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ARTICLE

A class of graphs of zero Turán density in a hypercube

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Abstract

For a graph H and a hypercube Q_n, $\mathrm{ex}(Q_n, H)$ is the largest number of edges in an H-free subgraph of Q_n. If $\lim_{n\to\infty} \mathrm{ex}(Q_n, H)/|E(Q_n)| > 0$, H is said to have a positive Turán density in a hypercube or simply a positive Turán density; otherwise, it has zero Turán density. Determining $\mathrm{ex}(Q_n, H)$ and even identifying whether H has a positive or zero Turán density remains a widely open question for general H. By relating extremal numbers in a hypercube and certain corresponding hypergraphs, Conlon found a large class of graphs, ones having so-called partite representation, that have zero Turán density. He asked whether this gives a characterisation, that is, whether a graph has zero Turán density if and only if it has partite representation. Here, we show that, as suspected by Conlon, this is not the case. We give an example of a class of graphs which have no partite representation, but on the other hand, have zero Turán density. In addition, we show that any graph whose every block has partite representation has zero Turán density in a hypercube.

Keywords: Hypercube; extremal; zero density; subgraph

2020 MSC Codes: Primary: 05C35; Secondary: 05C70, 05C75, 05D05

1. Introduction

A *hypercube* Q_n with a ground set X of size n is a graph on a vertex set $\{A : A \subseteq X\}$ and an edge set consisting of all pairs $\{A, B\}$, where $A \subseteq B$ and $|A| = |B| - 1$. Unless specified, $X = [n]$, where $[n] = \{1, \ldots, n\}$. We often identify vertices of Q_n with binary vectors that are indicator vectors of respective sets. If a graph is a subgraph of Q_n, for some n, it is called *cubical*. We denote the number of vertices and the number of edges in a graph H by $|H|$ and $||H||$, respectively. The degree of a vertex y in a graph H is denoted $d(y)$ or $d_H(y)$. A *block* in a graph is a maximal connected subgraph without a cut-vertex. Note that two distinct blocks in a graph share at most one vertex. We shall need the notion of layers. The ith *vertex layer* of Q_n, denoted V_i, is the set of vertices $\binom{[n]}{i}$, $i = 0, \ldots, n$. The ith *edge layer* L_i of Q_n is a graph induced by $V_i \cup V_{i-1}$, $i \in [n]$. A cycle of length n is denoted by C_n.

For a graph H, let the *extremal number* of H in Q_n, denoted $\mathrm{ex}(Q_n, H)$, be the largest number of edges in a subgraph G of Q_n such that there is no subgraph of G isomorphic to H. A graph H is said to have *zero Turán density in a hypercube* if $\mathrm{ex}(Q_n, H) = o(||Q_n||)$. Otherwise, we say that H has a *positive Turán density in a hypercube*. Note that by using a standard double counting argument, the sequence $\mathrm{ex}(Q_n, H)/||Q_n||$ is non-increasing; thus, the above density notions are well defined. When clear from context, we simply say Turán density instead of Turán density in a hypercube. The behaviour of the function $\mathrm{ex}(Q_n, H)$ is not well understood in general, and it is not even known what graphs have positive or zero Turán density. Currently, the only known cubical graphs of positive Turán density are those containing C_4 or C_6 as a subgraph, [8, 9], and one special graph of girth 8 [2]. Conlon [10] observed a connection between extremal numbers

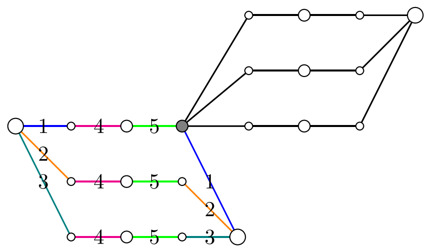

Figure 1. A graph $H(3)$ with the grey vertex being a main pole of the top theta graph and a subdivision vertex the bottom theta graph. The labelling of the edges and a corresponding colouring is a nice edge-colouring of one theta graph. Larger vertices correspond to the poles.

in a hypercube and classical extremal numbers for uniform hypergraphs. This connection proved that graphs in a large class, including some subdivisions, have zero Turán density, see [2]. For more results on extremal numbers in a hypercube, see [1, 5, 6, 16, 17].

A graph H has a *k-partite representation* $\mathcal{H}$, where $\mathcal{H}$ is k-partite k-uniform hypergraph, if for some n, H is isomorphic to a subgraph H' of the kth layer L_k of Q_n such that $V(H') \cap V_k$ is an edge set of $\mathcal{H}$. The graph H' is called a *representing graph*. If H has a k-partite representation for some k, we say that H has a *partite representation*. Here, a k-uniform hypergraph is k-partite if the vertex set can be partitioned into k parts such that each hyperedge has exactly one vertex in each part. Here, we omit the brackets and commas denoting sets, when clear from context, that is, for a set $\{1, 2\}$ we simply write 12. For example, if $H = C_8$, it has a 2-partite representation $\mathcal{H}$ with hyperedges 12, 23, 34, 14 corresponding to an C_8 with vertices 1, 12, 2, 23, 3, 34, 4, 14, 1, in order. For a k-uniform hypergraph $\mathcal{H}$, $\mathrm{ex}_k(n, \mathcal{H})$ denotes the largest number of edges in a k-uniform n-vertex hypergraph with no subgraph isomorphic to $\mathcal{H}$.

Using a theorem by Erdős [11], that states that $\mathrm{ex}_k(n, \mathcal{H}) = o_n(n^k)$ for any k-partite hypergraph $\mathcal{H}$, Conlon [10] proved that if a graph H has partite representation, then $\mathrm{ex}(Q_n, H) = o(||Q_n||)$. In the same paper, Conlon [10] asked whether cubical graphs that have no partite representation have positive Turán density. Here, we show that it is not the case, that is, having a partite representation is not a characterisation for zero Turán density in a hypercube. For that, we construct a family of cubical graphs that have no partite representation but have zero Turán density. These graphs are formed by two copies of so-called *theta-graphs* that we define below.

For a graph G, let $G(1)$ be a 1-subdivision of G, that is, a graph G' with a vertex set $V(G) \cup E(G)$ and an edge set $\{ue, ev : e = uv \in E(G)\}$. We call the vertices from $V(G)$ in G', the *poles* of G' and other vertices, the *subdivision vertices*. Let $K_{s,t}$ denote a complete bipartite graph with parts of sizes s and t, respectively. We shall be considering the 1-subdivision of $K_{q,2}$, $q \geq 3$. We shall call the two vertices of degree q in $K_{q,2}(1)$, the *main poles*. Note that $K_{q,2}(1)$ is also referred to as a *theta graph* with q legs of length 4. Here, the legs are paths with end points being main poles. We shall use a shorter notation $\Theta(q)$ for $K_{q,2}(1)$, when appropriate. If a graph G contains a subgraph isomorphic to H, we call such a subgraph a *copy of H*.

A characterisation by Havel and Moravek [14] states that a graph is cubical if and only if it has a nice edge-colouring. Here, an edge-colouring is *nice* if any cycle uses each colour an even number of times and each nontrivial path uses some colour an odd number of times.

Let $H(q)$ be a union of two copies of $\Theta(q)$ sharing exactly one vertex that is a main pole of one copy and a subdivision vertex in another copy, see Fig. 1.

Marquardt [15] showed that $H(q)$ has no partite representation for $q = 3$. Here, we give a slightly different proof of this fact for any $q \geq 3$ and show that $H(q)$ has zero Turán density. In doing so, we prove a result of an independent interest that gives a new class of cubical graphs of zero Turán density. Note that this class contains all previously known graphs of zero Turán density, as well as new such graphs.

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Theorem 1. *Let H be a graph whose every block has a partite representation. Then H has zero Turán density in a hypercube.*

Theorem 2. *For any $q \geq 3$ the graph $H(q)$ is cubical, has no partite representation, but has zero Turán density in a hypercube.*

2. Proofs of main results

We shall need the following lemmas. Lemma 3 gives a property of graphs with partite representations, formulated by Marquardt [15], we include it here for completeness. Lemma 4 is a slight variation of a result of Conlon [10], about an embedding of a graph in a layer in two different ways. Finally, Lemma 5 is the main lemma needed for Theorem 1. If a graph H has a k-partite representation with representing graph H', then the vertices of H corresponding to $V(H') \cap V_k$ are called *top vertices* with respect to this representation, and all other vertices are called *bottom vertices*. Note that if a graph has a k-partite representation, it has a $(k+1)$-partite representation that could be seen by simply adding a new element to every vertex of a representing graph.

Lemma 3. *Let H and G be connected graphs, each having a k-partite representation. Assume that $V(H) \cap V(G) = \{v\}$, where v is a top vertex for both H and G or a bottom vertex for both H and G. Then $H \cup G$ has a partite representation.*

Proof. Let H' and G' be copies of H and G that are subgraphs of the kth edge layers in hypercubes with ground sets X and Y, respectively, such that $V(H') \cap \binom{X}{k}$ and $V(H') \cap \binom{Y}{k}$ are the edge sets of k-partite hypergraphs with parts $U_1, \ldots, U_k$ and parts $W_1, \ldots, W_k$, respectively. Let u and w be respective copies of v in H' and G'. We shall specify X and Y in two cases below.

Assume first that v is a top vertex of H and of G. Let X and Y be chosen such that $X \cap Y = \emptyset$. Let F be an induced subgraph of a hypercube Q with ground set $X \cup Y$ and vertex set $\{x \cup w : x \in V(H')\} \cup \{u \cup y : y \in V(G')\}$. We see that F contains a copy F' of $H \cup G$ and is contained in the edge layer $2k$ of Q with the vertex $u \cup w$ playing a role of v. Moreover, $V(F') \cap \binom{X \cup Y}{2k}$ is an edge set of a $2k$-partite hypergraph with parts $U_1, \ldots, U_k, W_1, \ldots, W_k$.

Assume now that v is a bottom vertex of H and of G. Let X and Y be chosen such that $X \cap Y = [k-1]$ and $u = w = [k-1]$. Assume further that $i \in U_i$, $i \in W_i$, for each $i \in [k-1]$. Let F be an induced subgraph of a hypercube Q with ground set $X \cup Y$ and vertex set $V(H') \cup V(G')$. We see that F contains a copy F' of $H \cup G$ and is contained in the edge layer k of Q with vertex $v' = [k-1]$ playing a role of v. Moreover, $V(F') \cap \binom{X \cup Y}{k}$ is an edge set of a k-partite hypergraph with parts $U_1 \cup W_1, \ldots, U_k \cup W_k$. $\square$

Lemma 4. *Let Z be a connected bipartite graph with a partite representation. Fix one of the two partite set of Z arbitrarily and call its vertices odd. Then for any $\gamma > 0$ there is $n_1 = n_1(\gamma, Z)$ such that for any $n > n_1$ the following holds. Let L_j be the jth edge layer of Q_n, where $n/2 - n^{2/3} \leq j \leq n/2 + n^{2/3}$. Let $G \subseteq L_j$ be a graph such that $||G|| \geq \gamma ||L_j||$. Then there is a copy of Z with odd vertices in V_j and there is a copy of Z in G with odd vertices in V_{j-1}.*

Proof. Let Z have a k-partite representation in a hypercube Q_n with ground set $[n]$. We need the following notation. For any $x \in V(Q_n)$, let $\mathrm{Up}(x)$ be the *up set* of x, that is, $\mathrm{Up}(x) = \{y \subseteq [n] : x \subseteq y\}$. Note that $\mathrm{Up}(x)$ induces a graph isomorphic to Q_m, for $m = n - |x|$.

We need to consider four cases according to whether the odd vertices of Z correspond to k- or $(k-1)$-element sets in the representation and whether we are embedding the odd vertices in V_j or in V_{j-1}. Note, however, that we can assume that the odd vertices correspond to the k-element sets of the representation by selecting the vertices of other partite set of Z as odd vertices instead.

So, from now on, we assume first that the odd vertices of Z correspond to the k-element sets of the representation.

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We shall first a copy of Z in G with odd vertices in V_j.

Let $U_x = U_{x,k}$ be the intersection of G and the kth edge layer of the hypercube Q induced by $\mathrm{Up}(x)$, for each $x \in V_{j-k}$. Note that U_x is a subgraph of L_j, the jth layer of Q_n. We call a vertex y of U_x a *full vertex* if it has degree k in U_x, the largest possible degree. Let u_x be the number of full vertices in U_x. We shall argue that there is a vertex $x \in V_{j-k}$ such that u_x is large.

Let t be the number of k-edge stars in G with the centre in V_j. Then

$$t = \sum_{y \in V_j} \binom{d_G(y)}{k} \geq |V_j| \binom{\gamma ||L_j||/|V_j|}{k}.$$

Each such a star corresponds to a full vertex in U_x, for some x. Thus, $\sum_{x \in V_{j-k}} u_x \geq t$ and there is a vertex x such that

$$u_x \geq \frac{|V_j|}{|V_{j-k}|} \binom{\gamma ||L_j||/|V_j|}{k}.$$

Since $n/2 - n^{2/3} \leq j \leq n/2 + n^{2/3}$ and k is a fixed constant, we have that $||L_j||/|V_j| = \frac{n}{2}(1 + o(1))$ and $|V_j|/|V_{j-k}| \geq c'(k)$, so

$$u_x \geq c(k) n^k,$$

for positive constants $c(k)$ and $c'(k)$. Consider the full vertices in U_x. They correspond to u_x hyperedges in a k-uniform hypergraph with the vertex set $[n] - x$ of size $n - j + k = \frac{n}{2}(1 + o(1))$. By a theorem of Erdős [11], such a hypergraph contains any fixed k-partite k-uniform hypergraph, and thus in particular the one representing Z. Therefore, G contains a copy of Z with odd vertices in V_j.

Next, we shall find a copy of Z in G with odd vertices in V_{j-1}. To do this, we repeat the above argument by considering the vertices x in V_{j+k-1} and their downsets. Alternatively, we see that L_j corresponds to $L_{j'}$, a "symmetric" layer, where $j' = n + 1 - j$, $V_{j'}$ corresponds to V_{j-1}, and $V_{j'-1}$ corresponds to V_j. Thus, finding a copy of Z in $L_{j'}$ with odd vertices in $V_{j'}$ corresponds to finding a copy of Z in L_j with odd vertices in V_{j-1}. Since j' satisfies the same conditions as j, that is, $n/2 - n^{2/3} \leq j' \leq n/2 + n^{2/3}$, we thus could use the first part of the Lemma to obtain the second one. □

Lemma 5. *Let H be a connected bipartite graph with ℓ blocks, where every block has a partite representation. Then for any $\gamma > 0$ there is $n_0 = n_0(\gamma, H)$ such that for any $n > n_0$ the following holds. Let L_j be the jth edge layer of Q_n, where $n/2 - n^{2/3} \leq j \leq n/2 + n^{2/3}$. Let $G \subseteq L_j$ be a graph with $||G|| = \gamma ||L_j||$. Fix one of two partite sets of H arbitrarily and call its vertices odd. Then there is a copy of H in G with odd vertices in V_j and there is a copy of H in G with odd vertices in V_{j-1}.*

Proof. We shall prove the statement by induction on ℓ with the base case $\ell = 1$ directly following from Lemma 4.

If H has ℓ blocks, $\ell \geq 2$, such that each has a k-partite representation, let $H = H' \cup H''$, where H' and H'' share a single vertex v', and H'' is a block of H, that is, a leaf block. Let F be a graph that is a union of $q > |V(H')|$ copies $F_1, \ldots, F_q$ of H'' that pairwise share only a vertex corresponding to v'. Let $\gamma > 0$ and n_0 be sufficiently large (we shall specify how large later). Let $G \subseteq L_j$, $||G|| = \gamma ||L_j||$.

Assume first that v' is an odd vertex.

The idea of the proof is as follows. We shall first find many copies of H' in G for which the vertex corresponding to v' is in V_j. Then we shall find a copy F^* of F and a copy H^* of H' such that $V(F^*) \cap V(H^*) \cap V_j = \{v\}$, where v plays a role of v'. Finally, since q is large enough, we shall claim that there is a copy F_i^* of F_i for some i such that $V(F_i^*) \cap V(H^*) \cap V_{j-1} = \emptyset$. This will imply that $F_i^* \cup H^*$ is a copy of H. Next, we shall give the details of this argument:

First, we shall construct sets $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$, such that $V^{(3)} \subseteq V^{(2)} \subseteq V^{(1)} \subseteq V_j$ as follows: Assume that $n_0 > n_0(\gamma/4, H')$. Consider all copies of H' in G with a vertex playing a role of v' in V_j. Let $V^{(1)} \subseteq V_j$ be the set of all such vertices playing a role of v' in some copy of H' in G. Note that $\left|\left|G\left[\left(V_j - V^{(1)}\right) \cup V_{j-1}\right]\right|\right| < \frac{1}{4}\gamma||L_j||$, otherwise by induction we can find another copy of H' with a vertex $w \in V_j - V^{(1)}$ playing a role of v', contradicting the definition of $V^{(1)}$. Note, that for each $v \in V^{(1)}$, there could be several copies of H' with v playing a role of v'. We choose one such copy arbitrarily and denote it H'_v. Let

$$V^{(2)} = \left\{y \in V^{(1)} : d_G(y) \geq \frac{\gamma}{2}\frac{||L_j||}{|V_j|}\right\}.$$

The total number of edges of G not incident to $V^{(2)}$ is at most $\frac{1}{4}\gamma||L_j|| + \frac{1}{2}||G|| = \frac{3}{4}\gamma||L_j||$. Thus, $\left|\left|G\left[V^{(2)} \cup V_{j-1}\right]\right|\right| \geq \frac{\gamma}{4}||L_j||$. Since all vertices from V_j in L_j have the same degree,

$$|V^{(2)}| \geq \frac{\gamma}{4}|V_j|.$$

Let for each $v \in V^{(2)}$, $A_v \subseteq V_j$ and $B_v \subseteq V_{j-1}$ be sets of vertices such that $V(H'_v) = A_v \cup B_v \cup \{v\}$, $v \notin A_v$. Randomly colour each vertex in $V^{(2)}$ with red or blue independently with equal probability and colour the vertices in $V_j - V^{(2)}$ blue. We say that a vertex $v \in V^{(2)}$ is *good* if v is red and each vertex in A_v is blue. Then, the expected number of good v's is at least $|V^{(2)}|2^{-t}$, where $|A_v| = t - 1$. Thus, there is a set $V^{(3)} \subseteq V^{(2)}$, corresponding to a set of good v's, with $|V^{(3)}| \geq |V^{(2)}|2^{-t}$, such that for each $v \in V^{(3)}$, $A_v \cap V^{(3)} = \emptyset$.

We see that

$$\left|\left|G\left[V^{(3)} \cup V_{j-1}\right]\right|\right| \geq \frac{\gamma}{2}\frac{||L_j||}{|V_j|}|V^{(3)}| \geq \frac{\gamma}{2}\frac{||L_j||}{|V_j|}\frac{|V^{(2)}|}{2^t} \geq \gamma^2 2^{-t-3}||L_j||.$$

Consider the graph F defined above. By Lemma 3, F has a partite representation. Then by Lemma 4, there is a copy F^* of F in $G\left[V^{(3)} \cup V_{j-1}\right]$ with a vertex $v \in V^{(3)}$ corresponding to v'. Here, we assume that $n_0 > n_1\left(\gamma^2 2^{-t-3}, F\right)$. Let F_i^* be a respective copy of F_i, for each $i \in [q]$. By construction of $V^{(3)}$, there is a copy H^* of H' in G with all vertices except for v not in $V^{(3)}$. Let B_v be the set of vertices of H^* in V_{j-1}. We see that $V(F^*) \cap V(H^*) \cap V_j = \{v\}$. Since $q > |V(H')| > |B_v|$, there is $i \in [q]$, such that $V(F_i^*) \cap B_v \cap V_{j-1} = \emptyset$. Then $H^* \cup F_i^*$ is a copy of H in G.

The case when v' is not an odd vertex is treated similarly by first finding many copies of H' with a vertex corresponding to v' in V_{j-1}. □

Proof of Theorem 1. Let G' be a subgraph of Q_n such that $||G'|| = 2\gamma||Q_n||$, for some constant $\gamma > 0$ and sufficiently large n. By a standard argument, see for example Lemma 1 [4],

$$\sum_{i:|i-n/2|>n^{2/3}} \binom{n}{i} = o(2^n).$$

Since the degree of each vertex in Q_n is n, the total number of edges in G', incident to vertices in V_i's, for $i < n/2 - n^{2/3}$ or $i > n/2 + n^{2/3}$ is $o(n2^n) = o(||Q_n||) = o(||G||)$. Then there is $j \in \left\{n/2 - n^{2/3}, n/2 + n^{2/3}\right\}$ such that L_j contains at least $\gamma||L_i||$ edges of G'. Lemma 5 applied to $G = G' \cap L_j$ concludes the proof. □

Proof of Theorem 2. Let $H = H(q)$ be a union of two copies Θ_1 and Θ_2 of $\Theta(q)$ sharing exactly one vertex that is a main pole of Θ_2 and a subdivision vertex of Θ_1.

First, we need to check that $H(q)$ is cubical. Marquardt [15] gave an explicit embedding of $H(3)$ in a hypercube. We show that $H(q)$ is cubical by constructing its nice colouring by taking nice colourings of Θ_1 and Θ_2 using disjoint sets of colours. These colourings were given in [14] and we show one in Fig. 1.

Next, we shall show that $H(q)$ has no partite representation. The Hamming distance between two binary vectors u and v, denoted $d_H(u, v)$ is the number of positions where the vectors differ. In [2], it is shown that if $q \geq 3$ and v, v' are the main poles of a copy of $\Theta(q)$ embedded in a layer of a hypercube, then $d_H(v, v') = 2$. Let the main poles of Θ_1 and Θ_2 be denoted v_i, v'_i, respectively, $i = 1, 2$. Assume that H has a k-partite representation for some k, that is, $V(H) \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, for some n and $V(H) \cap \binom{[n]}{k}$ corresponds to the edges of a k-partite hypergraph, denote it by $\mathcal{H}$. We have that $d_H(v_i, v'_i) = 2$, $i = 1, 2$. Since a main pole of Θ_1 and a main pole of Θ_2 are adjacent, the main poles of Θ_1 are in one vertex layer and the main poles of Θ_2 are in another vertex layer, without loss of generality, $v_1, v'_1 \in \binom{[n]}{k}$ and $v_2, v'_2 \in \binom{[n]}{k-1}$. Assume further that v_1 and v'_1 equal to 10 and 01 in the first two positions of their binary representation and coincide on all other positions. Consider neighbours of v_1 in Θ_1. At least one of these neighbours, say w differs from v_1 in a position that is not one of the first two ones, say in the third position. Then, restricted to the first three position v_1 is 101, w is 100, and v'_1 is 011. Thus, the remaining two vertices on the v_1, v'_1 path, that contains w, must be equal to 110 and 010 in these three position. All vertices of this leg are the same in positions $4, \ldots, n$, say they are equal to 1 on a set of positions $A \subseteq \{4, \ldots, n\}$, $|A| = k - 2$. Thus, we have that $\mathcal{H}'$ contains hyperedges $\{1, 2\} \cup A$, $\{2, 3\} \cup A$, and $\{1, 3\} \cup A$. If $\mathcal{H}$ were to be k-partite, 1, 2, 3 and each element of A would belong to distinct parts, that is, to $k + 1$ parts, a contradiction. This shows that $H(q)$ has no partite representation.

Finally, we shall show that $H(q)$ has zero Turán density in a hypercube. Using Theorem 1 it is sufficient to show that $\Theta(q)$ has partite representation. Note that $\Theta(q) = K_{2,q}(1)$, a subdivision of $K_{2,q}$. We claim that $\mathcal{H} = K_{2,q}$ gives a 2-partite representation of $\Theta(q)$. Indeed, let $n = q + 2$ and let $\mathcal{H}$ be a hypergraph on a vertex set $[n]$ with the edge set $\{ij : i \in [2], j \in [n] \setminus [2]\}$. So, $\mathcal{H}$ is a bipartite graph, that is, 2-partite 2-uniform hypergraph. Let the edges of $\mathcal{H}$ correspond to subdivision vertices of $K_{2,q}(1)$. Let vertices v and v' correspond to the main poles of $\Theta(q)$ and vertices $1, \ldots, q$ correspond to other vertices. Then, the two parts of $\mathcal{H}$ are $\{v, v'\}$ and $\{1, \ldots, q\}$. This concludes the proof of Theorem 2. □

Conclusions

We showed that there are cubical graphs that have no partite representation and have zero Turán density in a hypercube. On the other hand, we proved that any graph whose every block has a partite representation has zero Turán density in a hypercube. This leads to a followup question:

Open question

Is it true that each cubical graph that is 2-connected and has zero Turán density in a hypercube has a partite representation?

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