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Maximal Subbundles of Rank 2 Vector Bundles on Projective Curves

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Abstract. Let E be a stable rank 2 vector bundle on a smooth projective curve X and $V(E)$ be the set of all rank 1 subbundles of E with maximal degree. Here we study the varieties (non-emptyness, irreducibility and dimension) of all rank 2 stable vector bundles, E, on X with fixed $\deg(E)$ and $\deg(L)$, $L \in V(E)$ and such that $\operatorname{card}\bigl(V(E)\bigr) \geq 2$ (resp. $\operatorname{card}\bigl(V(E)\bigr) = 2$).

0 Introduction

Let X be a smooth projective curve of genus g defined over an algebraically closed field $\mathbf{K}$. For every integer d, $M(X;2,d)$ will denote the scheme of all rank 2 stable vector bundles on X of degree d. It is known that $M(X;2,d)$ is irreducible, smooth and of dimension $4g-3$ (see *e.g.* the introduction of [12]). For every $E \in M(X;2,d)$ there is a unique integer $s(E)$ with $0 < s(E) \leq g$ and such that E has a line subbundle of degree $\bigl(d - s(E)\bigr)/2$ but no line subbundle of degree $> \bigl(d - s(E)\bigr)/2$ [12]. For historical reasons this integer $s(E)$ is often called the *C. Segre-M. Nagata-H. Lange-M. S. Narasimhan invariant of E*. For shortness we will call it the *Lange invariant* of E. By its very definition we have $s(E) \equiv d \bmod(2)$. Set $M(X;2,d,s) := \{E \in M(X;2,d) : s(E) = s\}$. We will see $M(X;2,d,s)$ as a locally closed subset of $M(X;2,d)$, and we will always use the corresponding reduced structure as scheme structure on $M(X;2,d,s)$. By [12, Prop. 3.1] for all integers d, s with $d \equiv s \bmod(2)$ and $0 < s \leq g-2$ the scheme $M(X;2,d,s)$ is a non-empty irreducible scheme of dimension $3g-2+s$. By [12, Prop. 3.3] for every integer s with $0 < s \leq g-2$ and every integer d with $d \equiv s \bmod(2)$ a general element of $M(X;2,d,s)$ has a unique line subbundle of degree $(d-s)/2$. For any $E \in M(X;2,d)$ let $V(E)$ be the set of all maximal degree rank 1 subbundles of E. Set $V(X;2,d,s;2) := \{E \in M(X;2,d,s) : \operatorname{card}\bigl(V(E)\bigr) \geq 2\}$ and $W(X;2,d,s;2) := \{E \in M(X;2,d,s) : \operatorname{card}\bigl(V(E)\bigr) = 2\}$. In the first section of this note we prove the following result.

Theorem 0.1 *Let X be a smooth projective curve of genus $g \geq 3$. For every integer s with $0 < s \leq g-2$ and every integer d the set $V(X;2,d,s;2)$ is an irreducible variety of dimension $2g+2s-1$ and $W(X;2,d,s;2)$ is a non-empty open subset of it.*

For every integer s with $0 < s \leq g-2$ we will construct bundles $E \in M(X;2,d,s)$ with $\operatorname{card}\bigl(V(E)\bigr) = 3$ (Proposition 1.6). For an upper bound for $\operatorname{card}\bigl(V(E)\bigr)$ for any $E \in M(X;2,d,s)$ with $\operatorname{card}\bigl(V(E)\bigr)$ finite, see [11]. The irreducibility of $V(X;2,d,s;2)$ was proved in [10]; the same paper contains the computation of its dimension. Only the non-emptyness of $W(X;2,d,s;2)$ is new. But our method is completely different: we use

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vector bundles, while in [10], [11] and [12] it was given and used the following very nice translation of these problems in terms of secant varieties and linear series; for proofs and more details, see [12, Sections 1 and 2].

Let E be a rank 2 stable vector bundle on X. Up to a twist by a line bundle we may assume that ω_X is a maximal degree line subbundle of E. Hence E fits in an exact sequence

$$0 \to \omega_X \to E \to L \to 0 \tag{1}$$

with $L \in \mathrm{Pic}(X)$ and $\deg(L) = 2g - 2 + s(E)$. Set $s := s(E)$. If $s \geq 3$ or E is general L is very ample. By Serre duality the extension (1) gives E as a point $e \in \mathbf{P}\big(H^0(X, L)\big)$. The very ample line bundle L induces an embedding $h_L \colon X \to \mathbf{P}\big(H^0(X, L)\big) \cong \mathbf{P}^{s+g-2}$. Set $Y := h_L(X) \cong X$. Let $Y^{(s)}$ be the s-th symmetric product of Y and $S^{(s-1)}(Y) \subset \mathbf{P}\big(H^0(X, L)\big)$ the $s-1$ secant variety of Y. We have $\dim\big(S^{(s-1)}(Y)\big) = 2s-1$. There is another subbundle S of E with $\deg(S) = 2g - 2$ if and only if $e \in S^{(s-1)}(Y)$. Let $p_1 \colon Y^{(s)} \times Y \to Y^{(s)}$ and $p_2 \colon Y^{(s)} \times Y \to Y$ be the projections. Set $W := \{(D, y) \in Y^{(s)} \times Y : D - y \geq 0\}$. Set $A := p_{1*}\Big(p_2^*\big(\mathbf{O}_Y(1)\big) \mid W\Big)$. Hence A is a rank s vector bundle on $Y^{(s)}$. A is called the *s-secant bundle of the pair* (X, L). There is a morphism $\gamma \colon \mathbf{P}(A) \to \mathbf{P}\big(H^0(X, L)\big)$ with $\gamma\big(\mathbf{P}(A)\big) = S^{(s-1)}(Y)$. The non-emptyness of $W(X; 2, d, s; 2)$ says that for general L the secant bundle map γ is birational. It is amusing that Theorem 0.1 is proved here only using vector bundle techniques, not the secant varieties.

Then we will consider the case $s = g - 1$. It is known that for every d with $d - g$ odd a general $E \in M(X; 2, d)$ has $s(E) = g - 1$ and it has finitely many maximal degree subbundles [12, Cor. 3.2]. For an open dense subset Ω of $M(X; 2, d)$ (with $d - g$ odd) there is an integer, δ, such that every $E \in \Omega$ has $s(E) = g - 1$ and exactly δ maximal degree subbundles. By [11] we have $\delta = 2^g$ (see also [6] and [9, Section 8]. If d is odd (*i.e.*, if g is even), there is a universal family $\pi \colon \mathbf{E} \to M(X; 2, d)$ and hence, taking for every $E \in \pi^{-1}(\Omega)$ the finite set $V(E)$ of its maximal degree subbundles, we obtain a finite degree δ covering $\alpha \colon T \to \Omega$. Let $G(X)$ be its Galois group; this is defined even if T is not irreducible, but we will see (Remark 0.3) that T is irreducible and hence $G(X)$ is the Galois group of the normalization of the field extension $\mathbf{K}(T) \setminus \mathbf{K}\big(M(X; 2, d)\big)$. $G(X)$ acts as permutation group of the fiber of α over the generic point of $M(X; 2, d)$ and hence it is a subgroup of the symmetric group S_δ. $G(X)$ is usually called the monodromy group of this problem. Obviously this monodromy group depends only on X and not on the congruence class of d modulo 2. Now assume d even, *i.e.*, g odd. Now $M(X; 2, d)$ is not a fine moduli scheme and hence there is no universal family of rank 2 vector bundles on it. However, there is still a universal family $\beta \colon \mathbf{P} \to M(X; 2, d)$ of projectivizations of rank 2 stable vector bundles. Since every maximal degree line subbundle of E corresponds to a suitable section of $\mathbf{P}(E)$, we may define δ, Ω and the finite covering $\alpha \colon T \to \Omega$ just using β. Hence we may define $G(X)$ if g is odd, too. At the end of Section 1 we will prove the following result.

Proposition 0.2 *Assume* $\mathrm{char}(\mathbf{K}) = 0$. *Let X be a smooth curve of genus g. Then $G(X)$ is at least double transitive.*

Remark 0.3 The transitivity of $G(X)$ is equivalent to the irreducibility of T.

In Section 2 we will use some ideas contained in [4] to prove the non-existence of rank 2 stable vector bundles on X with low $s(E)$ and with infinitely many rank 1 subbundles with maximal degree. D. Butler in [4] proved the following very nice result.

Theorem 0.4 (**[4, Theorem 1]**) *Assume* $\mathrm{char}(\mathbf{K}) = 0$. *Fix integers g, s with $s > 0$ and $g > s(2s-1)$. Let X be a smooth projective curve of genus g and E a rank 2 vector bundle on X with $s(E) = s$ and such that E has infinitely many maximal degree line subbundles. Then there exists a smooth curve C of genus $q > 0$, a covering $\pi \colon X \to C$ with* $\deg(\pi) > 1$, $L \in \mathrm{Pic}(X)$, *a rank 2 vector bundle F on C with* $s(F) = s(E)/\deg(\pi)$, $\pi^*(F) \cong E \otimes L$ *and such that for every maximal degree line subbundle, R, of $E \otimes L$ there exists a maximal degree line subbundle M of F with $\pi^*(M) \cong R$.*

We liked Theorem 0.4 but we liked even more Butler's proof of it, because we believe that it may be used in several other situations. In Section 2 we will prove the following result.

Proposition 0.5 *Assume* $\mathrm{char}(\mathbf{K}) \neq 2$. *Let X be a smooth curve of genus $g \geq 3$ with general moduli and E a rank 2 vector bundle on X such that E has infinitely many rank 1 subbundles with maximal degree. Then* $s(E) \geq (g-2)/3$.

Proposition 0.5 improves the lower bound on s given in [4, Remark on p. 31].

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1 Proofs of 0.1 and 0.2

Fix an integer s with $0 < s \leq g$. For every $E \in M(X;2,d,s)$ let $V(E)$ be the set of all line subbundles of E with maximal degree; $V(E)$ is in a natural way a Quot-scheme and hence it has a natural scheme structure; however, we will always consider $V(E)$ with the associated reduced structure; hence we will see $V(E)$ as a reduced projective scheme; there is a natural morphism $\pi_E \colon V(E) \to \mathrm{Pic}^{(d-s(E))/2}(X)$; by [12, Lemma 2.1], π_E is injective. For every $R \in \mathrm{Pic}^t(X)$ and every $E \in M(X;2,d,s)$ we have $E \otimes R \in M(X;2,d+2t,s)$. Hence instead of studying all schemes $M(X;2,d,s)$, $d \equiv s \bmod(2)$, it is sufficient to study all schemes $M(X;2,s,s)$. By definition every $E \in M(X,2,s,s)$ has a maximal degree line subbundle of degree 0. Fix an integer s with $0 < s \leq g$ and $L, M \in \mathrm{Pic}^0(X)$ with $L \neq M$. $V(s,L)$ will denote the subset of $M(X;2,s,s)$ formed by the stable bundles which have a subsheaf isomorphic to L; by definition of $M(X;2,s,s)$ such subsheaf is saturated and it is a maximal degree rank 1 subbundle. When $s \leq g-2$, $W(s,L)$ will denote the subset of $V(s,L)$ formed by the bundles with a unique degree 0 line subbundle (which is thus isomorphic to L). $V(s,L,M)$ will denote the set of all stable rank 2 vector bundles on X which have at least one subbundle isomorphic to L and one subbundle isomorphic to M. Since $L \neq M$, every $E \in V(s,L,M)$ contains a subsheaf isomorphic to $L \oplus M$. If $s \leq g-2$, $W(s,L,M)$ will denote the subset of $V(s,L,M)$ formed by the bundles with exactly two subbundles of degree 0; by definition of $V(s,L,M)$ one of these subbundles is

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isomorphic to L and the other one is isomorphic to M. All sets $V(s,L)$, $W(s,L)$, $V(s,L,M)$ and $W(s,L,M)$ are algebraic subsets of $M(X;2,s)$ and we will see them as schemes with the associated reduced structure. Tensoring any rank 2 vector bundle with $L \otimes M^*$ we obtain $V(s,M) \cong V(s,L)$ and $W(s,M) \cong W(s,L)$. Tensoring with any $R \in \mathrm{Pic}^0(X)$ we obtain $V(s,L,M) \cong V(s,L\otimes R, M\otimes R)$ and $W(s,L,M) \cong W(s,L\otimes R,M\otimes R)$ for all s, L and M. The proofs of [12, 3.1 and 3.3] show that for every $s \le g-2$ and every L, the scheme $V(s,L)$ is irreducible of dimension $2g-2+s$ and $W(s,L)$ is a non-empty open subset of $W(s,L)$.

For every integer $s > 0$ and every $R \in \mathrm{Pic}^0(X)$, set $D(R,s) := \{(P_1,\dots,P_s, Q_1,\dots,Q_s) \in X^{2s} : R \cong \mathbf{O}_X(\sum_{1\le i\le s} P_i - \sum_{1\le i\le s} Q_i)\}$. We will see $D(R,s)$ as a closed subset of the product X^{2s} with the reduced structure. We have a map $\pi(s)\colon X^{2s} \to \mathrm{Pic}^0(X)$ defined by $\pi(s)((P_1,\dots,P_s,Q_1,\dots,Q_s)) := \mathbf{O}_X(\sum_{1\le i\le s} P_i - \sum_{1\le i\le s} Q_i)$. Since X is complete, this map is proper. This map is surjective if and only if $2s \ge g$ (*e.g.* fix $P \in X$ and check by induction on t, $0 \le t < s$ that, with the notation of [8], $W_s-(s-t)P-W_t \ne W_s-(s-t-1)P-W_{t+1}$ as subset of $\mathrm{Pic}^0(X)$ and hence $\dim\big(W_s-(s-t)P-W_t\big)) = \min\{g, s+t\}$ for all s, t. For the same reason if $0 < 2s \le g$, $\dim(\mathrm{Im}\big(\pi(s)\big) = 2s$, *i.e.*, the map $\pi(s)$ is generically finite. Since $\pi(s)^{-1}(R) = D(R,s)$ for every $R \in \mathrm{Pic}^0(X)$, a simple computation of dimensions gives the following remark.

Remark 1.1

(i) If R is general in $\mathrm{Pic}^0(X)$ and $2s < g$ we have $D(R,s) = \varnothing$.
(ii) If R is general in $\mathrm{Pic}^0(X)$ and $g \le 2s \le 2g-4$ we have $\dim\big(D(R,s)\big) = 2s-g < s$.
(iii) If $0 < 2s \le g$ the set of all $R \in \mathrm{Pic}^0(X)$ with $D(R,s)$ finite and not empty has dimension $2s$.
(iv) For every integer s with $0 < 2s \le g$ and every integer $t > 0$ the set $A(s,t) := \{R \in \mathrm{Pic}^0(X) : D(R,s) \ne \varnothing$ and $D(R,s)$ has dimension $\ge t\}$ is a subset of $\mathrm{Pic}^0(X)$ with $\dim\big(A(s,t)\big) \le 2s-t-1$.

Let $\mathbf{E}(s,L,M)$ be the set of all isomorphism classes of bundles obtained from $L \oplus M$ making s positive elementary transformations. $\mathbf{E}(s,L,M)$ is parametrized in a natural way (but not one to one) by an irreducible variety of dimension $2s$: we choose s points of X and for each of these points we choose a positive elementary transformation supported by that point. First we study the case $s = 1$. Fix $P \in X$. There are three possibilities for a positive elementary transformation of $L \oplus M$ supported by P. Two of them are very special and they give bundles isomorphic to $L(P) \oplus M$ or $L \oplus M(P)$, *i.e.*, are the positive elementary transformations corresponding to the set $\mathbf{T}(1,L,M)$ considered below. The other one has an associated degree rank 2 bundle, F, with $\deg(F) = 1$ a bundle in which both L and M are saturated. Since $H^0\big(X, \mathrm{End}(L\oplus M)\big) = 2$ the group $\mathrm{Aut}(L\oplus M)$ acts transitively on the set of lines of $\mathbf{P}\big((L\oplus M) \mid \{P\}\big) \setminus \{\mathbf{P}(L \mid \{P\}) \cup \mathbf{P}(M \mid \{P\})\}$ which is the complement of 2 points in $\mathbf{P}^1$. This imples that the isomorphism class of F is uniquely determined by L, M and P, *i.e.*, that $\mathbf{E}(1,L,M) \setminus \mathbf{T}(1,L,M)$ is parametrized one to one by an irreducible variety of dimension 1. It is easy to check that any such F is stable and in particular simple. Hence this phenomenon does not occur for $s \ge 2$ and we see that $\mathbf{E}(s,L,M) \setminus \mathbf{T}(s,L,M)$ is parametrized by an irreducible variety of dimension $2s-1$.

Lemma 1.2 *Fix $L, M \in \mathrm{Pic}^0(X)$ with $L \ne M$ and an integer s with $0 < s \le g-2$.*

(a) The reduced scheme $V(s, L, M)$ is either empty or irreducible of dimension at most $2s - 1$.
(b) The reduced scheme $W(s, L, M)$ is either empty or irreducible of dimension at most $2s - 1$.

Proof Every $E \in V(s, L, M)$ is obtained from $L \oplus M$ making s positive elementary transformations, *i.e.*, $V(s, L, M) \subseteq \mathbf{E}(s, L, M)$. Vice versa, if $V(s, L, M) \neq \varnothing$, the openness of stability and the semicontinuity of the Lange invariant gives that a general element of $\mathbf{E}(s, L, M)$ is stable and with Lange invariant s. Hence either $V(s, L, M)$ is empty or it is irreducible. The same proof gives part (ii). ■

Now we will describe the set $\mathbf{T}(s, L, M)$ of isomorphism classes of all vector bundles, A, obtained from $L \oplus M$ making s positive elementary transformations and such that either L or M is not saturated in A; for instance if L is not saturated in A, then A is obtained from E taking $P \in X$, making the uniquely determined positive elementary transformation supported by P such that the saturation of L into the corresponding degree 1 rank 2 bundle is isomorphic to $L(P)$ and then making $s-1$ arbitrary positive elementary transformations. Hence $\mathbf{T}(s, L, M)$ is parametrized (*a priori* not necessarily one to one or even generically finite to one) by the union (not the disjoint union) of two varieties of dimension $2s - 1$; indeed the discussion on the set $\mathbf{E}(1, L, M) \setminus \mathbf{T}(1, L, M)$ made before shows that for every integer $s \geq 2$ $\mathbf{T}(s, L, M)$ is parametrized by the union of two varieties of dimension $2s - 2$. We have $\mathbf{T}(s, L, M) \neq \mathbf{E}(s, L, M)$ for every $s > 0$. Fix any vector bundle F obtained from $L \oplus M$ making s positive elementary transformations. Hence there are rank 2 subsheaves F_t, $0 \leq t \leq s$ of F with $F_0 = L \oplus M$, $F_t \subset F_{t+1}$ for $0 \leq t < s$, $F_s = F$ and $\deg(F_t) = t$ for every t. Let R be a subbundle of F with maximal degree. Set $m := \deg(R)$, $R_t := E_t \cap R$ and $m(t) := \deg(R_t)$. We have $m(t-1) \leq m(t) \leq m(t-1) + 1$, $1 \leq t \leq s$. We assume $F \notin \mathbf{T}(s, L, M)$. $F \in V(s, L, M)$ (or, equivalently, $F \in M(X; 2, s, s)$) if and only if for every such R we have $m \geq 0$, while $F \in W(s, L, M)$ if and only if for every such R we have $m < 0$. F is stable if and only if for every such R we have $2m < s$. If $F_{s-1} \notin V(s-1, L, M)$ (resp. $F_{s-1} \notin W(s-1, L, M)$) then $F \notin V(s, L, M)$ (resp. $F \notin W(s, L, M)$). If $F_{s-1} \in W(s-1, L, M)$ we have $\deg(R_{s-1}) < 0$ and this is true not only for R but for all maximal degree line subbundles of F different from L and M. Hence if $F_{s-1} \in W(s-1, L, M)$ and $F \notin \mathbf{T}(s, L, M)$, then $F \in V(s, L, M)$. Since $F \notin \mathbf{T}(s, L, M)$ there is a non-zero map $R_0 \to L$ and a non-zero map $R_0 \to M$, *i.e.*, there is an integer x with $0 < x \leq s$, $R_0 \in \mathrm{Pic}^{-X}(X)$ and effective degree x divisors D, D' with $L \cong R_0(D)$ and $M \cong R_0(D')$; in particular we have $L \otimes M^* \cong \mathbf{O}_X(D - D')$. By definition we have $x = m(0)$. Since $x \leq s$ we obtain the following result.

Lemma 1.3 *Fix $L, M \in \mathrm{Pic}^0(X)$ with $L \neq M$ and an integer s with $0 < s \leq g - 2$.*

(i) If $D(L \otimes M^*, s) = \varnothing$ we have $\mathbf{E}(s, L, M) \setminus \mathbf{T}(s, L, M) \subseteq W(s, L, M)$.
(ii) If $D(L \otimes M^*, s-1) = \varnothing$ we have $\mathbf{E}(s, L, M) \setminus \mathbf{T}(s, L, M) \subseteq V(s, L, M)$.

From now on we will study the case in which F is the general element of $\mathbf{E}(s, L, M)$.

Lemma 1.4 *Fix $L, M \in \mathrm{Pic}^0(X)$ with $L \neq M$ and an integer s with $0 < s \leq g - 2$. Assume $\dim\big(D(L \otimes M^*, t)\big) \leq t - 1$ for every integer t with $1 \leq t \leq s$. Then a general element of $\mathbf{E}(s, L, M)$ belongs to $W(s, L, M)$.*

Proof Fix a sequence of s general positive elementary transformations of $L \oplus M$, *i.e.*, fix rank 2 vector bundles F_t, $0 \le t \le s$ of F with $F_0 = L \oplus M$, $F_t \subset F_{t+1}$ for $0 \le t < s$, $\deg(F_t) = t$ for every t and such that for every integer t with $1 \le t \le s$, F_t is a "general" element of $\mathbf{E}(s, L, M)$. Set $F := F_t$. In order to obtain a contradiction we assume that F is not in $W(s, L, M)$. For every integer u with $0 \le u \le s$, let $B(u)$ be the set of all rank 1 subsheaves of $L \oplus M$ with degree $-u$ and not contained in L or in M. For all integers u, v, t with $0 \le v \le u$ and $0 < t \le s$, let $A(u, v, t)$ be the set of all elements of $B(u)$ whose saturation in F_t has degree at least $-v$. By assumption we have $\dim\bigl(B(u, u, 0)\bigr) \le u - 1$ for every integer u with $0 < u \le s$. By the generality of the positive elementary transformation giving F_{t+1} from F_t and induction on t we obtain that for all u, v, t with $t < s$ either $A(u, v, t) = \varnothing$ or $\dim\bigl(A(u, v, t + 1)\bigr) < \dim\bigl(A(u, v, t)\bigr)$. ■

Lemma 1.5 *Fix $L, M \in \mathrm{Pic}^0(X)$ with $L \ne M$ and an integer s with $0 < s \le g - 1$. Assume $\dim\bigl(D(L \otimes M^*, t)\bigr) \le t$ for every integer t with $1 \le t \le s$. Then a general element of $\mathbf{E}(s, L, M)$ belongs to $V(s, L, M)$.*

Notice that in the statement of Lemma 1.5 the case $s = g - 1$ is allowed. This case will be used to prove Theorem 0.2.

Proof of Theorem 0.1 We stress again the openness of stability and the semicontinuity of the Lange invariant. By Remark 1.1, Lemma 1.2 and Lemma 1.3 (the case $2s < g$) plus the discussion on the general element of $\mathbf{E}(s, L, M)$ (the case $2s \ge g$) we obtain the non-emptyness of $W(X; 2, d, s; 2)$ and hence of $V(X; 2, d, s; 2)$. The irreducibility of both schemes follows from the irreducibility of $\mathbf{E}(s, L, M)$ as in Lemma 1.2. To obtain $\dim\bigl(V(X; 2, d, s; 2)\bigr) = 2g + 2s - 1$ and $\dim\bigl(W(X; 2, d, s; 2)\bigr) = 2g + 2s - 1$ it is sufficient to show that every $E \in \bigl(V(X; 2, d, s; 2) \cap V(s, L, M)\bigr)$ is in a unique way obtained from $L \oplus M$ making s positive elementary transformations, *i.e.*, that the injective map of sheaves $L \oplus M \to E$ (which is assumed to exist) is unique. This is obvious if $E \in W(s, L, M)$, but it is true even assuming only $E \in V(s, L, M)$ because the maps $L \to E$ and $M \to E$ are uniquely determined by [9, Lemma 2.1], *i.e.*, by the injectivity of the map $\pi_E \colon V(E) \to \mathrm{Pic}^{(d-s(E))/2}(X)$. ■

An alternative proof of 0.1 could be given using in a more efficient way the set-up of [12], in particular the proofs of [12, 1.1, 2.3, Remark at p. 59, and 3.3], but we prefer to give this proof to obtain 0.2, too.

Now we will consider rank 2 vector bundles with exactly 3 maximal degree subbundles.

Proposition 1.6 *Fix $L, M \in \mathrm{Pic}^0(X)$ with $L \ne M$ and an integer s with $0 < s \le g - 2$. Assume $\dim\bigl(D(L \otimes M^*, t)\bigr) \le t - 1$ for every integer t with $1 \le t \le s$. Assume the existence of an integer v with $0 < v \le s$ and $D(L \otimes M^*, v) \ne \varnothing$. Then there exists $E \in \bigl(\mathbf{E}(s, L, M) \cap V(s, L, M)\bigr)$ such that E has exactly 3 line subbundles of degree 0, i.e., with $\mathrm{card}\bigl(V(E)\bigr) = 3$.*

Proof Let u be the minimal integer > 0 with $D(L \otimes M^*, u) \ne \varnothing$. Since $D(L \otimes M^*, u) \ne \varnothing$ and u is minimal with this property there exists $U \in \mathrm{Pic}^{-u}(X)$ such that there is an embedding of U into $L \oplus M$ as saturated subbundle. We take any such U which is a sufficiently

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general element of an irreducible component of $D(L \otimes M^*, u)$. We may take u positive elementary transformations of $L \oplus M$ such that the saturation, U', of U into the corresponding degree U rank 2 vector bundle, F, has degree 0. Furthermore, by the generality of U we may assume that U is the unique element of $D(L \otimes M^*, u)$ with this property. Then we take a degree s rank 2 vector bundle $E \in \mathbf{E}(s, L, M)$ obtained from F applying $s - u$ general positive elementary transformations. The proof of Lemma 1.5 shows that E has no line subbundle of positive degree and that L, M and U' are the unique line subbundles of E with degree 0. ■

Proof of 0.2 Just use Lemma 1.5 for $s = g - 1$ and the fact that $\mathrm{Pic}^0(X) \times \mathrm{Pic}^0(Y)$ is irreducible. A minor point: to check that a general $E \in M(X; 2, d, g-1)$ is contained in some $V(g-1, L, M)$ we need $\delta \neq 1$; by [11] we have $\delta = 2^g$. ■

2 Infinitely Many Subbundles

In this section we will prove Proposition 0.5. It seems to us an interesting problem to know for which integers $s < g$ a smooth genus g curve with general moduli has a rank 2 stable vector bundle E with $s(E) = s$ and such that E has infinitely many rank 1 subbundles with maximal degree $\deg(E) - 2s$.

Lemma 2.1 *Assume* $\mathrm{char}(\mathbf{K}) \neq 2$. *Let* $Y \subset \mathbf{P}^{g-1}$ *be the canonical embedding of a general smooth curve of genus* $g \geq 3$. *Assume the existence of an effective divisor* D *on* Y *with* $\deg(D) = s > 0$ *such that* $2D$ *spans a linear subspace* $\langle 2D \rangle$ *of dimension* $2s - 2$ *and the corresponding* g^1_{2s} *is base point free and complete. Then* $3s \leq g - 2$.

Proof Let U be the non-empty Zariski open dense subset of the moduli scheme M_g parametrizing curves without non-trivial automorphisms. On U there is a universal curve, C, and on C a universal scheme G^1_{2s} parametrizing the pencils of degree $2s$. Since $\rho(g, 1, 2s) \geq 1$, restricting U to a Zariski open subset (call it U, again) we may assume that G^1_{2s} is non-empty, smooth of dimension $\rho(g,1,2s) + 3g - 3$ (Brill-Noether theory (see *e.g.* [3, Ch. IV and Ch. V])) and connected [6] and hence irreducible; for the connectedness when $\mathrm{char}(\mathbf{K}) > 0$, see [6, Remark 2.8]; for the smoothness (and hence the irreducibility) for a general X when $\mathrm{char}(\mathbf{K}) > 0$, see [6]. The pair $(Y, 2D)$ corresponds to a base point free complete g^1_{2s} and hence, by the generality of Y, to an element of $\prod := G^1_{2s} \times_{M_g} C$ such that the corresponding map $f\colon Y \to \mathbf{P}^1$ has $2D$ as a fiber, $f^{-1}(o)$. We may even assume D reduced by the deformation theory of pencils. Since $\mathrm{char}(\mathbf{K}) \neq 2$ we may deform each point of $f^{-1}(o)_{red}$ independently inside the total space of $\prod$. Hence by the generality of Y we obtain that $\rho(g, 1, 2s) + 3g - 3 + 1 \geq 3g - 3 + s$, as wanted.

Proof of 0.5 Set $s := s(E)$. By [4, Prop 1.1], there is a one dimensional family of line bundles on X, say $\{L_t\}_{t \in T}$, with $L_t \in \mathrm{Pic}^{2s}(X)$ and L_t spanned and such that there is an effective degree s divisor D_t on X with $L_t \cong \mathbf{O}_X(2D_t)$. If $h^0(X, L_t) \geq 3$ for every t, then $\rho(g, 2, 2s) := g - 3(g + 2 - 2s) \geq 1$ by Brill-Noether theory because X has general moduli (see *e.g.* [3, Ch. V]). Hence we may assume $h^0(X, L_t) = 2$ for some $t \in T$ and hence $h^0\big(X, \mathbf{O}_X(D_t)\big) = 1$ for the corresponding t; we fix one such pair (L_t, D_t). Let $h_K\colon X \to$

$\mathbf{P}^{g-1}$ be the canonical embedding. By the geometric form of Riemann-Roch, $h_K(2D_t)$ spans a linear space of dimension $2s-2$. Hence the result follows from Lemma 2.1

Remark 2.2 The proof of the lower bound $(g+3)/4$ for $s(E)$ and any E with $V(E)$ infinite for X general given in [4, Remark on p. 31], uses only that X if "general" from the point of view of the Brill-Noether theory of pencils, *i.e.*, that if X has infinitely many base point free g^1_{2s}, then $\rho(g,1,2s) := g-2(g+1-2s) \geq 1$. By [2, Th. 2.6], this is true even if X is only assumed to be a general k-gonal curve for some integer k with $2 \leq k \leq g/2$. For the case in which X is a double covering of a curve of genus $q>0$, see Proposition 2.3.

Proposition 2.3 *Fix integers g, n, q, s with $n \geq 2$, $q \geq 0$ $g \geq 2$, $2g-2 \geq n(2q-2)$, $s>0$ and $g > 2ns + sq - n - 2s + 1$. Let X be a smooth curve of genus g such that there is a degree n morphism $\pi\colon X \to Y$ with Y a smooth curve of genus q. Assume that there is no factorization of π, say $\pi = \pi' \circ \pi''$ with $\deg(\pi'') > 1$ and $\deg(\pi') > 1$. Then for every rank 2 vector bundle E on X with $s(E) = s$ and with infinitely many maximal degree rank 1 subbundles there is $A \in \mathrm{Pic}(X)$ and a rank 2 vector bundle F on X with $E \otimes A \cong \pi^*(F)$ and such that for every maximal degree rank 1 subbundle L of E there is a rank 1 subbundle M of F with $\pi^*(M) \otimes A^* \cong L$.*

Proof Let E be a rank 2 vector bundle E on X with $s(E) = s$ and with infinitely many maximal degree rank 1 subbundles. By [4, 1.1], there are infinitely many base point free line bundles, L, on X with $h^0(X,L) \geq 2$ and $\deg(L) = 2s$. By the non-factorizability of the covering π and Castelnuovo-Severi inequality (see *e.g.* [1, Ch. 3]), if char($\mathbf{K}$) is arbitrary, just use that an integral curve $T \subset \mathbf{P}^1 \times Y$ with numerical equivalence class of type $(2s,n)$ has $p_a(T) = 1 + 2ns + sq - n - 2n < g$ by the adjunction formula (and hence X cannot be the normalization of T), for every such L there is $M \in \mathrm{Pic}^{2s/n}(Y)$ with $L \cong \pi^*(M)$ and $H^0(X,L) = \pi^*\bigl(H^0(X,M)\bigr)$. The proofs of [4, Prop. 1.4 and Th. 1] give the result. ∎

Remark 2.4 Note that if n is prime (and in particular if $n=2$) no degree n morphism $\pi\colon X \to Y$ has a factorization $\pi = \pi' \circ \pi''$ with $\deg(\pi') > 1$ and $\deg(\pi'') > 1$.

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