REGULARITY OF LOCALLY CONVEX SURFACES

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Interior estimates are derived for the $C^{2,\mu}$-Hölder norm of the radius vector $X \in C^{1,1}(\Omega)$ of a locally convex surface $\Sigma$ in terms of the first fundamental form $I_\Sigma$, the Gauss curvature $K$ and the integral $\int |H| d\sigma$. Here $H$ is the mean curvature of $\Sigma$. The coefficients $g_{ij}$ of $I_\Sigma$ are assumed to belong to the Hölder class $C^{2,\mu}(\Omega)$ for some $\mu$, $0 < \mu < 1$. A boundary condition is discussed which ensures an estimate for $\int |H| d\sigma$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\Omega$ be a domain in the $u = (u^1, u^2)$-plane. Consider a differential geometric, locally convex surface $\Sigma$, which is given by a radius vector $X$ of class $C^{1,1}(\Omega, \mathbb{R}^3)$ such that the unit normal

$$\nu = \frac{D_1 X \wedge D_2 X}{|D_1 X \wedge D_2 X|}$$

exists.

ASSUMPTION (A). Suppose that the coefficients $g_{ij}$ of the first fundamental form

$$I_\Sigma = D_1 X \cdot D_2 X \, du^i du^j$$
$$= g_{ij} du^i du^j$$

belong to the Hölder class $C^{2,\mu}(\Omega)$ for some $\mu$, $0 < \mu < 1$, such that

$$\|g_{ij}\|_{C^{2,\mu}(\Omega)} \leq a,$$

and

$$g, K \geq \frac{1}{c}.$$ 

Here

$$g = \det I_\Sigma = |D_1 X \wedge D_2 X|^2,$$

and

$$K = \frac{\det II_\Sigma}{\det I_\Sigma} = \frac{h}{g}.$$
is the Gauss curvature of \( \Sigma \), which, by the *theorema egregium*, depends only on the coefficients of \( I_\Sigma \) and their first and second derivatives.

\[
II_\Sigma = D_{ij}X \cdot \nu \, du^i du^j \\
= h_{ij} du^i du^j
\]

is the second fundamental form, which is defined almost everywhere.

**ASSUMPTION(B).** Suppose that

\[
\int_\Sigma |H| \, d\sigma \leq M,
\]

where

\[
H = \frac{h_{ij}g^{ij}}{2}, \quad [g^{ij}] = [g_{kl}]^{-1},
\]

is the mean curvature, and

\[
d\sigma = \sqrt{g} \, du
\]

is the area element of \( \Sigma \).

The main result of this note then reads as the following:

**THEOREM 1.** The radius vector \( X \) belongs to the Hölder class \( C^{r,\mu}_{loc}(\Omega) \). For each subset \( \Omega' \), which is compactly contained in \( \Omega \), there is an estimate of the form

\[
\|D^2X\|_{C^{r,\mu}(\Omega')} \leq C,
\]

where the constant \( C \) only depends on \( \mu, a, c, M, \) and \( \text{dist}(\Omega', \partial\Omega) \).

The regularity part of Theorem 1 follows from the regularity theory for elliptic Monge–Ampère equations [27] via the Darboux equation (5) (see for example, Nirenberg [18]). The \( C^{2,\mu}_{loc} \)-estimates follow from [24, 26], if \( \Sigma \) is a graph or a closed surface. These results can also be derived from the prescribed Gauss curvature equation

\[
\det D^2z = K \left(1 + |Dz|^2\right)^2
\]

(compare Sabitov [20] for the regularity and [22, 23] for the *a priori* estimates for graphs and closed surfaces). The prescribed Gauss curvature equation (2) is particularly useful when the regularity requirements regarding \( I_\Sigma \) are weakened to the extent that the Gauss curvature \( K \) is only pinched between two positive numbers (see Heinz [6], Nikolaev and Shefel’ [16, 17]).

The regularity statement can be considered a variation of regularity theorems of Alexandrow [1] and Pogorelov [19]. That it is sharp follows from Sabitov and Shefel’ [21], who investigated the connections between the regularity of a surface and its metric.
The case of closed surfaces is of particular interest because of Weyl's embedding problem (see Weyl [29], Lewy [14], Nirenberg [18], Heinz [5, 22, 23]).

The purpose of the present note however is to provide the stated local $C^{2,\mu}$-estimates for the radius vector $X$ of a locally convex surface, thus improving those of Heinz [7], which require additional regularity assumptions regarding both the radius vector $X(u)$ and the coefficients $g_{ij}$ of the first fundamental form. The approach, which is due to Heinz [7], consists of introducing conjugate isothermal parameters, that is, of constructing a conformal map $x = x(u)$ with respect to the second fundamental form of $\Sigma$.

The present estimates rest on sharp estimates for the Jacobian of the Darboux system (8), which is satisfied by the inverse mapping $u = u(x)$. These estimates were derived in [25], generalising classical theorems of Lewy [12, 13] (see also Efimow [4]) and Heinz [7, 8].

The a priori constant in (1) does depend on the integral $\int |H| \, d\sigma$, because suitable Riemannian metrics on the unit disc with positive Gauss–Kronecker curvature can be embedded in Euclidean 3-space such that $\int |H| \, d\sigma$ is arbitrarily large (see Theorem 3 of Heinz [9]).

If $\Sigma$ is a closed convex surface, then the integral $\int |H| \, d\sigma$ can be estimated because of Minkowski's integral formula

\[ \int_{\Sigma} H \, d\sigma = - \int_{\Sigma} K \nu \cdot X \, d\sigma, \]

which holds for orientable closed surfaces (Minkowski [15], Herglotz [10], Efimow [4], Heinz [7]). A careful investigation of the proof of (3), (which we take from Klingenberg [11], p.106, instead of proving (3) like in [7]) shows that an a priori estimate for $\int |H| \, d\sigma$ can also be derived if $\Sigma$ is attached to the unit sphere $S^2$ of order 1:

**Proposition 2.** Suppose that $X \in C^{1,1}(\overline{B}) \cap C^2_{loc}(B)$, $B = \{|u| < 1\}$, is the radius vector of a locally convex surface, which satisfies the boundary condition

\[ |X| = 1, \quad \frac{\partial |X|}{\partial n} = 0 \quad \text{for} \quad |u| = 1. \]

Here $n$ is the outward pointing normal to $\partial B = S^1$. Then there is an estimate of the form

\[ \int_{\Sigma} |H| \, d\sigma \leq C(a, \kappa), \]

where

\[ |g_{ij}| \leq a, \quad K \leq \kappa. \]
2. THE DARBOUX EQUATION AND THE REGULARITY PROOF

Let
\[ \rho = \rho(u) = X \cdot X_0, \quad X_0 = \nu(u_0), \]
for some \( u_0 \in \Omega \). The Gauss equations
\[ D_{ij}X = \Gamma^k_{ij}D_kX + h_{ij}, \]
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kt} (D_j g_{it} + D_i g_{jt} - D_t g_{ij}), \]
then imply that
\[ \det[D_{ij}\rho - \Gamma^k_{ij}D_k\rho] = h(\nu \cdot X_0)^2 \]
\[ = K(D_1 XD_2 X X_0)^2 \]
\[ = K \left[ g - \frac{g^{ij}}{g} D_i \rho D_j \rho \right]. \]

(5) is the Darboux equation, which is elliptic in a neighbourhood \( N \) of \( u_0 \in \Omega \), because \( \Sigma \) is then a convex graph over a plane perpendicular to \( \nu(u_0) \). The regularity theory for elliptic Monge–Ampère equations, in particular Theorem 1 of [27], yields the regularity \( \rho \in C^{2,\mu}_{\text{loc}}(N) \). To translate this into the regularity \( X \in C^{2,\mu}_{\text{loc}}(N, \mathbb{R}^3) \), consider the three \( 3 \times 3 \)-systems
\[ X_0 \cdot D_{ij}X = D_{ij} \rho, \]
\[ D_kX \cdot D_{ij}X = \frac{1}{2} (D_j g_{ik} + D_i g_{jk} - D_k g_{ij}) \]
(which can easily be derived from the Gauss equations). By (5), the determinant of the coefficient matrix is
\[ X_0 D_1 XD_2 X = \sqrt{g - \frac{g^{ij}}{g} D_i \rho D_j \rho} \neq 0, \]
and the statement \( X \in C^{2,\mu}(\Omega, \mathbb{R}^3) \) of Theorem 1 follows from Cramer’s rule. \( \square \)

3. CONJUGATE ISOThERMAL PARAMETERS AND THE DARBOUX SYSTEM

**Lemma 3.** Let \( a_{ij} \) be functions of class \( C^1(\Omega) \) such that
\[ \Delta = \det[a_{ij}] > 0. \]
Let $\overline{B}_R = \overline{B}_R(u_0) \subset \Omega$. Then there exists a homeomorphism $u = u(x)$ from $\overline{B} = \{|z| \leq 1\}$ onto $\overline{B}_R$ of class $C^{1,\mu}_{\text{loc}}(B)$ with $u(0) = u_0$, which satisfies the system

$$D_{\alpha} \left( \sqrt{\Delta} D_{\alpha} u^k \right) = D_{\alpha} \left( \Delta u^k \right) Du^1 \wedge Du^2.$$  

Furthermore

$$Du^1 \wedge Du^2 = D_1 u^1 D_2 u^2 - D_2 u^1 D_1 u^2 \neq 0,$$

and

$$\sqrt{\Delta} a^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}.$$

This lemma is contained in Lemma 2 of [26], which in turn is an improved version of Lemma 2 of Heinz [5]. The proof is by mapping the disc $\overline{B}_R(u_0)$ conformally with respect to the metric

$$ds^2 = a_{ij} du^i du^j$$

onto the unit disc $\overline{B} = \{|z| \leq 1\}$.

**PROPOSITION 4.** Suppose that $\Sigma$ is a locally convex surface with radius vector $X \in C^{2,\mu}(\Omega, \mathbb{R}^3)$ for some $\mu$, $0 < \mu < 1$, and let $\overline{B}_R = \overline{B}_R(u_0) \subset \Omega$. Then there exist conjugate isothermal parameters $x = (x^1, x^2)$, that is, there exists a homeomorphism $u = u(x)$ from $\overline{B} = \{|z| \leq 1\}$ onto $\overline{B}_R$ of class $C^{1,\mu}_{\text{loc}}(B)$ with $u(0) = u_0$, and

$$Du^1 \wedge Du^2 = D_1 u^1 D_2 u^2 - D_2 u^1 D_1 u^2 > 0$$

such that the following conformality relations hold:

$$\sqrt{g} K h^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}.$$

Furthermore $u$ satisfies the Darboux system

$$\Delta_K u^k = D_{\alpha} \left( \sqrt{\Delta} D_{\alpha} u^k \right) + \sqrt{\Delta} K_{ij}^k Du^i \cdot Du^j = 0 \quad (k = 1, 2).$$

**PROOF:** Assume first that $X \in C^3(\Omega, \mathbb{R}^3)$ so that $II_{\Sigma} \in C^1(\Omega, \mathbb{R}^3)$ and consider the differential form

$$ds^2 = \frac{1}{\sqrt{g}} II_{\Sigma} = \frac{h_{ij}}{\sqrt{g}} du^i du^j.$$

Lemma 3 yields the existence of the parameters $x = (x^1, x^2)$ which satisfy the conformality relations (7). According to (6),

$$D_{\alpha} \left( \sqrt{\Delta} D_{\alpha} u^k \right) = D_{\alpha} \left( \sqrt{g} K h^{\alpha k} \right) Du^1 \wedge Du^2,$$

that is,

$$D_{\alpha} \left( \sqrt{\Delta} D_{\alpha} u^1 \right) = \left[ D_1 \left[ \frac{h_{22}}{\sqrt{g}} \right] - D_2 \left[ \frac{h_{12}}{\sqrt{g}} \right] \right] Du^1 \wedge Du^2,$$

$$D_{\alpha} \left( \sqrt{\Delta} D_{\alpha} u^2 \right) = \left[ D_2 \left[ \frac{h_{11}}{\sqrt{g}} \right] - D_1 \left[ \frac{h_{12}}{\sqrt{g}} \right] \right] Du^1 \wedge Du^2.$$
By invoking the Codazzi–Mainardi equations

\[ D_j h_{ik} - D_i h_{jk} = \Gamma^t_{jk} h_{it} - \Gamma^t_{ik} h_{jt} \]

and

\[ D_k g = 2 g (\Gamma^1_{ik} + \Gamma^2_{ik}), \]

which follow from

\[ D_k g_{ij} = \Gamma^t_{ik} g_{lj} + \Gamma^t_{jk} g_{li}, \]

one computes

\[
D_1 \left[ \frac{h_{22}}{\sqrt{g}} \right] - D_2 \left[ \frac{h_{12}}{\sqrt{g}} \right] = \frac{1}{\sqrt{g}} \left( \Gamma^t_{12} h_{2t} - \Gamma^t_{22} h_{1t} - h_{22} (\Gamma^1_{11} + \Gamma^2_{21}) + h_{12} (\Gamma^1_{12} + \Gamma^2_{22}) \right)
\]

\[ = \frac{1}{\sqrt{g}} \left( -\Gamma^1_{11} h_{22} + 2 \Gamma^1_{12} h_{12} - \Gamma^2_{22} h_{11} \right) \]

\[ = -\sqrt{k} \Gamma^1_{ij} \frac{Du^i}{Du^1} \wedge \frac{Du^j}{Du^2}, \]

and

\[
D_2 \left[ \frac{h_{11}}{\sqrt{g}} \right] - D_1 \left[ \frac{h_{12}}{\sqrt{g}} \right] = \frac{1}{\sqrt{g}} \left( \Gamma^t_{21} h_{1t} - \Gamma^t_{11} h_{2t} - h_{11} (\Gamma^1_{12} + \Gamma^2_{22}) + h_{12} (\Gamma^1_{11} + \Gamma^2_{12}) \right)
\]

\[ = \frac{1}{\sqrt{g}} \left( -\Gamma^2_{11} h_{22} + \Gamma^2_{12} h_{12} - \Gamma^1_{22} h_{11} \right) \]

\[ = -\sqrt{k} \Gamma^2_{ij} \frac{Du^i}{Du^1} \wedge \frac{Du^j}{Du^2}. \]

The statement remains true if \( X \in C^{2,\mu}(\Omega, \mathbb{R}^3) \). This is seen by essentially repeating the approximation argument in the proof of Lemma 2 of [26]: Let \( \{X^{(n)}\}_{n=1}^\infty \) be \( C^3(\Omega, \mathbb{R}^3) \)-mappings which approximate the radius vector \( X \) and its first and second derivatives uniformly in \( \overline{B}_R \). The regularity theory for linear equations (see [25]) yields local \( C^{1,\mu} \)-estimates for the approximating mappings \( \{u^{(n)}\}_{n=1}^\infty \), because \( K = h/g \in C^{\mu}(\Omega) \), and since the conformality relations for \( u^{(n)} \) imply that

\[
\int_B \left| Du^{(n)} \right|^2 dx \leq C \int_B Du^1 \wedge Du^2 dx
\]

\[ = C \int_{\overline{B}_R} du
\]

\[ = CR^2. \]

Hence there exists a limit mapping \( u = u(x) \), which is univalent because the inverses \( x^{(n)} = x^{(n)}(u) \) are equicontinuous in \( \overline{B}_R \) by the Courant–Lebesgue lemma.
This is true because the conformality relations for \( u^{(n)} \) also imply that

\[
\int_{B_R} \left| Dz^{(n)} \right|^2 \, du \leq C \int_{B_R} Dz^1 \wedge Dz^2 \, du
\]

\[
= C \int_B \, dx
\]

\[
= C.
\]

In order to conclude that \( u = u(x) \) is a diffeomorphism from \( B \) onto \( BR \), consider the integrability conditions for the inverses \( x^{(n)} = x^{(n)}(u) \), the elliptic system

\[
D_j \left( h^{ij(n)} D_i z^{(n)} \right) = 0,
\]

which has Hölder continuous coefficients. Then there are \( C^{1, \mu}_{loc} \)-estimates for \( \{x^{(n)}\}_{n=1}^\infty \), and the limit mapping \( x = x(u) \) is therefore of class \( C^{1, \mu}_{loc}(BR) \cap C^0(\overline{BR}) \), which is the inverse of \( u = u(x) \). This in turn implies the nonvanishing of \( Du^1 \wedge Du^2 \) and the relations (7) are therefore satisfied. \( \square \)

4. A PRIORI ESTIMATES FOR LOCALLY CONVEX SURFACES

**Lemma 5.** Let \( \Sigma \) be a locally convex surface with radius vector \( X \in C^{2, \mu}(\Omega, \mathbb{R}^3) \) for some \( \mu, \ 0 < \mu < 1 \). Suppose that

\[
|g_{ij}| \leq a,
\]

\[
g, \ K \geq \frac{1}{c}.
\]

Then the mapping \( u = u(x), \ x \in B \), from Proposition 4, satisfies the estimate

\[
\int_B |Du|^2 \, dx \leq C(a, c) \int_{\Sigma} |H| \, d\sigma.
\]  

**Proof:** The mean curvature \( H \) of \( \Sigma \) can be estimated by the conformality relations (7):

\[
|H| = \left| \frac{h_{ij}g^{ij}}{2} \right|
\]

\[
= \left| \frac{h}{2g} \frac{g_{ij}h^{ij}}{2} \right|
\]

\[
= \frac{1}{2} \sqrt{\frac{K}{g} g_{ij} \frac{Du^i \cdot Du^j}{|Du^1 \wedge Du^2|}}
\]

\[
\geq \frac{1}{2} \sqrt{\frac{K}{g} \frac{2}{2a} \frac{|Du|^2}{|Du^1 \wedge Du^2|}}
\]

\[
\geq \frac{1}{4ac} \frac{|Du|^2}{|Du^1 \wedge Du^2|}.
\]
Therefore
\[ \int_B |Du|^2 \, dx \leq 4ac \int_{B_R(u_0)} |H| \, du \]
\[ \leq 4a\sqrt{c} \int_\Omega |H| \, d\sigma. \]

PROOF OF THEOREM 1 (of the a priori estimates): Let \( \overline{B}_R = \overline{B}_R(u_0) \subset \Omega \). Consider the homeomorphism \( u = u(x) \) from \( \overline{B} \) onto \( \overline{B}_R \) from Proposition 4. Now \( u \) is of class \( C_{loc}^{1,\mu}(B) \) and its Dirichlet integral is estimated by (9). Furthermore

(10) \[ \sqrt{g} K h^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}, \]
(11) \[ \Delta_K u = 0. \]

Suppose now, without loss of generality, that \( B_R(u_0) = B = \{|u| < 1\} \) (otherwise consider the mapping \( \frac{1}{R} u(x) - u_0 \)). The Main Theorem of [25] can then be applied to the system (11) to give the following estimates in any disc \( B_\rho = \{|x| < \rho\}, \ 0 < \rho < 1 \):

(12) \[ \|u\|_{C^{1,\mu}(B_\rho)} \leq C(\ldots, \rho), \]
(13) \[ Du^1 \wedge Du^2 \geq c(\ldots, \rho) > 0. \]

By taking \( \rho = 1/2 \), and then also taking into account that we assumed that \( B_R(u_0) = B \), the relations (10) yield the bounds

\[ |h_{ij}(u_0)| \leq C(\ldots, R), \]

from which, in \( \Omega' \),

\[ |h_{ij}| \leq C(\ldots, \text{dist}(\Omega', \partial\Omega)). \]

Furthermore the functions \( h_{ij}(u(x)) \) satisfy Hölder estimates of the form

\[ [h_{ij}]_{B_\rho} \leq C(\ldots, \rho, R) \]

in each \( B_\rho = \{|x| < \rho\} \). In order to translate this into estimates for \( h_{ij}(u) \), note that the estimate (9) for the Dirichlet integral of \( u \) implies, by the Courant Lebesgue lemma, that there exists a \( \rho = \rho(a, c, \mu, R), \ 0 < \rho < 1 \), such that \( z \in B_\rho \) if \( u \in B_{R/2}(u_0) \). Since

\[ x_k = \int_0^1 Dz_k(u_0 + \tau(u - u_0)) \cdot (u - u_0) \, d\tau, \]
the estimates (12,13) yield a dilation inequality of the form

$$|x| \leq C(\ldots, R)|u - u_0|$$

if \( u \in B_{R/2}(u_0) \), and therefore

$$|h_{ij}(u) - h_{ij}(u_0)| \leq C(\ldots, R)|u - u_0|^{\mu},$$

which implies the Hölder estimates

$$[h_{ij}]^{\mu'}_{\mu} \leq C(\ldots, \text{dist}(\Omega', \partial\Omega)).$$

A priori estimates for the second derivatives of the radius vector \( X \) follow from the Gauss equations as required.

5. PROOF OF PROPOSITION 2

If \( X \in C^3(B) \), then

$$D_i(\sqrt{g}K h^{ij} D_j X) = 2\sqrt{g}K \nu.$$  

This formula is easily shown to hold in Fermi coordinates (see [11], pp.104, 106). By dotting with \( X \) and integrating over \( B_\rho \), \( 0 < \rho < 1 \), it follows that

$$2\int_{\Sigma_\rho} K \nu \cdot X \, d\sigma = -\int_{B_\rho} \sqrt{g}K g_{ij} h^{ij} du + \frac{1}{2}\int_{\partial B_\rho} \sqrt{g}K h^{ij} D_j |X|^2 n_i \, ds$$

$$= -2\int_{\Sigma_\rho} H \, d\sigma + \frac{1}{2}\int_{\partial B_\rho} \sqrt{g}K h^{ij} D_j |X|^2 n_i \, ds.$$  

This relation holds true if \( X \in C^3(B) \). By letting \( \rho \to 1 \) and incorporating the boundary condition (4), it follows that

$$\int_{\Sigma} K \nu \cdot X \, d\sigma = -\int_{\Sigma} H \, d\sigma.$$  

Finally, \( K > 0 \) implies the required estimate

$$\int_{\Sigma} |H| \, d\sigma \leq C(a, \kappa) |X|
\leq C(a, \kappa).$$
REFERENCES


[6] E. Heinz, ‘Über die Differentialungleichung $0 < \alpha \leq rt - s^2 \leq \beta < \infty$’, *Math. Z.* 72 (1959), 107–126.


[20] I. Kh. Sabitov, ‘The regularity of convex regions with a metric that is regular in the


