Proceedings of the Edinburgh Mathematical Society (2007) **50**, 197–215 ©
DOI:10.1017/S0013091503000326 Printed in the United Kingdom

SPECTRAL SELF-AFFINE MEASURES IN $\mathbb{R}^N$

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(Received 24 April 2003)

Abstract The aim of this paper is to investigate and study the possible spectral pair $(\mu_{M,D}, \Lambda(M,S))$ associated with the iterated function systems $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$ and $\{\psi_s(x) = M^*x + s\}_{s\in S}$ in $\mathbb{R}^n$. For a large class of self-affine measures $\mu_{M,D}$, we obtain an easy check condition for $\Lambda(M,S)$ not to be a spectrum, and answer a question of whether we have such a spectral pair $(\mu_{M,D}, \Lambda(M,S))$ in the Eiffel Tower or three-dimensional Sierpinski gasket. Further generalization of the given condition as well as some elementary properties of compatible pairs and spectral pairs are discussed. Finally, we give several interesting examples to illustrate the spectral pair conditions considered here.

Keywords: spectral measure; iterated function system (IFS); compatible pair; transfer operator

2000 *Mathematics subject classification:* Primary 28A80
Secondary 42C05; 46E30

1. Introduction

Let μ be a probability measure of compact support on $\mathbb{R}^n$. We call μ a *spectral measure* if there exists a set $\Lambda \subset \mathbb{R}^n$ such that the set of complex exponentials $\{e(\lambda \cdot x) : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$, where $e(\lambda \cdot x) = \mathrm{e}^{2\pi \mathrm{i}\lambda\cdot x}$. The set Λ is then called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. A spectral measure often has more than one spectrum (not translates of each other; see [**10**, Example 2.9 (a)]). It is known that

(i) $\{e(\lambda \cdot x) : \lambda \in \Lambda\}$ is orthonormal in $L^2(\mu)$ if and only if

$$\hat{\mu}(\lambda_1 - \lambda_2) = 0, \quad \forall \lambda_1, \lambda_2 \in \Lambda,\ \lambda_1 \neq \lambda_2; \tag{1.1}$$

(ii) $\{e(\lambda \cdot x) : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$ if and only if

$$\sum_{\lambda\in\Lambda} |\hat{\mu}(\xi - \lambda)|^2 = 1, \quad \forall \xi \in \mathbb{R}^n. \tag{1.2}$$

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We shall study the spectral self-affine measure μ associated with iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$ and its dual IFS $\{\psi_s(x) = M^*x + s\}_{s\in S}$, where $M \in M_n(\mathbb{Z})$ is an expanding integer matrix (i.e. all entries are integers and all the eigenvalues of M have moduli greater than 1), D and S are finite subsets of $\mathbb{Z}^n$ of the same cardinality, $|D| = |S|$. Our self-affine measure, which is denoted by $\mu_{M,D}$, is the unique probability measure μ satisfying the self-affine identity

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1} \tag{1.3}$$

and is supported on $T := T(M,D)$, where $T(M,D)$ is the attractor (or invariant set) of the IFS $\{\phi_d\}_{d\in D}$. Such a $T(M,D)$ is the unique non-empty compact set satisfying

$$T = \bigcup_{d\in D}\phi_d(T) \tag{1.4}$$

and is given by

$$T(M,D) := \biggl\{\sum_{j=1}^{\infty} M^{-j}d_j : d_j \in D\biggr\}. \tag{1.5}$$

Corresponding to the dual IFS $\{\psi_s\}_{s\in S}$, we use $\Lambda(M,S)$ to denote the expansive orbit of 0 under $\{\psi_s\}$, that is

$$\Lambda(M,S) := \biggl\{\sum_{j=0}^{k-1} M^{*j}s_j : k \geqslant 1 \text{ and } s_j \in S\biggr\}. \tag{1.6}$$

We shall focus our attention on the following question: under what conditions is $\Lambda(M,S)$ a spectrum for $\mu_{M,D}$?

It is known that certain self-affine measures $\mu_{M,D}$ in $\mathbb{R}^n$ have an orthonormal basis for $L^2(\mu_{M,D})$ consisting of complex exponentials. This was first observed by Jorgensen and Pedersen [**5**] and studied further by Strichartz [**9**,**10**]. More recently, Łaba and Wang [**7**] have established a large class of spectral Cantor measures associated with IFS, and given a necessary and sufficient condition on the spectrum of such a measure in dimension 1, which extends the studies by Jorgensen and Pedersen [**5**] and Strichartz [**10**]. The studies in [**5**,**7**,**9**,**10**] also leave several open problems which have motivated the present research.

The aim of this paper is to investigate and study the possible spectral pair $(\mu_{M,D}, \Lambda(M,S))$ in $\mathbb{R}^n$. We first present some elementary properties of a compatible pair. We then extend the result of [**7**], and obtain an easy check condition for $(\mu_{M,D}, \Lambda(M,S))$ not to be a spectral pair. Using this condition, we answer a question considered in [**4**, **5**, **9**, **10**] by showing that, in the Eiffel Tower or three-dimensional Sierpinski gasket, the corresponding $(\mu_{M,D}, \Lambda(M,S))$ is not a spectral pair. Further generalization of the given condition is also discussed. Since the known examples of the spectral pair $(\mu_{M,D}, \Lambda(M,S))$ considered in the previous papers require $M = \operatorname{diag}[r, r, \ldots, r]$, we construct a spectral pair in which M is not of this form, and show that the spectral pair is invariant under the $\mathbb{Z}$-similarity of matrix M. We also give several examples in the final section to illustrate the spectral pair conditions considered here.

2. Compatible pairs

The concept of compatible pairs, following the terminology of [**10**], plays an important role in the study of spectral measure. In this section we present certain of their elementary properties.

Let B and S be finite subsets of $\mathbb{R}^n$ of the same cardinality q. We say (B, S) is a *compatible pair* if the $q \times q$ matrix

$$H_{B,S} := [q^{-1/2} e(b \cdot s)]_{b \in B,\, s \in S} \tag{2.1}$$

is unitary, i.e. $H_{B,S} H^*_{B,S} = I_q$. Note that we use '$*$' to denote the conjugated transpose.

It follows from this definition that if (B, S) is a compatible pair, then so is (S, B), and vice versa. Furthermore, we can translate either B or S and obtain another compatible pair. Thus, we may assume without essential loss of generality that 0 belongs to both B and S. Also note that, if B is given, there may not be any S such that $H_{B,S}$ is unitary. Take, for example, $B = \{0, 1, 3\}$. We see that no subset $S \subset \mathbb{R}$ can be found such that (B, S) is a compatible pair. This is because the equation $z^3 + z + 1 = 0$ has no roots on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

For each finite subset $A \subset \mathbb{R}^n$ of the cardinality $|A|$, we define its *symbol* by

$$m_A(t) := \frac{1}{|A|} \sum_{a \in A} \mathrm{e}^{-2\pi \mathrm{i} a \cdot t}.$$

Let

$$\delta_A(x) := \frac{1}{|A|} \sum_{a \in A} \delta(x - a),$$

where $\delta(x - a)$ denotes a Dirac delta function (point measure) at a. Then

$$\hat{\delta}_A(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i} x \cdot \xi} \, \mathrm{d}\delta_A(x) = \frac{1}{|A|} \sum_{a \in A} \int \mathrm{e}^{-2\pi \mathrm{i} x \cdot \xi} \, \mathrm{d}\delta(x - a) = m_A(\xi). \tag{2.2}$$

Proposition 2.1. *Let* $B, S \subset \mathbb{R}^n$ *be finite sets of the same cardinality. Then the following statements are equivalent:*

(i) (B, S) *is a compatible pair;*

(ii) $(RB, R^{*-1} S)$ *is a compatible pair for any non-singular matrix* $R \in M_n(\mathbb{R})$*;*

(iii) $m_B(s_1 - s_2) = 0$*, for any distinct* $s_1, s_2 \in S$*;*

(iv) $\sum_{s \in S} |m_B(\xi + s)|^2 = 1$*, for all* $\xi \in \mathbb{R}^n$*;*

(v) (δ_B, S) *is a spectral pair.*

The proof of Proposition 2.1 can be found, explicitly or implicitly, in [**5**, **7**, **10**].

Proposition 2.2. *Let* $D, S \subset \mathbb{Z}^n$ *and* $M \in M_n(\mathbb{Z})$ *with* $|\det(M)| > 1$ *such that* $(M^{-1} D, S)$ *is a compatible pair. Then the following statements hold.*

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(i) *The elements in D are distinct modulo M (i.e. $d_i - d_j \notin M\mathbb{Z}^n$ for distinct $d_i, d_j \in D$), and the elements in S are distinct modulo M^*.*

(ii) *Suppose that $\hat{D}, \hat{S} \subset \mathbb{Z}^n$ such that $\hat{D} \equiv D(\text{mod}\, M)$ and $\hat{S} \equiv S(\text{mod}\, M^*)$. Then $(M^{-1}\hat{D}, \hat{S})$ is a compatible pair.*

(iii) *Define $D_k = D + MD + \cdots + M^{k-1}D$ and $S_k = S + M^*S + \cdots + M^{*(k-1)}S$. Then $(M^{-k}D_k, S_k)$ is a compatible pair.*

Proof. If there exist distinct i and j such that $d_i - d_j \in M\mathbb{Z}^n$, then $d_i = d_j + Mz_{ij}$ for $z_{ij} \in \mathbb{Z}^n$. Therefore,

$$e(M^{-1}d_i \cdot s_k) = e(M^{-1}d_j \cdot s_k)e(z_{ij} \cdot s_k) = e(M^{-1}d_j \cdot s_k) \tag{2.3}$$

holds for $k = 1, 2, \ldots, q$. This shows that the matrix $H_{M^{-1}D,S}$ has two identical rows i and j, so it cannot be unitary: a contradiction. Hence, the elements in D are distinct modulo M. Similarly, $H_{M^{-1}D,S}$ will have two identical columns if the elements in S are not distinct modulo M^*; again, this is a contradiction. This proves (i).

Let $\hat{D} = \{\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_q\} \equiv D(\text{mod}\, M)$ and $\hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_q\} \equiv S(\text{mod}\, M^*)$. Then $\hat{d}_i = d_i + Mz_i$ and $\hat{s}_j = s_j + M^*\tilde{z}_j$ for $z_i, \tilde{z}_j \in \mathbb{Z}^n$, $1 \leqslant i, j \leqslant q$. Since

$$\begin{aligned} e(M^{-1}\hat{d}_i \cdot \hat{s}_j) &= e(M^{-1}d_i \cdot s_j)e(M^{-1}d_i \cdot M^*\tilde{z}_j)e(z_i \cdot s_j)e(z_i \cdot M^*\tilde{z}_j) \\ &= e(M^{-1}d_i \cdot s_j), \end{aligned} \tag{2.4}$$

the conclusion (ii) follows immediately from the definition.

Finally, (iii) is a special case of [**10**, Lemma 2.5]. □

Note that if we let $M \in M_n(\mathbb{Z})$ with $|\det(M)| = m > 1$ and $B = \{b_1, b_2, \ldots, b_m\}$ be a complete residue system $(\text{mod}\, M)$, then $\{b_j + M\mathbb{Z}^n\}_{j=1}^m$ constitutes a partition of $\mathbb{Z}^n$. Under the assumptions of Proposition 2.2, we also have $|D| = |S| \leqslant |\det(M)|$.

The following proposition may be considered as a generalization of the Gaussian compatible pair (see [**10**, Example 2.4 (a)]).

Proposition 2.3. *Let $M \in M_n(\mathbb{Z})$ with $|\det(M)| = m > 1$. Assume that D is a complete residue system $(\text{mod}\, M)$. If S is a complete residue system $(\text{mod}\, M^*)$, then $(M^{-1}D, S)$ is a compatible pair.*

Proof. Let $D = \{d_1, d_2, \ldots, d_m\}$ and $S = \{s_1, s_2, \ldots, s_m\}$. Then (see [**3**, Lemma 5.1])

$$\frac{1}{m}\sum_{j=1}^{m} e(M^{-1}d_j \cdot k) = \begin{cases} 1 & \text{if } k \in M^*\mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In view of the fact that $s_p - s_q \in M^*\mathbb{Z}^n$ if and only if $p = q$, it follows from (2.5) that, for any distinct $s_p, s_q \in S$,

$$m_{M^{-1}D}(s_p - s_q) = \frac{1}{m}\sum_{j=1}^{m} e(M^{-1}d_j \cdot (s_p - s_q)) = 0. \tag{2.6}$$

Therefore, the desired result follows from Proposition 2.1. □

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Note that, in Proposition 2.3, one possible choice of D and S is given by the formula

$$D := \mathbb{Z}^n \cap M[0,1)^n \quad \text{and} \quad S := \mathbb{Z}^n \cap M^*[0,1)^n.$$

3. Conditions for spectral pairs

Let $M \in M_n(\mathbb{Z})$ be expanding, and let D and S be finite subsets of $\mathbb{Z}^n$. Jorgensen and Pedersen [**5**] proved that if $(M^{-1}D, S)$ is a compatible pair, then the following conclusions hold.

(i) $\{e(\lambda \cdot x) : \lambda \in \Lambda(M,S)\}$ is orthogonal in $L^2(\mu_{M,D})$.

(ii) Let $Q(\xi) := \sum_{\lambda \in \Lambda(M,S)} |\hat{\mu}_{M,D}(\xi + \lambda)|^2$. Then $Q(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^n$ (the Bessel inequality).

(iii) $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair $\iff$ $Q(\xi) = 1$, for all $\xi \in \mathbb{R}^n$ (the Parseval identity).

(iv) The function $Q(\xi)$ has an entire analytic extension to $\mathbb{C}^n$ which is of linear exponential growth in the imaginary direction. Furthermore, it satisfies the functional identity

$$Q(\xi) = \sum_{s \in S} |m_D(M^{*-1}(\xi + s))|^2 Q(M^{*-1}(\xi + s)), \quad \xi \in \mathbb{R}^n. \tag{3.1}$$

The right-hand side of (3.1) is known as the Ruelle transfer operator C defined on function Q. Equation (3.1) is equivalent to $C(Q)(\xi) = Q(\xi)$. From Proposition 2.1 (iv), $C(1) = 1$. The method of [**5**] is based on the identification of a function space which contains $Q(\xi) - 1$, then showing that C is strictly contractive and, as a consequence, that $Q(\xi) \equiv 1$, when the axioms hold.

Note that, from the self-affine identity (1.3),

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi).$$

The above conclusion (i) implies (1.1), i.e. $\Lambda(M,S) \subseteq \Lambda(M,S) - \Lambda(M,S) \subseteq Z(\hat{\mu}_{M,D}) \cup \{0\}$ (recall that $0 \in S$ is assumed), where $Z(\hat{\mu}_{M,D})$ is the zero set of $\hat{\mu}_{M,D}$. Strichartz [**9**] provided a different way to verify $Q(\xi) \equiv 1$ by using an approach that is reminiscent of the Cohen criterion in wavelet theory. In fact, Strichartz [**9**] proved the following.

Theorem 3.1. *Let $M \in M_n(\mathbb{Z})$ be expanding, D and S be finite subsets of $\mathbb{Z}^n$ such that $(M^{-1}D, S)$ is a compatible pair. Suppose that the zero set $Z(m_{M^{-1}D}(t))$ is disjoint from the set $T(M^*, S)$. Then $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair.*

Furthermore, Strichartz [**10**] extended the construction of a spectral pair to a larger class of measures and spectra. Łaba and Wang [**7**] studied spectral Cantor measures in $\mathbb{R}$. They found that a compatible pair automatically yields a spectral measure [**7**, Theorem 1.2], and give the following necessary and sufficient condition for $\Lambda(N,S)$ to be a spectrum [**7**, Theorem 1.3].

Theorem 3.2. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Let $S \subset \mathbb{Z}$ with $0 \in S$ such that $(N^{-1}D, S)$ is a compatible pair. Then $(\mu_{N,D}, \Lambda(N, S))$ is not a spectral pair if and only if there exist $s_j^* \in S$ and non-zero integers η_j, $0 \leqslant j \leqslant m-1$, such that $\eta_{j+1} = N^{-1}(\eta_j + s_j^*)$ for all $0 \leqslant j \leqslant m-1$ (with $\eta_m := \eta_0$ and $s_m^* := s_0^*$).*

The study in [**7**] also leaves several questions unanswered. For example, does Theorem 3.2 or something similar hold in higher dimensions? In the following, we modify the techniques in [**7**] to extend and simplify the condition of Theorem 3.2.

Theorem 3.3. *Let $D, S \subset \mathbb{Z}^n$ be finite subsets of the same cardinality. Suppose that $M \in M_n(\mathbb{Z})$ is expanding and that $(M^{-1}D, S)$ is a compatible pair. If there exists a non-zero $\tilde{s} \in \hat{S} \equiv S (\operatorname{mod} M^*)$ such that*

$$(M^* - I)^{-1}\tilde{s} \in \mathbb{Z}^n, \tag{3.2}$$

then for any $\hat{D} \equiv D(\operatorname{mod} M)$, $(\mu_{M,\hat{D}}, \Lambda(M, \hat{S}))$ is not a spectral pair. In particular, $\Lambda(M, \hat{S})$ is not a spectrum for $\mu_{M,D}$.

Proof. Let $\eta = (M^* - I)^{-1}\tilde{s} \in \mathbb{Z}^n$. Then $\eta = M^{*-1}(\eta + \tilde{s})$ and

$$M^{*l}\eta = \eta + \sum_{j=0}^{l-1} M^{*j}\tilde{s}, \quad \forall l \in \mathbb{N}. \tag{3.3}$$

From Propositions 2.1 (iv) and 2.2 (ii), for any $\hat{D} \equiv D(\operatorname{mod} M)$,

$$\sum_{\hat{s} \in \hat{S}} |m_{\hat{D}}(M^{*-1}(\xi + \hat{s}))|^2 = 1, \quad \forall \xi \in \mathbb{R}^n. \tag{3.4}$$

In view of the fact that $m_{\hat{D}}(M^{*-1}(\eta + \tilde{s})) = m_{\hat{D}}(\eta) = 1$, taking $\xi = \eta$ in (3.4), we have, for $\hat{s} \in \hat{S}$,

$$m_{\hat{D}}(M^{*-1}(\eta + \hat{s})) = \begin{cases} 1 & \text{if } \hat{s} = \tilde{s}, \\ 0 & \text{if } \hat{s} \neq \tilde{s}. \end{cases} \tag{3.5}$$

For any $\lambda = \hat{s}_0 + M^*\hat{s}_1 + \cdots + M^{*k}\hat{s}_k \in \Lambda(M, \hat{S})$, it follows from (3.5) that

$$\begin{aligned} m_{\hat{D}}(M^{*-1}(\eta + \lambda)) &= m_{\hat{D}}(M^{*-1}(\eta + \hat{s}_0) + \hat{s}_1 + M^*\hat{s}_2 + \cdots + M^{*(k-1)}\hat{s}_k) \\ &= m_{\hat{D}}(M^{*-1}(\eta + \hat{s}_0)) \\ &= \begin{cases} 1 & \text{if } \hat{s}_0 = \tilde{s}, \\ 0 & \text{if } \hat{s}_0 \neq \tilde{s}. \end{cases} \end{aligned} \tag{3.6}$$

If $\hat{s}_0 \neq \tilde{s}$, then, from (3.6),

$$\hat{\mu}_{M,\hat{D}}(\eta + \lambda) = \prod_{j=1}^{\infty} m_{\hat{D}}(M^{*-j}(\eta + \lambda)) = 0. \tag{3.7}$$

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In the case when $\hat{s}_0 = \tilde{s}$, i.e. $\lambda = \tilde{s} + M^*\hat{s}_1 + \cdots + M^{*k}\hat{s}_k$, we consider the second factor of $\hat{\mu}_{M,\hat{D}}(\eta + \lambda)$, and use (3.3) and (3.5) to conclude that

$$\begin{aligned} m_{\hat{D}}(M^{*-2}(\eta+\lambda)) &= m_{\hat{D}}(M^{*-2}(\eta + \tilde{s} + M^*\hat{s}_1) + \hat{s}_2 + M^*\hat{s}_3 + \cdots + M^{*(k-2)}\hat{s}_k) \\ &= m_{\hat{D}}(M^{*-1}(\eta + \hat{s}_1)) \\ &= \begin{cases} 1 & \text{if } \hat{s}_1 = \tilde{s}, \\ 0 & \text{if } \hat{s}_1 \neq \tilde{s}. \end{cases} \end{aligned} \tag{3.8}$$

If $\hat{s}_1 \neq \tilde{s}$, then, from (3.8), $\hat{\mu}_{M,\hat{D}}(\eta + \lambda) = 0$. When $\hat{s}_1 = \tilde{s}$, i.e.

$$\lambda = \tilde{s} + M^*\tilde{s} + M^{*2}\hat{s}_2 + \cdots + M^{*k}\hat{s}_k,$$

by the same argument, we consider the third factor of $\hat{\mu}_{M,\hat{D}}(\eta + \lambda)$, and use (3.3) and (3.5) to obtain $\hat{\mu}_{M,\hat{D}}(\eta + \lambda) = 0$ if $\hat{s}_2 \neq \tilde{s}$. After k steps, we see that if one of $\hat{s}_j$ in $\{\hat{s}_0, \hat{s}_1, \dots, \hat{s}_k\}$ is not equal to $\tilde{s}$, then $\hat{\mu}_{M,\hat{D}}(\eta + \lambda) = 0$.

Suppose that $\lambda = \tilde{s} + M^*\tilde{s} + M^{*2}\tilde{s} + \cdots + M^{*k}\tilde{s}$. We consider the $(k+2)$th factor of $\hat{\mu}_{M,\hat{D}}(\eta + \lambda)$ and conclude that

$$\begin{aligned} m_{\hat{D}}(M^{*-(k+2)}(\eta+\lambda)) &= m_{\hat{D}}\left(M^{*-(k+2)}\left(\eta + \sum_{j=0}^{k} M^{*j}\tilde{s}\right)\right) \\ &= m_{\hat{D}}(M^{*-1}(\eta + 0)) \\ &= \begin{cases} 1 & \text{if } 0 = \tilde{s}, \\ 0 & \text{if } 0 \neq \tilde{s}, \end{cases} \end{aligned} \tag{3.9}$$

which gives the desired result that $\hat{\mu}_{M,\hat{D}}(\eta + \lambda) = 0$ for any $\lambda \in \Lambda(M, \hat{S})$. Since

$$Q_1(\eta) := \sum_{\lambda \in \Lambda(M,\hat{S})} |\hat{\mu}_{M,\hat{D}}(\eta + \lambda)|^2 = 0,$$

we know from the Parseval identity that $(\mu_{M,\hat{D}}, \Lambda(M, \hat{S}))$ is not a spectral pair. In the special case when $\hat{D} = D$, we also find that $\Lambda(M, \hat{S})$ is not a spectrum for $\mu_{M,D}$. This completes the proof of Theorem 3.3. □

From Theorem 3.3, we can further prove the following more general result.

Theorem 3.4. *Let $D, S \subset \mathbb{Z}^n$ be finite subsets of the same cardinality. Suppose that $M \in M_n(\mathbb{Z})$ is expanding and $(M^{-1}D, S)$ is a compatible pair. If there exist $\tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{p-1} \in \hat{S} \equiv S (\operatorname{mod} M^*)$ for some $p \geqslant 1$ such that*

$$(M^{*p} - I)^{-1}\{\tilde{s}_0 + M^*\tilde{s}_1 + \cdots + M^{*(p-1)}\tilde{s}_{p-1}\} \in \mathbb{Z}^n \setminus \{0\}, \tag{3.10}$$

then, for any $\hat{D} \equiv D (\operatorname{mod} M)$, $(\mu_{M,\hat{D}}, \Lambda(M, \hat{S}))$ is not a spectral pair.

Proof. Let $\hat{D}_p := \hat{D} + M\hat{D} + \cdots + M^{p-1}\hat{D}$ and $\hat{S}_p := \hat{S} + M^*\hat{S} + \cdots + M^{*(p-1)}\hat{S}$. Then, from Proposition 2.2 (ii) and (iii), $(M^{-p}\hat{D}_p, \hat{S}_p)$ is a compatible pair. Also, the condition (3.10) is equivalent to

$$(M^{*p} - I)^{-1}\tilde{s} \in \mathbb{Z}^n \setminus \{0\},$$

where $\tilde{s} = \tilde{s}_0 + M^*\tilde{s}_1 + \cdots + M^{*(p-1)}\tilde{s}_{p-1} \in \hat{S}_p$. Applying Theorem 3.3 to M^p, $\hat{D}_p$ and $\hat{S}_p$ instead of M, $\hat{D}$ and $\hat{S}$, we find that $(\mu_{M^p,\hat{D}_p}, \Lambda(M^p, \hat{S}_p))$ is not a spectral pair. In view of the fact that

$$\hat{\mu}_{M^p,\hat{D}_p} = \hat{\mu}_{M,\hat{D}} \quad \text{and} \quad \Lambda(M^p, \hat{S}_p) = \Lambda(M, \hat{S}), \tag{3.11}$$

we thus deduce that $(\mu_{M,\hat{D}}, \Lambda(M, \hat{S}))$ is not a spectral pair. □

It should be pointed out that $(M^{*p} - I)^{-1}\hat{S}_p$ containing a non-zero integer point for some $p \in \mathbb{N}$ is equivalent to the condition (3.10). For any $x \in (M^{*p} - I)^{-1}\hat{S}_p$, we know that there exist $\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_{p-1} \in \hat{S}$ such that

$$\begin{aligned} x &= M^{*-p}(x + \hat{s}_0 + M^*\hat{s}_1 + \cdots + M^{*(p-1)}\hat{s}_{p-1}) \\ &= f_{\hat{s}_{p-1}} \circ f_{\hat{s}_{p-2}} \circ \cdots \circ f_{\hat{s}_1} \circ f_{\hat{s}_0}(x), \end{aligned}$$

i.e. x is a fixed point of finite composition $f_{\hat{s}_{p-1}} \circ f_{\hat{s}_{p-2}} \circ \cdots \circ f_{\hat{s}_1} \circ f_{\hat{s}_0}$ of members of the IFS $\{f_{\hat{s}}(x) = M^{*-1}(x + \hat{s})\}_{\hat{s} \in \hat{S}}$. Hence, $(M^{*p} - I)^{-1}\hat{S}_p \subseteq T(M^*, \hat{S})$. Since $T(M^*, \hat{S})$ is compact, in order to find out whether or not (3.10) holds we need only to check a finite number of integers in $T(M^*, \hat{S})$ having the form (3.10).

At the end of this section, we consider the following two special cases: the dimensions $n = 1$ and the case $|D| = |S| = |\det(M)|$.

Case 1 ($n = 1$). From the sufficient condition of Theorem 3.2,

$$\left.\begin{aligned} N\eta_1 &= \eta_0 + s_0^*, \\ N^2\eta_2 &= N\eta_1 + Ns_1^* = \eta_0 + s_0^* + Ns_1^*, \\ &\ \vdots \\ N^m\eta_m &= \eta_0 + s_0^* + Ns_1^* + \cdots + N^{m-1}s_{m-1}^*. \end{aligned}\right\} \tag{3.12}$$

Since $\eta_m = \eta_0$, the above condition (3.12) is equivalent to

$$\left.\begin{aligned} \eta_0 &= (N^m - 1)^{-1}\{s_0^* + Ns_1^* + \cdots + N^{m-2}s_{m-2}^* + N^{m-1}s_{m-1}^*\}, \\ \eta_1 &= (N^m - 1)^{-1}\{s_1^* + Ns_2^* + \cdots + N^{m-2}s_{m-1}^* + N^{m-1}s_0^*\}, \\ &\ \vdots \\ \eta_{m-1} &= (N^m - 1)^{-1}\{s_{m-1}^* + Ns_0^* + \cdots + N^{m-2}s_{m-3}^* + N^{m-1}s_{m-2}^*\}, \end{aligned}\right\} \tag{3.13}$$

where $s_j^* \in S$ and η_j are non-zero integers, $0 \leqslant j \leqslant m - 1$. Observe that any one of the conditions in (3.13) is just (3.10) if we let the dimensions $n = 1$, $M = N$, $p = m$

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and $\hat{S} = S$ in (3.10). Applying the same argument to any one of the non-zero integers η_j in (3.13) gives $\hat{\mu}_{N,D}(\eta_j + \lambda) = 0$ for all $\lambda \in \Lambda(N, S)$, where $0 \leqslant j \leqslant m-1$. Hence, from Theorems 3.2 and 3.4, we obtain the following necessary and sufficient condition for $\Lambda(N, S)$ to be a spectrum.

Theorem 3.5. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Let $S \subset \mathbb{Z}$ with $0 \in S$ such that $(N^{-1}D, S)$ is a compatible pair. Then $(\mu_{N,D}, \Lambda(N, S))$ is not a spectral pair if and only if there exist $s_0^*, s_1^*, \dots, s_{m-1}^* \in S$, $m \geqslant 1$, such that*

$$(N^m - 1)^{-1}\{s_0^* + Ns_1^* + \cdots + N^{m-1}s_{m-1}^*\} \in \mathbb{Z} \setminus \{0\}. \tag{3.14}$$

Note that, under the assumptions of Theorem 3.5, the above discussion shows that the condition (3.14) is equivalent to $\eta_j \in \mathbb{Z} \setminus \{0\}$ for all $0 \leqslant j \leqslant m-1$, where η_j is defined by (3.13). As a consequence of Theorem 3.5, we also get the following.

Corollary 3.6. *Let $N \in \mathbb{Z}$ with $|N| > 2$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Suppose that $S \subset \mathbb{Z}$ with $0 \in S$ such that $(N^{-1}D, S)$ is a compatible pair. If $S \subseteq [2 - |N|, |N| - 2]$, then $(\mu_{N,D}, \Lambda(N, S))$ is a spectral pair.*

Proof. For any $s_0^*, s_1^*, \dots, s_{m-1}^* \in S$, $m \geqslant 1$, it follows from $|s_j^*| \leqslant |N| - 2$ that

$$-(|N|^m - 1) < s_0^* + Ns_1^* + \cdots + N^{m-1}s_{m-1}^* < (|N|^m - 1), \tag{3.15}$$

which means that $(N^m - 1)^{-1}\{s_0^* + Ns_1^* + \cdots + N^{m-1}s_{m-1}^*\}$ cannot be a non-zero integer. Hence, the desired conclusion follows from Theorem 3.5. □

Case 2 ($|D| = |S| = |\det(M)|$). For $M \in M_n(\mathbb{Z})$ with $|\det(M)| > 1$ and for the finite subsets $D, S \subset \mathbb{Z}^n$ with $|D| = |S| = |\det(M)|$, it follows from Propositions 2.2 and 2.3 that $(M^{-1}D, S)$ is a compatible pair if and only if D is a complete residue system $(\operatorname{mod} M)$ and S is a complete residue system $(\operatorname{mod} M^*)$. In the case when $M \in M_n(\mathbb{Z})$ is expanding and D is a complete residue system $(\operatorname{mod} M)$, we can show the following.

Theorem 3.7. *Let $M \in M_n(\mathbb{Z})$ be expanding and D be a complete residue system $(\operatorname{mod} M)$. Then $\mu_{M,D}$ is a spectral measure.*

Proof. From a result in [**12**, Theorem 3.4], there exists a full-rank lattice $\Gamma \subseteq \mathbb{Z}^n$ such that $T(M, D)$ tiles $\mathbb{R}^n$ by Γ-translation. Equivalently, from a result in [**2**], $\{e(\lambda \cdot x) : \lambda \in \Gamma^*\}$ is an orthogonal basis for $L^2(T(M, D))$ in the sense of Lebesgue measure μ_{L}, where Γ^* is the dual lattice of Γ. This also says that $T(M, D)$ is a *spectral set* with one of its *spectra* Γ^*. Therefore, we have the Parseval identity

$$\sum_{\gamma^* \in \Gamma^*} |\hat{\chi}_{T(M,D)}(\xi - \gamma^*)|^2 = (\mu_{\mathrm{L}}(T(M, D)))^2, \quad \forall \xi \in \mathbb{R}^n, \tag{3.16}$$

where $\hat{\chi}_{T(M,D)}$ is the Fourier transform of the characteristic function $\chi_{T(M,D)}$. It can be shown from (1.3) and (1.4) that $\hat{\chi}_{T(M,D)}(\xi) = \mu_{\mathrm{L}}(T(M, D))\hat{\mu}_{M,D}(\xi)$. So we deduce from (3.16) that

$$\sum_{\gamma^* \in \Gamma^*} |\hat{\mu}_{M,D}(\xi - \gamma^*)|^2 = 1, \quad \forall \xi \in \mathbb{R}^n. \tag{3.17}$$

Hence, Γ^* is a spectrum for $\mu_{M,D}$ and $\mu_{M,D}$ is a spectral measure. □

Therefore, under the condition that $(M^{-1}D, S)$ is a compatible pair and $|D| = |S| = |\det(M)|$, we know that $\mu_{M,D}$ is a spectral measure. Is $\Lambda(M, S)$ a spectrum for $\mu_{M,D}$? In the following we provide a necessary and sufficient condition for $(\mu_{M,D}, \Lambda(M, S))$ to be a spectral pair.

We first introduce the following generalization of *p-adic integers* (see [**11**]). Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix and let $S \subset \mathbb{Z}^n$ be a complete residue system $(\operatorname{mod} M^*)$ with $0 \in S$. Then, for each $x_0 \in \mathbb{Z}^n$, there is a unique canonical representation of x_0 in the form $\sum_{i=0}^{\infty} M^{*i} s_i$, where $s_i \in S$, which will be abbreviated as $s_0 s_1 s_2 \cdots$ and called the *M^*-adic address* of $x_0 \in \mathbb{Z}^n$. It follows from $\mathbb{Z}^n = S + M^*\mathbb{Z}^n$ that the ith entry s_i, $i = 0, 1, \dots$, in the M^*-adic address of x_0 is the unique element of S such that $s_i \equiv x_i (\operatorname{mod} M^*)$, where $x_{i+1} = M^{*-1}(x_i - s_i)$ (*recursive algorithm*). The M^*-adic address $s_0 s_1 s_2 \cdots$ of x_0 is called *finite* if $s_i = 0$ for all i sufficiently large. It can be shown that the M^*-adic address of every point $x_0 \in \mathbb{Z}^n$ is finite if and only if $\Lambda(M, S) = \mathbb{Z}^n$. If a point $x_0 \in \mathbb{Z}^n$ has an M^*-adic address with repeating string $s_{i+1} \cdots s_{i+q}$, we say that x_0 has a *repeating address*. Although a point $x_0 \in \mathbb{Z}^n$ may not have a finite M^*-adic address, it can be proved that the M^*-adic address of any point in $\mathbb{Z}^n$ is repeating [**11**, Lemma 2].

Lemma 3.8. *Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix and let $S \subset \mathbb{Z}^n$ be a complete residue system* $(\operatorname{mod} M^*)$ *with $0 \in S$. Then $(I - M^{*r})^{-1} S_r$ contains no non-zero integer point for $r = 1, 2, \dots$ if and only if $\Lambda(M, S) = \mathbb{Z}^n$.*

Proof. Suppose first that $(I - M^{*r})^{-1} S_r$ contains no non-zero integer point for $r = 1, 2, \dots$. We need to show that $\Lambda(M, S) = \mathbb{Z}^n$ or, equivalently, the M^*-adic address of every point $x_0 \in \mathbb{Z}^n$ is finite. If the M^*-adic address of some point $y \in \mathbb{Z}^n$ is not finite, then $y \in \mathbb{Z}^n$ has a repeating address where the repetition is not zero. We assume that

$$y = s_0 s_1 s_2 \cdots s_{q-1} \overbrace{s_q s_{q+1} \cdots s_{q+r-1}} \overbrace{s_q s_{q+1} \cdots s_{q+r-1}} \cdots$$

for some $q \geqslant 1$ and $r \geqslant 1$. Then

$$y - y_0 = M^{*q} x,$$

where

$$y_0 = s_0 s_1 s_2 \cdots s_{q-1} \quad \text{and} \quad x = \overbrace{s_q s_{q+1} \cdots s_{q+r-1}} \overbrace{s_q s_{q+1} \cdots s_{q+r-1}} \cdots .$$

So $x = M^{*-q}(y - y_0) \in \mathbb{Z}^n$ consists of that portion of y that repeats from the beginning. Also $(I - M^{*r})x = \sum_{i=0}^{r-1} M^{*i} s_{q+i} \in S_r$, and therefore $(I - M^{*r})^{-1} S_r$ contains a non-zero point $x \in \mathbb{Z}^n$ which does not have finite address: a contradiction. Hence, $\Lambda(M, S) = \mathbb{Z}^n$.

On the other hand, if $\Lambda(M, S) = \mathbb{Z}^n$, then $(I - M^{*r})^{-1} S_r$ contains no non-zero integer point for $r = 1, 2, \dots$. For if $(I - M^{*r})^{-1} S_r$ contains a non-zero point $x \in \mathbb{Z}^n$ for some r, then $(I - M^{*r})x = \sum_{i=0}^{r-1} M^{*i} s_i$ with $s_i \in S$. The point $\tilde{x} \in \mathbb{Z}^n$ whose infinite address consists of the repeated digits $s_0, s_1, \dots, s_{r-1}$ satisfies the same equation: $(I - M^{*r})\tilde{x} = \sum_{i=0}^{r-1} M^{*i} s_i$. Hence, $x = \tilde{x}$ has a repeating (non-finite) address; again this is a contradiction. This proves Lemma 3.8. □

Theorem 3.9. *Let $M \in M_n(\mathbb{Z})$ be expanding and let $D, S \subset \mathbb{Z}^n$ be finite subsets with $|D| = |S| = |\det(M)|$ and $0 \in D \cap S$. Suppose that $(M^{-1}D, S)$ is a compatible pair. Then $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair if and only if*

$$\Lambda(M,S) = \mathbb{Z}^n \quad \text{and} \quad \mu_{\mathrm{L}}(T(M,D)) = 1.$$

Proof. Suppose that $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair. It follows from Theorem 3.4 that $(I - M^{*p})^{-1}S_p$ contains no non-zero integer point for $p = 1, 2, \ldots$. By Lemma 3.8, we have $\Lambda(M,S) = \mathbb{Z}^n$. This in turn leads to

$$\sum_{\lambda \in \mathbb{Z}^n} |\hat{\mu}_{M,D}(\xi - \lambda)|^2 = \sum_{\lambda \in \Lambda(M,S)} |\hat{\mu}_{M,D}(\xi - \lambda)|^2 = 1, \quad \forall \xi \in \mathbb{R}^n, \tag{3.18}$$

which yields

$$\sum_{\lambda \in \mathbb{Z}^n} |\hat{\chi}_{T(M,D)}(\xi - \lambda)|^2 = (\mu_{\mathrm{L}}(T(M,D)))^2, \quad \forall \xi \in \mathbb{R}^n. \tag{3.19}$$

Hence, $\{e(\lambda \cdot x) : \lambda \in \mathbb{Z}^n\}$ is an orthogonal basis for $L^2(T(M,D))$. Equivalently, $T(M,D)$ tiles $\mathbb{R}^n$ by $\mathbb{Z}^n$-translations or $\mu_{\mathrm{L}}(T(M,D)) = 1$.

On the other hand, it follows from $\mu_{\mathrm{L}}(T(M,D)) = 1$ that (3.19) holds, which gives (3.18) by $\Lambda(M,S) = \mathbb{Z}^n$ and $\hat{\chi}_{T(M,D)}(\xi) = \hat{\mu}_{M,D}(\xi)$. This shows that $\mu_{M,D}$ is a spectral measure with spectrum $\mathbb{Z}^n$. Therefore, we have the desired spectral pair $(\mu_{M,D}, \Lambda(M,S))$. □

Note that, in Theorem 3.9, either one of the two conditions $\Lambda(M,S) = \mathbb{Z}^n$ and $\mu_{\mathrm{L}}(T(M,D)) = 1$ cannot guarantee that $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair. For example,

$$M = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 2 \end{pmatrix} \right\},$$

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \right\},$$

$(M^{-1}D, S)$ is a compatible pair. Since $T(M^*, S)$ is the fractal red cross, $T(M^*, S) \cap \mathbb{Z}^2 = \{(0,0)^{\mathrm{T}}\}$ [**1**, p. 58–59], we have $\Lambda(M,S) = \mathbb{Z}^2$. Furthermore, since the characteristic polynomial of M is irreducible over $\mathbb{Q}$, it follows from [**12**, Lemma 3.1 and Theorem 5.3] that $\mu_{\mathrm{L}}(T(M,D)) = 2$. Hence, $(\mu_{M,D}, \Lambda(M,S))$ is not a spectral pair. Examples 4.5 and 4.7 in the next section illustrate the other case, in which $\mu_{\mathrm{L}}(T(M,D)) = 1$ but $\Lambda(M,S) \neq \mathbb{Z}^n$, and thus $\Lambda(M,S)$ is not a spectrum for $\mu_{M,D}$.

4. Remarks and examples

We first consider the $\mathbb{Z}$-similarity of the expanding integer matrices. Two matrices $M, M_1 \in M_n(\mathbb{Z})$ are $\mathbb{Z}$-similar, denoted by $M \sim M_1$, if there exists a unimodular matrix $P \in M_n(\mathbb{Z})$ (i.e. P is invertible and $P^{-1} \in M_n(\mathbb{Z})$) such that $P^{-1}MP = M_1$. The $\mathbb{Z}$-similarity is an equivalent relationship; its equivalence classes are called $\mathbb{Z}$-similar classes.

Let $D, S \subset \mathbb{Z}^n$ be finite subsets of the same cardinality. Let $M, M_1 \in M_n(\mathbb{Z})$ be two expanding matrices such that $M \sim M_1$ as above. Define $D_1 := P^{-1}D$, $S_1 := P^*S$. Then from Proposition 2.1 (ii), $(M_1^{-1}D_1, S_1)$ is a compatible pair if and only if $(M^{-1}D, S)$ is a compatible pair. Also, from (1.5) and (1.6), we have

$$T(M_1, D_1) = P^{-1}T(M, D), \qquad \Lambda(M_1, S_1) = P^*\Lambda(M, S) \tag{4.1}$$

and

$$\hat{\mu}_{M_1,D_1}(\xi) = \prod_{j=1}^{\infty} m_{D_1}(M_1^{*-j}\xi) = \prod_{j=1}^{\infty} m_D(M^{*-j}(P^{*-1}\xi)) = \hat{\mu}_{M,D}(P^{*-1}\xi). \tag{4.2}$$

Therefore,

$$\sum_{\lambda \in \Lambda(M_1,S_1)} |\hat{\mu}_{M_1,D_1}(\xi + \lambda)|^2 = \sum_{\lambda \in \Lambda(M,S)} |\hat{\mu}_{M,D}(P^{*-1}\xi + \lambda)|^2. \tag{4.3}$$

It follows from (1.2) or the Parseval identity that $(\mu_{M,D}, \Lambda(M, S))$ is a spectral pair if and only if $(\mu_{M_1,D_1}, \Lambda(M_1, S_1))$ is a spectral pair. In this sense, we have the following.

Proposition 4.1. *Spectral pairs and compatible pairs are invariant under the $\mathbb{Z}$-similarity.*

Note that, in the above discussion, it is not necessary that the elements of matrices and sets are integer. The usual similarity over $\mathbb{R}$ and sets in $\mathbb{R}^n$ will yield the same result. For example, given the expanding matrix $M \in M_n(\mathbb{R})$ and finite set $D \subset \mathbb{R}^n$ (not necessarily an integer matrix and integer set), the self-affine measure $\mu_{M,D}$ also has the following property.

Proposition 4.2. *Let $M \in M_n(\mathbb{R})$ be expanding, $D \subset \mathbb{R}^n$ be a finite set and $\Lambda \subset \mathbb{R}^n$ be a discrete set. Suppose that $P \in M_n(\mathbb{R})$ is non-singular with $PM = MP$. Then $(\mu_{M,D}, \Lambda)$ is a spectral pair if and only if $(\mu_{M,P^{-1}D}, P^*\Lambda)$ is a spectral pair.*

In fact, Proposition 4.2 follows from the same discussion as above by taking $M_1 = M$ simply.

The $\mathbb{Z}$-similar classification of expanding integer matrices was first studied by Lagarias and Wang [**8**]. They showed that there are only six $\mathbb{Z}$-similar classes of 2×2 integer matrices with $|\det(M)| = 2$. That is, the following proposition holds.

Proposition 4.3. *Let $M \in M_2(\mathbb{Z})$ be expanding. If $\det(M) = -2$, then*

$$M \sim \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}. \tag{4.4}$$

If $\det(M) = 2$, then M is $\mathbb{Z}$-similar to one of the following matrices:

$$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad \pm\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \pm\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}. \tag{4.5}$$

The complete classifications for $M \in M_2(\mathbb{Z})$ with $|\det(M)| = 3, 4, 5$ are given in [**6**]. Most of the expanding integer matrices in our example come from these $\mathbb{Z}$-similar classes such as (4.5).

Example 4.4. The Eiffel Tower (or three-dimensional Sierpinski gasket):

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$
$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$(M^{-1}D, S)$ is a compatible pair and $(M^* - I)^{-1} = I$. Obviously, each non-zero element $\tilde{s} \in S$ satisfies $(M^* - I)^{-1}\tilde{s} \in \mathbb{Z}^3$. Therefore, Theorem 3.3 shows that $(\mu_{M,D}, \Lambda(M, S))$ is not a spectral pair (a question that cannot be answered in [**4, 5, 9, 10**]).

For the general cases,

$$\left.\begin{aligned} M &= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}r \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2}r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}r \end{pmatrix} \right\}, \\ S &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad r \in \mathbb{Z}, \ |r| \geqslant 2, \end{aligned}\right\} \tag{4.6}$$

$(M^{-1}D, S)$ is a compatible pair. It is known that if $r = -2$ or $|r| \geqslant 4$ and even, the corresponding $(\mu_{M,D}, \Lambda(M, S))$ is a spectral pair, while if $|r| \geqslant 2$ and odd, for example, $r = 3$, then $\{e(\lambda \cdot x) : \lambda \in \Lambda(M, S)\}$ is not orthonormal in $L^2(\mu_{M,D})$. The case $r = 2$ was an open question for a long time. The discussion here provides an answer. Since $D \subset \mathbb{Z}^3$ for $r \in \mathbb{Z}$ and even, one cannot expect that a compatible pair $(M^{-1}D, S)$ automatically yields an orthonormal system $\{e(\lambda \cdot x) : \lambda \in \Lambda(M, S)\}$ in $L^2(\mu_{M,D})$ without the condition $D \subset \mathbb{Z}^n$.

Example 4.5.

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \qquad S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Observe that

$$\hat{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\} \equiv S (\operatorname{mod} M^*) \quad \Longleftrightarrow \quad k_1 \in \mathbb{Z}, \ k_2 \in 2\mathbb{Z} + 1.$$
$$\hat{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_3 \\ k_4 \end{pmatrix} \right\} \equiv D (\operatorname{mod} M) \quad \Longleftrightarrow \quad k_3 \in 2\mathbb{Z} + 1, \ k_4 \in \mathbb{Z}.$$

It follows from Theorem 3.3 that, for each $k_1 \in \mathbb{Z}$ and $k_2 \in 2\mathbb{Z}+1$, the orthonormal system $\{e(\lambda \cdot x) : \lambda \in \Lambda(M, \hat{S})\}$ is not complete in $L^2(\mu_{M,\hat{D}})$ for any $k_3 \in 2\mathbb{Z}+1$ and $k_4 \in \mathbb{Z}$. However, since $M^{2k} = 2^k I$, $k \in \mathbb{Z}$, $T(M, D) = [0, 1] \times [0, 1]$ and

$$\hat{\mu}_{M,D}(\xi_1, \xi_2) = \mathrm{e}^{-\pi(\xi_1+\xi_2)\mathrm{i}} \frac{\sin(\pi\xi_1)}{\pi\xi_1} \frac{\sin(\pi\xi_2)}{\pi\xi_2} = \int_{T(M,D)} \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x}\, \mathrm{d}x, \tag{4.7}$$

$\mu_{M,D}$ is simply the restriction of the Lebesgue measure μ_{L} to $T(M, D)$ (this is a general conclusion if $\mu_{\mathrm{L}}(T(M, D)) = 1$; see the proofs of Theorems 3.7 and 3.9). It is known that $\{e(\lambda \cdot x) : \lambda \in \mathbb{Z}^2\}$ is an orthonormal basis for $L^2([0, 1]^2)$. So $\mu_{M,D}$ is a spectral measure; one of the spectra for $\mu_{M,D}$ is $\mathbb{Z}^2$. This shows that even if $(\mu_{M,D}, \Lambda(M, \hat{S}))$ is not a spectral pair, we cannot assert that $\mu_{M,D}$ is not a spectral measure. The only assertion is that $\Lambda(M, \hat{S})$ is not a spectrum for $\mu_{M,D}$ if $k_1 \in \mathbb{Z}$ and $k_2 \in 2\mathbb{Z}+1$ (note that, in this case, $\Lambda(M, \hat{S})$ is a subset of $\mathbb{Z}^2$).

Example 4.6.

$$M = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \qquad S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Observe that the condition (3.10) of Theorem 3.4 is not satisfied if $\hat{S} = S$. We shall show that $(\mu_{M,D}, \Lambda(M, S))$ is a spectral pair by applying Theorem 3.1 or Theorem 3.9.

In fact, it follows from $M^2 = -2I$ that $T(M^*, S) = [-\frac{1}{3}, \frac{2}{3}] \times [-\frac{2}{3}, \frac{1}{3}]$. The zero set $Z(m_{M^{-1}D})$ of $m_D(M^{*-1}t) = \mathrm{e}^{\pi \mathrm{i} t_2/2} \cos(\frac{1}{2}\pi t_2)$ is $\{(t_1, 2k+1) : t_1 \in \mathbb{R},\ k \in \mathbb{Z}\}$, which is disjoint from the set $T(M^*, S)$. Hence, $(\mu_{M,D}, \Lambda(M, S))$ is a spectral pair by Theorem 3.1. On the other hand, since $T(M^*, S)$ contains no non-zero integer point, we have $\Lambda(M, S) = \mathbb{Z}^n$ by Lemma 3.8. One can check $\Lambda(M, S) = \mathbb{Z}^n$ and $\mu_{\mathrm{L}}(T(M, D)) = 1$ directly to obtain the spectral pair $(\mu_{M,D}, \Lambda(M, S))$ by Theorem 3.9.

Note that this is the first example of the spectral pair $(\mu_{M,D}, \Lambda(M, S))$ in which M is not a diagonal matrix. Of course, $\mathbb{Z}$-similar classes of matrix M give such spectral pairs. Also see the spectral pair produced by (4.13), below. Furthermore, compared with Example 4.5, $T(M, D)$ is a unit square $[-\frac{2}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{2}{3}]$, $\mu_{M,D}$ is the restriction of the Lebesgue measure to $T(M, D)$ but $\Lambda(M, S) = \mathbb{Z}^2$. In both examples, $(M^{-1}D, S)$ is a Gaussian compatible pair.

In the case when

$$\hat{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2k - 3k_1 + 1 \\ 2k + 1 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3k_1 \\ 3(2k + 1) \end{pmatrix} \right\}, \quad k, k_1 \in \mathbb{Z},$$

we have $\hat{S} \equiv S (\operatorname{mod} M^*)$. It follows from Theorem 3.4 that, for any $\hat{D} \equiv D (\operatorname{mod} M)$, $\Lambda(M, \hat{S})$ is not a spectrum for $\mu_{M,\hat{D}}$. In particular, $(\mu_{M,D}, \Lambda(M, \hat{S}))$ is not a spectral pair compared with the above established fact that $(\mu_{M,D}, \Lambda(M, S))$ is a spectral pair but $S \equiv \hat{S} (\operatorname{mod} M^*)$.

Example 4.7. (a) Twin dragon:

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \qquad S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

https://doi.org/10.1017/S0013091503000324 Published online by Cambridge University Press

Since $(M^* - I)^{-1} \in M_2(\mathbb{Z})$, for any $\hat{S} \equiv S (\operatorname{mod} M^*)$, i.e.

$$\hat{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\}, \quad k_2 - k_1 \in 2\mathbb{Z} + 1,$$

it follows from Theorem 3.3 that the orthonormal system $\{e(\lambda \cdot x) : \lambda \in \Lambda(M, \hat{S})\}$ is not complete in $L^2(\mu_{M,\hat{D}})$ for any $\hat{D} \equiv D (\operatorname{mod} M)$.

(b) Terdragon:

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}.$$

Since $(M^* - I)^{-1} \in M_2(\mathbb{Z})$, for any $\hat{S} \equiv S (\operatorname{mod} M^*)$, i.e.

$$\hat{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 + k_1 + k_2 \\ 2k_2 - k_1 \end{pmatrix}, \begin{pmatrix} 2 + k_3 + k_4 \\ 2k_4 - k_3 \end{pmatrix} \right\}, \quad k_j \in \mathbb{Z}, \ 1 \leqslant j \leqslant 4,$$

it follows from Theorem 3.3 that, for any $\hat{D} \equiv D (\operatorname{mod} M)$, $(\mu_{M,\hat{D}}, \Lambda(M, \hat{S}))$ is not a spectral pair.

(c) Shark-jawed parallelogram [**1**, p. 45]:

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad S = D.$$

Since $(M^* - I)^{-1} \in M_2(\mathbb{Z})$, for any $\hat{S} \equiv S (\operatorname{mod} M^*)$, i.e.

$$\hat{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2k_1 + 1 \\ 2k_2 + k_1 \end{pmatrix}, \begin{pmatrix} 2k_3 + 1 \\ 2k_4 + k_3 + 1 \end{pmatrix}, \begin{pmatrix} 2k_5 \\ 2k_6 + k_5 + 1 \end{pmatrix} \right\},$$

where $k_j \in \mathbb{Z}$, $1 \leqslant j \leqslant 6$, it follows from Theorem 3.3 that the orthonormal system $\{e(\lambda \cdot x) : \lambda \in \Lambda(M, \hat{S})\}$ is not complete in $L^2(\mu_{M,\hat{D}})$ for any $\hat{D} \equiv D (\operatorname{mod} M)$.

Examples (a)–(c) have the same property that $(M^* - I)^{-1} \in M_2(\mathbb{Z})$. This prevents us getting an orthonormal basis in $L^2(\mu_{M,D})$ from the compatible pair. However, it follows from Theorem 3.7 that in Example 4.7 there does exist an orthonormal basis in $L^2(\mu_{M,D})$, i.e. $\mu_{M,D}$ is a spectral measure and one of the spectra is $\mathbb{Z}^2$. One can compare this case with Example 4.5. On the other hand, since $\det(M^* - I) = \pm 1$, it follows from Lemma 3.8 that there is no set $S \subset \mathbb{Z}^2$ that is a complete residue system $(\operatorname{mod} M^*)$ such that $\Lambda(M, S) = \mathbb{Z}^2$.

It should be pointed out that, in (c), if we take

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad D = S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \tag{4.8}$$

then the corresponding result still holds. In fact, in this interesting case, we have

$$\begin{aligned}\hat{\mu}_{M,D}(\xi) &= \prod_{j=1}^{\infty} \tfrac{1}{2}(1+\mathrm{e}^{-2\pi\mathrm{i}\xi_1/2^j}) \\ &= \mathrm{e}^{-\pi\xi_1\mathrm{i}} \prod_{j=1}^{\infty} \cos\left(\frac{\pi\xi_1}{2^j}\right) \\ &= \mathrm{e}^{-\pi\xi_1\mathrm{i}} \frac{\sin(\pi\xi_1)}{\pi\xi_1}, \end{aligned} \tag{4.9}$$

where $\xi = (\xi_1, \xi_2)^{\mathrm{T}} \in \mathbb{R}^2$, and a familiar infinite product formula for $\cos(x/2^j)$ (the Euler–Vieta formula) is used as in (4.7). The zero set of $\hat{\mu}_{M,D}(\xi)$ is

$$Z(\hat{\mu}_{M,D}) = \{(\xi_1,\xi_2)^{\mathrm{T}} \in \mathbb{R}^2 : \xi_1 \in \mathbb{Z}\setminus\{0\},\ \xi_2 \in \mathbb{R}\}.$$

Incidentally, we have the known identity

$$\sum_{\gamma\in\mathbb{Z}} \left|\frac{\sin(\pi(\gamma+t))}{\pi(\gamma+t)}\right|^2 = 1, \quad \forall t \in \mathbb{R}. \tag{4.10}$$

Now define Λ by

$$\Lambda := \left\{ \begin{pmatrix} \lambda_1 \\ \beta(\lambda_1) \end{pmatrix} \in \mathbb{R}^2 : \lambda_1 \in \mathbb{Z} \right\}, \tag{4.11}$$

where $\beta : \mathbb{Z} \to \mathbb{R}$ is an arbitrary single-valued function.

It follows from (4.9)–(4.11) that

$$\sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D}(\xi-\lambda)|^2 = \sum_{\lambda_1\in\mathbb{Z}} \left|\frac{\sin(\pi(\xi_1-\lambda_1))}{\pi(\xi_1-\lambda_1)}\right|^2 = 1, \quad \forall \xi \in \mathbb{R}^2. \tag{4.12}$$

Hence, $\mu_{M,D}$ is a spectral measure and one of the spectra is given by (4.11). In the case (4.8), both $\Lambda(M,S)$ and $\mathbb{Z}^2$ are not spectra for $\mu_{M,D}$, but the points $\{(\lambda_1,0)^{\mathrm{T}} \in \mathbb{R}^2 : \lambda_1 \in \mathbb{Z}\}$ on the x-axis form a spectrum for $\mu_{M,D}$, compared with the Lebesgue-measure case, in which each spectrum for a spectral set cannot be contained in any proper affine subspace of $\mathbb{R}^n$ (see [**2**, Remark (3), p. 109]).

Example 4.8. We consider the problem in [**10**, Example 2.9 (b)]. In dimension 1, let $M = 4$ and $D = \{0, 2\}$. From a result in [**5**, Corollary 5.9], we see that $\mu_{M,D}$ is a spectral measure. Choose $S = \{0, 5\}$ and $\tilde{S} = \{0, 3\}$. Then $(M^{-1}D, S)$ and $(M^{-1}D, \tilde{S})$ are compatible pairs. Strichartz [**10**, Example 2.9 (b)] showed that $\Lambda(M,S)$ is a spectrum for $\mu_{M,D}$. However, if one tries to replace 5 by 3 in the choice of S, then Strichartz's theorem does not apply. This leaves the question of whether or not $\Lambda(M,\tilde{S})$ is a spectrum for $\mu_{M,D}$ in [**10**]. In fact, from Theorem 3.3 above, one can easily show that the answer to this question is negative.

Note that from Proposition 4.2 and (4.1), we know that $\Lambda(M,\tilde{S})$ is not a spectrum for $\mu_{M,D}$ if and only if $2\Lambda(M,\tilde{S})$ (where $2\Lambda(M,\tilde{S}) = \Lambda(M,2\tilde{S})$) is not a spectrum for $\mu_{M,\frac{1}{2}D}$. So, by letting $N_1 = M = 4$, $D_1 = \frac{1}{2}D = \{0,1\}$, $S_1 = 2\tilde{S} = \{0,6\}$, the condition (3.14) is satisfied for the compatible pair $(N_1^{-1}D_1, S_1)$ with $m = 1$ and $s_0^* = 6$, while the sufficient condition of Theorem 3.2 is satisfied with $m = 1$, $\eta_0 = 2$ and $s_0^* = 6$. We also get the negative answer to the question from Theorem 3.5 or Theorem 3.2.

4.1. Concluding remarks

We now present several concluding remarks on the problems that existent in the process of dealing with spectral self-affine measures.

The combination of Proposition 2.3 and Corollary 3.6 yields a large number of new spectral pairs in dimension 1. Of course, most of these spectral pairs cannot be obtained by Theorem 3.1 in a simple manner. This is because the condition of Theorem 3.1 can be very difficult to check. Example 3.1 in [**7**] addresses the condition of Theorem 3.1 by Strichartz. We note that this example is not exact. In fact, in the case when $N = 5$, $D = \{0, \pm 2, \pm 11\}$ and $S = \{0, \pm 1, \pm 2\}$, it follows from Proposition 2.3 and Corollary 3.6 that $\Lambda(N, S)$ is a spectrum for $\mu_{N,D}$. Since

$$m_D\left(\frac{t}{5}\right) = \frac{1}{5}\left(1 + 2\cos\frac{4\pi t}{5} + 2\cos\frac{22\pi t}{5}\right),$$

the first non-negative zero of $m_D(\frac{1}{5}t)$ is located in the interval $[0.60, 0.62]$. Hence, the zero set $Z(m_{5^{-1}D})$ of $m_D(\frac{1}{5}t)$ is disjoint from the set $T(N,S) = [-\frac{1}{2}, \frac{1}{2}]$. By Theorem 3.1, we also find that $(\mu_{N,D}, \Lambda(N,S))$ is a spectral pair. To illustrate the superiority of our result, the minor change in this example will do. Consider $N = 5$, $D = \{0, \pm 7, \pm 11\}$ and $S = \{0, \pm 1, \pm 2\}$. It follows from Proposition 2.3 and Corollary 3.6 that $\Lambda(N,S)$ is a spectrum for $\mu_{N,D}$. Since

$$m_D\left(\frac{t}{5}\right) = \frac{1}{5}\left(1 + 2\cos\frac{14\pi t}{5} + 2\cos\frac{22\pi t}{5}\right),$$
$$m_D(0) > 0, \quad m_D\left(\frac{1}{22}\right) = \frac{1}{5}\left(1 - 2 + 2\cos\frac{7\pi}{11}\right) < 0,$$

we see that $m_D(\frac{1}{5}t)$ has a zero in $[0, \frac{5}{22}]$, which is contained in $T(N,S) = [-\frac{1}{2}, \frac{1}{2}]$. Theorem 3.1 fails to yield the desired conclusion.

Theorems 3.3 and 3.4 only give sufficient conditions for a set not to be a spectrum, due to technical reasons (see the open problems in [**7**]). We do not know whether the hypotheses in Theorems 3.3 or 3.4 are necessary. Compared with Theorem 3.5, some condition which is similar to $\gcd(D) = 1$ should be added to Theorems 3.3 and 3.4 in order to obtain the necessity. But, compared with Theorem 3.9, this condition seems to be superfluous. This is due to the fact that, in the case when $|D| = |\det(M)|$, the higher-dimensional generalization of the condition $\gcd(D) = 1$ is $\mathbb{Z}[M,D] = \mathbb{Z}^n$, and the condition $\mu_{\mathrm{L}}(T(M,D)) = 1$ implies that $\mathbb{Z}[M,D] = \mathbb{Z}^n$, where $\mathbb{Z}[M,D]$ is the smallest M-invariant sublattice of $\mathbb{Z}^n$ containing the difference set $D - D$. In the case when $|D| < |\det(M)|$, the lattice $\mathbb{Z}[M,D]$ in the spectral pair $(\mu_{M,D}, \Lambda(M,S))$ may have different properties: either full rank or non-full rank. For instance, let M, D and S be given by (4.6), where $r=-2$ or $|r|\geqslant 4$ and even. Then the corresponding $\mathbb{Z}[M,D]$ is a full-rank lattice. On the other hand, if we consider

$$M=\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix},\qquad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\},\qquad S=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \tag{4.13}$$

then $(\mu_{M,D},\Lambda(M,S))$ is a spectral pair but $\mathbb{Z}[M,D]$ is not a full-rank lattice. To see this, we first note that if M, D and S are given by (4.13), then $(M^{-1}D,S)$ is a compatible pair and

$$\hat{\mu}_{M,D}(\xi)=\mathrm{e}^{\pi\xi_1\mathrm{i}/3}\frac{\sin(\pi\xi_1)}{\pi\xi_1},\quad \xi=(\xi_1,\xi_2)^{\mathrm{T}}\in\mathbb{R}^2. \tag{4.14}$$

So the Λ given by (4.11) is one of the spectra for $\mu_{M,D}$. We then see that

$$\Lambda(M,S)=\left\{\sum_{j=0}^{k-1}\begin{pmatrix}(-2)^j s_{1,j}\\ 2^j s_{2,j}\end{pmatrix}:\begin{pmatrix}s_{1,j}\\ s_{2,j}\end{pmatrix}\in S,\ k\geqslant 1\right\}, \tag{4.15}$$

and

$$\left\{\sum_{j=0}^{k-1}(-2)^j s_{1,j}:s_{1,j}\in\{0,1\},\ k\geqslant 1\right\}=\mathbb{Z}. \tag{4.16}$$

This shows that $\Lambda(M,S)$ has the form (4.11), where $\beta(\lambda_1):\mathbb{Z}\to\{0,-1,-2,\dots\}$ is a certain single-valued function. Hence, $(\mu_{M,D},\Lambda(M,S))$ is a spectral pair and $\mathbb{Z}[M,D]=\{(k,0)^{\mathrm{T}}:k\in\mathbb{Z}\}$ is not a full-rank lattice in $\mathbb{R}^2$.

In the Eiffel Tower or three-dimensional Sierpinski gasket, we have shown that the condition $(M^*-I)^{-1}\in M_n(\mathbb{Z})$ or $\det(M^*-I)=\pm 1$ led us to the conclusion that the orthogonal system $\{e(\lambda\cdot x):\lambda\in\Lambda(M,S)\}$ is not complete in $L^2(\mu_{M,D})$. Thus, we cannot obtain the orthogonal basis from the compatible pair in this case. Even so, $\mu_{M,D}$ may be a spectral measure.

Acknowledgements. I am indebted to Professor K. S. Lau for his valuable suggestions. I also thank Professor Y. Wang for a helpful discussion on the subject during his visit to The Chinese University of Hong Kong.

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