w(2R)} [s^{\sigma+1} - w^{\sigma+1}(R)]^{-\frac{1}{m}} ds \geq C_2 \log 2.$$

We observe that, by the change of variable $s = w(R)t$,

$$[v(R)]^{-\frac{\sigma+1-m}{m}} \int_1^\infty (t^{\sigma+1} - 1)^{-\frac{1}{m}} dt \geq C_2 \log 2, \quad R \geq R_0. \tag{3.6}$$

On the other hand, from (3.2) and $\sigma + 1 - m > 0$, we have

$$\lim_{R\to\infty} [v(R)]^{-\frac{\sigma+1-m}{m}} \int_1^\infty (t^{\sigma+1} - 1)^{-\frac{1}{m}} dt = 0,$$

which contradicts (3.6). This completes the proof. ∎

REMARK. When $m = 2$, Theorem 1 was proved by [2, Theorem 3.1], [3, Theorem 2.1], and [10, Theorem 3.4].

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4. **Proof of Theorem 2.** In this section we assume that $m > N$ and (1.4) holds. Then, there exists a positive constant c such that

$$p(x) \geq \frac{c}{1 + |x|^{N + \frac{\sigma(m-N)}{m-1}}}, \quad x \in \mathbb{R}^N.$$

Define a function q by

$$q(r) = \frac{c}{1 + r^{N + \frac{\sigma(m-N)}{m-1}}}, \quad r \geq 0.$$

Then q satisfies (2.2) and there exist constants $C_0 > 0$ and $R_0 > 0$ such that

$$q(r) \geq C_0 r^{-N - \frac{\sigma(m-N)}{m-1}}, \quad r \geq R_0. \tag{4.1}$$

The proof of Theorem 2 is decomposed into several steps.

LEMMA 4.1. *Let v be a positive entire solution of (2.1). Then*

$$\lim_{r \to \infty} \frac{v(r)}{r^{\frac{m-N}{m-1}}} = \infty. \tag{4.2}$$

PROOF. From (2.4) we observe that, for $r > 1$,

$$\begin{aligned} v(r) &\geq \int_1^r \left(s^{1-N} \int_0^1 t^{N-1} q(t) v^\sigma(t)\, dt \right)^{\frac{1}{m-1}} ds \\ &\geq \left(\int_0^1 s^{N-1} q(s) v^\sigma(s)\, ds \right)^{\frac{1}{m-1}} \int_1^r s^{-\frac{N-1}{m-1}}\, ds. \end{aligned}$$

Then, we obtain

$$v(r) \geq C_1 r^{\frac{m-N}{m-1}}, \quad r \geq R_0, \tag{4.3}$$

for some constant $C_1 > 0$. From (2.3), we have

$$v'(r) \geq r^{-\frac{N-1}{m-1}} \left(\int_{R_0}^r s^{N-1} q(s) v^\sigma(s)\, ds \right)^{\frac{1}{m-1}}, \quad r \geq R_0.$$

By virtue of (4.1) and (4.3), we find that

$$\begin{aligned} v'(r) &\geq (C_0 C_1^\sigma)^{1/(m-1)} r^{-\frac{N-1}{m-1}} \left(\int_{R_0}^r s^{-1}\, ds \right)^{\frac{1}{m-1}} \\ &\geq C_2 r^{-\frac{N-1}{m-1}} (\log r)^{\frac{1}{m-1}}, \quad r \geq R_0, \end{aligned}$$

for some $C_2 > 0$. This implies that $\lim_{r\to\infty} v'(r)/r^{-\frac{N-1}{m-1}} = \infty$. By L'Hôspital's rule, we conclude that (4.2) holds. ∎

LEMMA 4.2. *Let v be a positive entire solution of (2.1). Let $w(r) = v(r)/r^{\frac{m-N}{m-1}}$ and $\lambda = \sigma/(m-1)$. Then, for some $C > 0$,*

$$w(2r) \geq C[w(r)]^\lambda, \quad r \geq R_0. \tag{4.4}$$

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PROOF. Integrating (2.1) twice over $[R, r]$, $(r > R \geq R_0)$, we have

$$v(r) \geq \int_R^r s^{-\frac{N-1}{m-1}} \left(\int_R^s t^{N-1} q(t) v^\sigma(t)\, dt \right)^{\frac{1}{m-1}} ds, \quad r > R \geq R_0.$$

Putting $r = 2R$ in the above, we obtain

$$v(2R) \geq \int_R^{2R} s^{-\frac{N-1}{m-1}} \left(\int_R^s t^{N-1} q(t) v^\sigma(t)\, dt \right)^{\frac{1}{m-1}} ds, \quad R \geq R_0.$$

Since (4.1) holds and $v(r)$ is increasing, it follows that

$$v(2R) \geq C_0^{1/(m-1)} [v(R)]^{\frac{\sigma}{m-1}} \int_R^{2R} s^{-\frac{N-1}{m-1}} \left(\int_R^s t^{-1-\frac{\sigma(m-N)}{m-1}}\, dt \right)^{\frac{1}{m-1}} ds, \quad R \geq R_0.$$

We therefore have

$$v(2R) \geq C_0^{1/(m-1)} R^{\frac{m-N}{m-1}(1-\frac{\sigma}{m-1})} [v(R)]^{\frac{\sigma}{m-1}} \int_1^2 t^{-\frac{N-1}{m-1}} \left(\int_1^t \tau^{-1-\frac{\sigma(m-N)}{m-1}}\, d\tau \right)^{\frac{1}{m-1}} dt$$

for $R \geq R_0$. This implies (4.4) with

$$C = C_0^{1/(m-1)} \int_1^2 t^{-\frac{N-1}{m-1}} \left(\int_1^t \tau^{-1-\frac{\sigma(m-N)}{m-1}}\, d\tau \right)^{\frac{1}{m-1}} dt.$$

∎

LEMMA 4.3. *Let v be a positive entire solution of (2.1). Then, for any $k \in N$,*

(4.5) $$\lim_{r\to\infty} \frac{v(r)}{r^k} = \infty.$$

PROOF. From Lemmas 4.1 and 4.2, we see that $\lim_{r\to\infty} w(r) = \infty$ and

(4.6) $$w(2r) \geq C[w(r)]^\lambda, \quad r \geq R_0,$$

where $w(r) = v(r)/r^{\frac{m-N}{m-1}}$ and $\lambda = \sigma/(m-1)$. Choose $R_1 \geq R_0$ so large that

(4.7) $$C^{\frac{1}{\lambda-1}} w(r) \geq 2, \quad r \geq R_1.$$

From (4.6) we observe that, for any $\ell \in N$,

(4.8) $$w(2^\ell r) \geq C^{1+\lambda+\cdots+\lambda^\ell} [w(r)]^{\lambda^\ell} = C^{-\frac{1}{1-\lambda}} [C^{\frac{1}{\lambda-1}} w(r)]^{\lambda^\ell}, \quad r \geq R_1.$$

Let $r \geq 2R_1$. Then we can find $\ell = \ell(r) \in N$ and $R_2 \in [R_1, 2R_1)$ such that $2^\ell R_1 \leq r < 2^{\ell+1} R_1$ and $r = 2^\ell R_2$. We notice here that

(4.9) $$\ell(r) \geq \frac{\log r - \log R_1 - \log 2}{\log 2}.$$

From (4.7) and (4.8), we have

$$w(r) = w(2^{\ell(r)} R_2) \geq C^{-\frac{1}{\lambda-1}} [C^{\frac{1}{\lambda-1}} w(R_2)]^{\lambda^\ell} \geq C^{-\frac{1}{\lambda-1}} H\big(\ell(r)\big), \quad r \geq 2R_1,$$

where $H(\alpha) = 2^{\lambda^\alpha}$. By virtue of (4.9), we easily see that $\lim_{r\to\infty} H\big(\ell(r)\big)/r^k = \infty$ for any $k \in N$. Then we have, for any $k \in N$,

$$\lim_{r\to\infty} \frac{w(r)}{r^k} = \infty.$$

Thus we obtain (4.5). ∎

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PROOF OF THEOREM 2. Suppose to the contrary that inequality (1.1) admits a positive entire solution u. Then, by Lemma 2.1, there exists a positive entire solution v of (2.1). We note that v satisfies

$$(r^{N-1}|v'|^{m-2}v')' = r^{N-1}\Big(q(r)[v(r)]^{\frac{\sigma-m+1}{2}}\Big)v^{\frac{\sigma+m-1}{2}}, \quad r > 0. \tag{4.10}$$

From (4.1) and Lemma 4.3, we observe that

$$\lim_{r\to\infty} r^m\Big(q(r)[v(r)]^{\frac{\sigma-m+1}{2}}\Big) = \infty. \tag{4.11}$$

Since $(\sigma + m - 1)/2 > m - 1$, by virtue of Theorem 1 we can show that (4.10) has no positive entire solutions. Thus, we have a contradiction. This completes the proof of Theorem 2. ∎

5. **Proof of Theorem 3.** Only a sketch of the proof of Theorem 3 is given here, since a parallel argument to that of Theorem 2 is valid.

Assume that $m = N$ and (1.5) holds. Then, there exists a positive constant c such that

$$p(x) \geq \frac{c}{1 + |x|^m\big(\log(1 + |x|)\big)^{\sigma+1}}, \quad x \in \mathbb{R}^N.$$

Put

$$q(r) = \frac{c}{1 + r^m\big(\log(1 + r)\big)^{\sigma+1}}, \quad r \geq 0.$$

We then show that q satisfies (2.2) and there exist constants $C_0 > 0$ and $R_0 > 0$ such that

$$q(r) \geq C_0 r^{-m}(\log r)^{-\sigma-1}, \quad r \geq R_0. \tag{5.1}$$

LEMMA 5.1. *Let v be a positive entire solution of (2.1). Then*

$$\lim_{r\to\infty} \frac{v(r)}{\log r} = \infty. \tag{5.2}$$

PROOF. From (2.4) we observe that, for $r > 1$,

$$v(r) \geq \int_1^r \Big(s^{1-m}\int_0^1 t^{m-1}q(t)v^\sigma(t)\,dt\Big)^{\frac{1}{m-1}} ds \geq \Big(\int_0^1 s^{m-1}q(s)v^\sigma(s)\,ds\Big)^{\frac{1}{m-1}} \int_1^r s^{-1}\,ds.$$

Then, we obtain

$$v(r) \geq C_1 \log r, \quad r \geq R_0, \tag{5.3}$$

for some constant $C_1 > 0$. From (2.3), we have

$$v'(r) \geq r^{-1}\Big(\int_{R_0}^r s^{N-1}q(s)v^\sigma(s)\,ds\Big)^{\frac{1}{m-1}}, \quad r > R_0.$$

By virtue of (5.1) and (5.3), we find that

$$\begin{aligned} v'(r) &\geq (C_0C_1^\sigma)^{1/(m-1)}r^{-1}\Big(\int_{R_0}^r s^{-1}(\log s)^{-1}\,ds\Big)^{\frac{1}{m-1}} \\ &\geq C_2 r^{-1}\big(\log(\log r)\big)^{\frac{1}{m-1}}, \quad r \geq R_0, \end{aligned}$$

for some $C_2 > 0$. This implies that $\lim_{r\to\infty} rv'(r) = \infty$. By L'Hôspital's rule, we conclude that (5.2) holds. ∎

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LEMMA 5.2. *Let v be a positive entire solution of (2.1). Let $w(r) = v(r)/\log r$ and $\lambda = \sigma/(m-1)$. Then, for some $C > 0$,*

$$w(r^2) \geq C[w(r)]^{\lambda}, \quad r \geq R_0. \tag{5.4}$$

PROOF. Integrating (2.1) twice over $[R, r]$, $(r > R \geq R_0)$, we have

$$v(r) \geq \int_R^r s^{-1}\Big(\int_R^s t^{m-1}q(t)v^{\sigma}(t)\,dt\Big)^{\frac{1}{m-1}}\,ds, \quad r > R \geq R_0.$$

Putting $r = R^2$ in the above, we obtain

$$v(R^2) \geq \int_R^{R^2} s^{-1}\Big(\int_R^s t^{m-1}q(t)v^{\sigma}(t)\,dt\Big)^{\frac{1}{m-1}}\,ds, \quad R \geq R_0.$$

Since (5.1) holds and $v(r)$ is increasing, it follows that

$$v(R^2) \geq C_0^{1/(m-1)}[v(R)]^{\frac{\sigma}{m-1}}\int_R^{R^2} s^{-1}\Big(\int_R^s t^{-1}(\log t)^{-\sigma-1}\,dt\Big)^{\frac{1}{m-1}}\,ds, \quad R \geq R_0,$$

and hence, we obtain

$$v(R^2) \geq C_0^{1/(m-1)}(\log R)^{-\frac{\sigma}{m-1}+1}[v(R)]^{\frac{\sigma}{m-1}}\int_1^2\Big(\int_1^t \tau^{-\sigma-1}\,d\tau\Big)^{\frac{1}{m-1}}\,dt, \quad R \geq R_0.$$

This implies (5.5) with

$$C = \frac{1}{2}C_0^{1/(m-1)}\int_1^2\Big(\int_1^t \tau^{-\sigma-1}\,d\tau\Big)^{\frac{1}{m-1}}\,dt.$$

■

LEMMA 5.3. *Let v be a positive entire solution of (2.1). Then, for any $k \in N$,*

$$\lim_{r\to\infty}\frac{v(r)}{(\log r)^k} = \infty. \tag{5.5}$$

PROOF. Let $w(r) = v(r)/\log r$ and

$$z(s) = w(r), \quad s = \log r.$$

Then, from Lemmas 5.1 and 5.2, we see that $\lim_{s\to\infty} z(s) = \infty$ and

$$z(2s) \geq C[z(s)]^{\lambda}, \quad s \geq S_0, \tag{5.6}$$

where $S_0 = \log R_0$. Hence exactly as in the proof of Theorem 2, we can show that

$$\lim_{s\to\infty}\frac{z(s)}{s^k} = \infty,$$

for any $k \in N$, which implies

$$\lim_{r\to\infty}\frac{w(r)}{(\log r)^k} = \infty.$$

Thus we obtain (5.5). ■

The final stage of the proof of Theorem 3 is the same as that of Theorem 2. So we leave the proof to the reader.

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