

NOTES ON LOCALLY COMPACT CONNECTED TOPOLOGICAL LATTICES

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It was shown in (2) that if

- (1) L is a locally compact connected topological lattice and if
- (2) L is topologically contained in R^2 , the Euclidean plane,

then each compact subset of L has an upper bound and a lower bound in L . It was also asked whether this result could be proved without assuming condition (2). In this note, we show that this result continues to hold if condition (2) is weakened to: L is finite-dimensional.

In (11), it was shown that the centre of a compact topological lattice is totally disconnected. We shall prove that this result is also true even in a locally compact, locally convex topological lattice with 0 and 1. This yields that any locally compact topological Boolean algebra is totally disconnected.

Finally, we shall give a necessary and sufficient condition for a topological lattice to admit enough continuous lattice homomorphisms into I , the closed unit interval, to distinguish points.

The terminology and notation used in this note is the same as in (1; 2; 5). It is well known that any locally compact connected topological lattice is chain-wise connected, which means that for any pair a, b with $a < b$ there exists a compact connected chain from a to b .

THEOREM 1. *If L is a locally compact connected topological lattice of finite dimension, then each compact subset of L is bounded.*

Proof. We recall from (1) that a locally compact connected topological lattice is locally convex, and from (9) that its codimension is not less than its breadth. It was also shown in (6) that if L is a locally compact and locally convex topological lattice of finite breadth, then for a neighbourhood U of a point p in L , there exist a neighbourhood V of p and a closed interval $[s, t]$ ($= s \vee (t \wedge L)$) with $s \leq t$ such that $V \subset [s, t] \subset U$. Now let us begin the proof of the theorem. Let A be a compact subset of L . For every $a \in L$, consider L as a neighbourhood of a . Choose a neighbourhood $V(a)$ of a and a closed interval $[s(a), t(a)]$ such that $V(a) \subset [s(a), t(a)]$. Clearly

$$\{V(a) \mid a \in A\}$$

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is also an open covering of A . Having a finite open sub-covering $\{V(a_i)\}$ of $\{V(a)\}$, we can easily see that A is bounded by the elements $\inf s(a_i)$ and $\sup t(a_i)$ in L .

The following corollaries are immediate.

COROLLARY 1. *If L is a locally compact and locally convex topological lattice of finite breadth, then each compact subset is bounded.*

COROLLARY 2. *If L is a locally compact connected metric topological lattice of finite dimension, then L is simply connected i.e., the fundamental group π_1 of L is trivial. (See 2, Theorem 5.)*

An element a of a lattice L is *neutral* if and only if every triple $\{a, x, y\}$ in L generates a distributive sublattice of L . An element is in the *centre* of a lattice with 0 and 1 if and only if it is neutral and complemented. It is well known that the centre of a lattice with 0 and 1 forms a Boolean lattice. It is also well known that the connected component of an element in a topological lattice L forms a sublattice of L .

THEOREM 2. *If L is a locally compact, locally convex topological lattice with 0 and 1, then the centre of L is totally disconnected.*

Proof. Let C be the centre of L and let E be the connected component of C containing an element c . Choose neighbourhoods U , V , and W of c , in L , such that U^* (where $*$ denotes topological closure) is compact, V is convex and $W \vee W \subset V \subset U^*$, $W \wedge W \subset V \subset U^*$. We now show that $W \cap E = \{c\}$. Assume that there is an element $d' \neq c$ in $W \cap E$. There are two cases to consider.

Case 1. $c \vee d' \neq c$. Let $d = c \vee d'$, thus $c < d$. Furthermore,

$$d \in W \vee W \subset V,$$

and $d \in E$, since E is a sublattice. Hence $d \in V \cap E$. Since V is convex, $[c, d] \subset V$.

Case 2. $c \vee d' = c$. In this case let $d = d'$, so that $d < c$. Then

$$d \in W \cap E \subset V \cap E,$$

and $[d, c] \subset V$. The cases are entirely analogous and thus we shall only consider the first case. Note that $[c, d]$ is a compact sublattice since it is a closed subset of U^* . Further, a little verification shows that since C is a sublattice of L , we have

$$\{c \vee (d \wedge L)\} \cap C = c \vee (d \wedge C) \supset c \vee (d \wedge E).$$

Note that $c \vee (d \wedge C)$ is a connected sub-Boolean topological lattice containing more than one point. Further, it is contained in the centre of the compact lattice $[c, d]$ ($= c \vee (d \wedge L)$). In fact, every element in $c \vee (d \wedge C)$ is relatively complemented in the closed interval $[c, d]$ and is a neutral element

of $[c, d]$ since it is neutral in L . Now we recall that the centre of a compact topological lattice is totally disconnected (11); this contradiction completes the proof.

We note that compactness implies local convexity in a topological lattice, and hence Theorem 2 yields the theorem in (11).

COROLLARY 3. *If L is a locally compact topological Boolean lattice, then L is totally disconnected.*

Proof. It suffices to show that the connected component E of zero 0 in L is $\{0\}$ itself. Suppose that E contains an element a other than 0 . Then we have $a \wedge E = [0, a]$, since if $x \in [0, a]$, then $x = x \wedge a \in x \wedge E$. Since $x \wedge E$ is connected, and contains 0 , we have $x \in E$, hence $x \in a \wedge E$. Thus $a \wedge E$ is a locally compact connected non-degenerate topological Boolean lattice in its relative topology. Therefore it is locally convex. It must be totally disconnected by Theorem 2. This is a contradiction.

A non-degenerate closed interval $K = [a, b]$ (i.e., $a \neq b$) in a topological lattice L is called I -reducible if and only if there exists at least one non-constant continuous lattice homomorphism of K into the closed unit interval I with the usual lattice operations and the usual topology. Let $\text{Hom}(L, I)$ denote the collection of continuous lattice homomorphisms of a topological lattice L into I .

LEMMA 1. *Let L be a topological lattice. Then $\text{Hom}(L, I)$ distinguishes points if and only if L is distributive and each non-degenerate closed interval in L is I -reducible.*

Proof. Suppose that $\text{Hom}(L, I)$ distinguishes points and let $[a, b]$ be a non-degenerate interval of L . Choose $\phi \in \text{Hom}(L, I)$ such that $\phi(a) \neq \phi(b)$. Then clearly the restriction of ϕ on $[a, b]$ is non-constant, and it is in $\text{Hom}([a, b], I)$. Thus the interval $[a, b]$ is I -reducible. We now show that L is distributive. Since $\text{Hom}(L, I)$ distinguishes points, the evaluation mapping: $L \rightarrow I^{\text{Hom}(L, I)}$ is a lattice monomorphism. Thus L must be distributive.

Conversely, for $a, b \in L$ with $a \neq b$, either $[a, a \vee b]$ or $[a \wedge b, a]$ is non-degenerate. Assume that $[a, a \vee b]$ is I -reducible. Let $a \vee b = c$. For a non-constant mapping $f \in \text{Hom}([a, c], I)$ we define $F: L \rightarrow I$ by

$$F(x) = f(a \vee (c \wedge x));$$

then $F \in \text{Hom}(L, I)$. Further, $F(a) \neq F(b)$ since $F(a) \neq F(c)$.

Recently Lawson (10) gave an example of a compact connected metrizable distributive topological lattice L with $\text{Hom}(L, I)$ consisting of constant mappings only, i.e., $\text{Hom}(L, I)$ does not separate points.

THEOREM 3. *If L is a locally compact connected distributive topological lattice, and if each non-degenerate closed interval in L has a finite-dimensional non-degenerate closed subinterval, then $\text{Hom}(L, I)$ distinguishes points.*

Proof. It is enough to show that any finite-dimensional non-degenerate closed interval $K = [a, b]$ in L is an I -reducible interval. We recall (3; 5) that $\text{Ca}([a, b]) \leq$ the breadth of $[a, b] \leq$ the dimension of $[a, b]$ ($= n > 0$), where $\text{Ca}([a, b])$ denotes the number of atoms of the centre of $[a, b]$. Now consider the set $\{m \mid \text{Ca}(x, y) = m \text{ for some } [x, y] \subset [a, b]\}$. Let m be the maximal positive integer of the set and let $m = \text{Ca}([x, y])$ for some $[x, y] \subset [a, b]$. Then the interval $[x, y]$ is isomorphic with a Cartesian product of m non-degenerate compact connected chains (see 7). Therefore, since $m \geq 1$, $[x, y]$ contains a non-degenerate compact connected chain, which is also a closed interval of L . It was established in (12) that for any compact connected chain C , $\text{Hom}(C, I)$ is point-separating. Thus C is, of course, I -reducible if C is non-degenerate.

From the proof of Theorem 3, the following corollary is immediate.

COROLLARY 4 (L. W. Anderson). *If L is a locally compact connected distributive topological lattice of finite breadth, then $\text{Hom}(L, I)$ separates points.*

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