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HILBERT-KUNZ MULTIPLICITY OF THREE-DIMENSIONAL LOCAL RINGS  
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Abstract. In this paper, we investigate the lower bound $s_{HK}(p, d)$ of Hilbert-Kunz multiplicities for non-regular unmixed local rings of Krull dimension $d$ containing a field of characteristic $p > 0$. Especially, we focus on three-dimensional local rings. In fact, as a main result, we will prove that $s_{HK}(p, 3) = 4/3$ and that a three-dimensional complete local ring of Hilbert-Kunz multiplicity $4/3$ is isomorphic to the non-degenerate quadric hypersurface $k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$ under mild conditions. Furthermore, we pose a generalization of the main theorem to the case of $\dim A \geq 4$ as a conjecture, and show that it is also true in case $\dim A = 4$ using the similar method as in the proof of the main theorem.

Introduction

Let $A$ be a commutative Noetherian ring containing an infinite field of characteristic $p > 0$ with unity. In [15], Kunz proved the following theorem, which gives a characterization of regular local rings of positive characteristic.

Kunz’ Theorem. ([15]) Let $(A, m, k)$ be a local ring of characteristic $p > 0$. Then the following conditions are equivalent:

1. $A$ is a regular local ring.
2. $A$ is reduced and is flat over the subring $A^p = \{a^p : a \in A\}$. In other words, the Frobenius map $F : A \to A (a \mapsto a^p)$ is flat.
3. $l_A(A/m^{[q]}) = q^d$ for any $q = p^e$, $e \geq 1$, where $m^{[q]} = \langle a^q : a \in m \rangle$ and $l_A(M)$ denotes the length of an $A$-module $M$.

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Furthermore, in [16], Kunz observed that \( l_A(A/m^{[q]})/q^d \) \((q = p^e)\) is a reasonable measure for the singularity of a local ring. Based on the idea of Kunz, Monsky [18] proved that there exists a constant \( c = c(A) \) such that

\[
l_A(A/m^{[q]}) = cq^d + O(q^{d-1})
\]

and defined the notion of *Hilbert-Kunz multiplicity* by \( e_{HK}(A) = c \). In 1990’s, Han and Monsky [10] have given an algorism to compute the Hilbert-Kunz multiplicity for any hypersurface of Briskorn-Fermat type

\[
A = k[X_0, \ldots, X_n]/(X_0^{d_0} + \cdots + X_n^{d_n}).
\]

See e.g. [1], [2], [4], [24] about the other examples. Hochster and Huneke [11] have given a “Length Criterion for Tight Closure” in terms of Hilbert-Kunz multiplicity (see Theorem 1.8) and indicated the close relation between tight closure and Hilbert-Kunz multiplicity. In [22], the authors proved a theorem which gives a characterization of regular local rings in terms of Hilbert-Kunz multiplicity:

**Theorem A.** ([22, Theorem 1.5]) \( (A, m, k) \) be an unmixed local ring of positive characteristic. Then \( A \) is regular if and only if \( e_{HK}(A) = 1 \).

Many researchers have tried to improve this theorem. For example, Blickle and Enescu [3] recently proved the following theorem:

**Theorem B.** (Blickle-Enescu [3]) \( (A, m, k) \) be an unmixed local ring of characteristic \( p > 0 \). Then the following statements hold:

1. If \( e_{HK}(A) < 1 + \frac{1}{p^e} \), then \( A \) is Cohen-Macaulay and \( F \)-rational.
2. If \( e_{HK}(A) < 1 + \frac{1}{p^d} \), then \( A \) is regular.

So it is natural to consider the following problem:

**Problem C.** Let \( d \geq 2 \) be any integer. Determine the lower bound \( s_{HK}(p, d) \) of Hilbert-Kunz multiplicities for \( d \)-dimensional non-regular unmixed local rings of characteristic \( p \). Also, characterize the local rings \( A \) for which \( e_{HK}(A) = s_{HK}(p, d) \) holds.

In case of one-dimensional local rings, it is easy to answer to this problem. In fact, \( s_{HK}(p, 1) = 2 \); \( e_{HK}(A) = 2 \) if and only if \( e(A) = 2 \). In case of two-dimensional Cohen-Macaulay local rings, the authors [23] have given a complete answer to this problem. Namely, we have \( s_{HK}(p, 2) = \frac{3}{2} \) by the theorem below.
Theorem D. (see also Corollary 2.6) Let \((A, \mathfrak{m}, k)\) be a two-dimensional Cohen-Macaulay local ring of positive characteristic. Put \(e = e(A)\), the multiplicity of \(A\). Then the following statements hold:

1. \(e_{HK}(A) \geq \frac{e+1}{2}\).
2. Suppose that \(k = \overline{k}\). Then \(e_{HK}(A) = \frac{e+1}{2}\) holds if and only if the associated graded ring \(\text{gr}_\mathfrak{m}(A)\) is isomorphic to the Veronese subring \(k[X, Y]^{(e)}\).

In the following, let us explain the organization of this paper. In Section 1, we recall the notions of Hilbert-Kunz multiplicity and tight closure etc. and gather several fundamental properties of them. In particular, Goto-Nakamura’s theorem (Theorem 1.9) is important because it plays a central role in the proof of the main result (Theorem 3.1).

In Section 2, we give a key result to estimate Hilbert-Kunz multiplicities for local rings of lower dimension. Indeed, Theorem 2.2 is a refinement of the argument in [23, Section 2]. Also, the point of our proof is to estimate \(l_{A}(\mathfrak{m}^{[q]} / \mathfrak{J}^{[q]})\) (where \(J\) is a minimal reduction of \(\mathfrak{m}\)) using volumes in \(\mathbb{R}^d\).

In Section 3, we prove the following theorem as the main result in this paper.

**Theorem 3.1.** Let \((A, \mathfrak{m}, k)\) be a three-dimensional unmixed local ring of characteristic \(p > 0\). Then the following statements hold.

1. If \(A\) is not regular, then \(e_{HK}(A) \geq \frac{4}{3}\).
2. Suppose that \(k = \overline{k}\) and \(\text{char } k \neq 2\). Then the following conditions are equivalent:
   - (a) \(e_{HK}(A) = \frac{4}{3}\).
   - (b) \(\hat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)\).

Also, we study lower bounds on \(e_{HK}(A)\) for local rings \(A\) having a given (small) multiplicity \(e\). In particular, we will prove that any three-dimensional unmixed local ring \(A\) with \(e_{HK}(A) < 2\) is \(F\)-rational.

In Section 4, we consider a generalization of Theorem 3.1 and pose the following conjecture:

**Conjecture 4.2.** Let \(d \geq 1\) be an integer and \(p > 2\) a prime number. Put

\[ A_{p,d} := \mathbb{F}_p[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2). \]
Let \((A, \mathfrak{m}, k)\) be a \(d\)-dimensional unmixed local ring with \(k = \mathbb{F}_p\). Then the following statements hold.

(1) If \(A\) is not regular, then \(e_{HK}(A) \geq e_{HK}(A_{p,d}) \geq 1 + \frac{c_d}{d!}\) (see 4.2 for the definition of \(c_d\)). In particular, \(s_{HK}(p, d) = e_{HK}(A_{p,d})\).

(2) If \(e_{HK}(A) = e_{HK}(A_{p,d})\), then the \(\mathfrak{m}\)-adic completion \(\widehat{A}\) of \(A\) is isomorphic to \(A_{p,d}\) as local rings.

Also, we prove that this is true in case of \(\dim A = 4\). Namely we will prove the following theorem.

**Theorem 4.3.** Let \((A, \mathfrak{m}, k)\) be a four-dimensional unmixed local ring of characteristic \(p > 0\). Also, suppose that \(k = \mathbb{F}_p\) and \(\text{char } k \neq 2\). Then \(e_{HK}(A) \geq \frac{5}{4}\) if \(e(A) \geq 3\). Also, the following statements hold.

(1) If \(A\) is not regular, then \(e_{HK}(A) \geq e_{HK}(A_{p,4}) = \frac{29p^2 + 15}{24p^2 + 12}\).

(2) The following conditions are equivalent:

(a) Equality holds in (1).

(b) \(e_{HK}(A) < \frac{5}{4}\).

(c) \(\widehat{A}\) is isomorphic to \(A_{p,4}\).

### §1. Preliminaries

Throughout this paper, let \(A\) be a commutative Noetherian ring with unity. Furthermore, we assume that \(A\) has a positive characteristic \(p\), that is, it contains a prime field \(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}\), unless otherwise specified. For every positive integer \(e\), let \(q = p^e\). If \(I\) is an ideal of \(A\), then \(I^{[q]} = (a^q : a \in I)A\). Also, we fix the following notation: \(l_A(M)\) (resp. \(\mu_A(M)\)) denotes the length (resp. the minimal number of generators) of \(M\) for any finitely generated \(A\)-module \(M\).

First, we recall the notion of Hilbert-Kunz multiplicity (see [15], [16], [18]). Also, see [17] or [20] for usual multiplicity.

**Definition 1.1.** (multiplicity, Hilbert-Kunz multiplicity) Let \((A, \mathfrak{m}, k)\) be a local ring of characteristic \(p > 0\) with \(\dim A = d\). Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(A\), and let \(M\) be a finitely generated \(A\)-module. The (Hilbert-Samuel) multiplicity \(e(I, M)\) of \(I\) with respect to \(M\) is defined by

\[
e(I, M) = \lim_{n \to \infty} \frac{d!}{n^d} l_A(M/I^n M).
\]
The Hilbert-Kunz multiplicity $e_{HK}(I, M)$ of $I$ with respect to $M$ is defined by

$$e_{HK}(I, M) = \lim_{q \to \infty} \frac{l_A(M/I^{[q]}M)}{q^d}.$$ 

By definition, we put $e(I) = e(I, A)$ (resp. $e_{HK}(I) = e_{HK}(I, A)$) and $e(A) = e(m)$ (resp. $e_{HK}(A) = e_{HK}(m)$).

We recall several basic results on Hilbert-Kunz multiplicity.

**Proposition 1.2.** (Fundamental properties (cf. [13], [15], [16], [18], [22])) Let $(A, m, k)$ be a local ring of positive characteristic. Let $I, I'$ be $m$-primary ideals of $A$, and let $M$ be a finitely generated $A$-module. Then the following statements hold.

1. If $I \subseteq I'$, then $e_{HK}(I) \geq e_{HK}(I')$.
2. $e_{HK}(A) \geq 1$.
3. $\dim M < d$ if and only if $e_{HK}(I, M) = 0$.
4. If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated $A$-modules, then 

   $$e_{HK}(I, M) = e_{HK}(I, L) + e_{HK}(I, N).$$

5. (Associative formula)

   $$e_{HK}(I, M) = \sum_{p \in \text{Assh}(A)} e_{HK}(I, A/p) \cdot l_{A/p}(M_p),$$

   where $\text{Assh}(A)$ denotes the set of prime ideals $p$ of $A$ with $\dim A/p = \dim A$.

6. If $J$ is a parameter ideal of $A$, then $e_{HK}(J) = e(J)$. In particular, if $J$ is a minimal reduction of $I$ (i.e., $J$ is a parameter ideal which is contained in $I$ and $I^{r+1} = JJ^r$ for some integer $r \geq 0$), then $e_{HK}(J) = e(I)$.

7. If $A$ is regular, then $e_{HK}(I) = l_A(A/I)$.

8. (Localization) $e_{HK}(A_p) \leq e_{HK}(A)$ holds for any prime ideal $p$ such that $\dim A/p + \text{height } p = \dim A$.

9. If $x \in I$ is $A$-regular, then $e_{HK}(I) \leq e_{HK}(I/xA)$.
If \((A, \mathfrak{m}) \to (B, \mathfrak{n})\) is a flat local ring homomorphism such that \(B/\mathfrak{m}B\) is a field, then \(e_{\text{HK}}(I) = e_{\text{HK}}(IB)\).

**Remark 1.** Also, the similar result as above (except (6), (7)) holds for usual multiplicities.

Let \((A, \mathfrak{m}, k)\) be any local ring of positive dimension. The associated graded ring \(\text{gr}_\mathfrak{m}(A)\) of \(A\) with respect to \(\mathfrak{m}\) is defined as follows:

\[
\text{gr}_\mathfrak{m}(A) := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}.
\]

Then \(G = \text{gr}_\mathfrak{m}(A)\) is a homogeneous \(k\)-algebra such that \(\mathfrak{M} := G_+\) is the unique homogeneous maximal ideal of \(G\). If \(\text{char} A = p > 0\) and \(\text{dim} A = d\), then \(G_{\mathfrak{M}}\) is also a local ring of characteristic \(p\) with \(\text{dim} G_{\mathfrak{M}} = d\).

**Proposition 1.3.** ([22, Theorem (2.15)]) Let \((A, \mathfrak{m}, k)\) be a local ring of positive characteristic. Let \(G = \text{gr}_\mathfrak{m}(A)\) the associated graded ring of \(A\) with respect \(\mathfrak{m}\) as above. Then \(e_{\text{HK}}(A) \leq e_{\text{HK}}(G_{\mathfrak{M}}) \leq e(A)\).

**Remark 2.** We use the same notation as in the above proposition. Although \(e(A) = e(G_{\mathfrak{M}})\) always holds, \(e_{\text{HK}}(A) = e_{\text{HK}}(G_{\mathfrak{M}})\) seldom holds.

**Proposition 1.4.** (cf. [13]) Let \((A, \mathfrak{m}, k)\) be a local ring of positive characteristic with \(d = \text{dim} A\). Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(A\). Then

\[
\frac{e(I)}{d!} \leq e_{\text{HK}}(I) \leq e(I).
\]

Also, if \(d \geq 2\), then the inequality in the left-hand side is strict; see [9].

We say that a local ring \(A\) is **unmixed** if \(\text{dim} \widehat{A}/\mathfrak{p} = \text{dim} \widehat{A}\) holds for any associated prime ideal \(\mathfrak{p}\) of \(\widehat{A}\). The following theorem is an analogy of Nagata’s theorem ([20, (40.6)]), which is a starting point in this article.

**Theorem 1.5.** ([22, Theorem (1.5)]) Let \((A, \mathfrak{m}, k)\) be an unmixed local ring of positive characteristic. Then \(A\) is regular if and only if \(e_{\text{HK}}(A) = 1\).

It is not so easy to compute Hilbert-Kunz multiplicities in general. However, one has simple formulas for them in case of quotient singularities and in case of binomial hypersurfaces; see below or [4, Theorem 3.1].
Theorem 1.6. (cf. [22, Theorem (2.7)]) Let \((A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n})\) be a module-finite extension of local domains of positive characteristic. Then for every \(\mathfrak{m}\)-primary ideal \(I\) of \(A\), we have
\[
e_{HK}(I) = \frac{e_{HK}(IB)}{[Q(B) : Q(A)]} \cdot [B/\mathfrak{n} : A/\mathfrak{m}],
\]
where \(Q(A)\) denotes the fraction field of \(A\).

Now let us see some examples of Hilbert-Kunz multiplicities which are given by the above formula. First, we consider the Veronese subring \(A\) defined by
\[
A = k[[X_1^{i_1} \cdots X_d^{i_d} : i_1, \ldots, i_d \geq 0, \sum i_j = r]].
\]

Applying Theorem 1.6 to \(A \hookrightarrow B = k[[X_1, \ldots, X_d]]\), we get
\[
(1.1) \quad e_{HK}(A) = \frac{1}{r} \left( \binom{d + r - 1}{r - 1} \right).
\]
In particular, if \(d = 2, r = e(A)\), then \(e_{HK}(A) = \frac{e(A) + 1}{2}\).

Next, we consider the homogeneous coordinate rings of quadric hypersurfaces in \(\mathbb{P}_k^n\). Let \(k\) be a field of characteristic \(p > 2\), and let \(R\) be the homogeneous coordinate ring of the hyperquadric \(Q\) defined by \(q = q(X, Y, Z, W)\). Put \(\mathfrak{m} = R_+\), the unique homogeneous maximal ideal of \(R\), and \(A = R_{\mathfrak{m}} \otimes_k \frac{k}{\mathfrak{m}}\). By suitable coordinate transformation, we may assume that \(\hat{A}\) is isomorphic to one of the following rings:

\[
(1.2) \quad \begin{cases} k[[X, Y, Z, W]]/(X^2), & \text{if rank}(q) = 1, \\
 k[[X, Y, Z, W]]/(X^2 - YZ), & \text{if rank}(q) = 2, \\
 k[[X, Y, Z, W]]/(XY - ZW), & \text{if rank}(q) = 3.
\end{cases}
\]

Then \(e_{HK}(A) = 2, \frac{3}{2}, \text{ or } \frac{4}{3}\), respectively.

In order to state other important properties of Hilbert-Kunz multiplicity, the notion of tight closure is very important. See [11], [12], [13] for definition and the fundamental properties of tight closure. In particular, the notion of \(F\)-rational ring is essential in our argument.

Definition 1.7. ([6], [11], [12]) Let \((A, \mathfrak{m}, k)\) be a local ring of positive characteristic. We say that \(A\) is weakly \(F\)-regular (resp. \(F\)-rational) if every ideal (resp. every parameter ideal) is tightly closed. Also, \(A\) is \(F\)-regular (resp. \(F\)-rational) if any local ring of \(A\) is weakly \(F\)-regular (resp. \(F\)-rational).
Note that an $F$-rational local ring is normal and Cohen-Macaulay. Hochster and Huneke have given the following criterion of tight closure in terms of Hilbert-Kunz multiplicity.

**Theorem 1.8.** (Length Criterion for Tight Closure (cf. [11, Theorem 8.17])) Let $I \subseteq J$ be $m$-primary ideals of a local ring $(A, m, k)$ of positive characteristic.

1. If $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$.
2. Suppose that $A$ is excellent, reduced and equidimensional. Then the converse of (1) is also true.

The following theorem plays an important role in studying Hilbert-Kunz multiplicities for non-Cohen-Macaulay local rings.

**Theorem 1.9.** (Goto-Nakamura [8]) Let $(A, m, k)$ be an equidimensional local ring which is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$. Then

1. If $J$ is a parameter ideal of $A$, then $e(J) \geq l_A(A/J^*)$.
2. Suppose that $A$ is unmixed. If $e(J) = l_A(A/J^*)$ for some parameter ideal $J$, then $A$ is $F$-rational (hence is Cohen-Macaulay).

The next corollary is well-known in case of Cohen-Macaulay local rings (e.g. see [13]).

**Corollary 1.10.** Let $(A, m, k)$ be an unmixed local ring of characteristic $p > 0$. Suppose that $e(A) = 2$. Then $\hat{A}$ is $F$-rational if and only if $e_{HK}(A) < 2$. When this is the case, $A$ is an $F$-rational hypersurface.

**Proof.** Since any Cohen-Macaulay local ring of multiplicity 2 is a hypersurface, it suffices to prove the first statement.

We may assume that $A$ is complete and $k$ is infinite. We can take a minimal reduction $J$ of $m$. First, suppose that $e_{HK}(A) < 2$. Then we show that $A$ is Cohen-Macaulay, $F$-rational. By Goto-Nakamura’s theorem, we have $2 = e(J) \geq l_A(A/J^*)$. If equality does not hold, then $l_A(A/J^*) = 1$, that is, $J^* = m$. Then $e_{HK}(A) = e_{HK}(J^*) = e_{HK}(J) = e(J) = 2$ by Proposition 1.2. This is a contradiction. Hence $e(J) = l_A(A/J^*)$. By Goto-Nakamura’s theorem again, we obtain that $A$ is Cohen-Macaulay, $F$-rational.
Conversely, suppose that \( A \) is a complete \( F \)-rational local ring. Then since \( A \) is Cohen-Macaulay and \( J^* = J \neq m \), we have \( e_{HK}(A) < e_{HK}(J) = e(J) = 2 \) by the Length Criterion for Tight Closure.

The next question is open in general. However, we will show that it is true for \( \text{dim} A \leq 3 \); see Section 3.

**Question 1.11.** If \( A \) is an unmixed local ring with \( e_{HK}(A) < 2 \), then is it \( F \)-rational?

### §2. Estimate of Hilbert-Kunz multiplicities

In this section, we will prove the key result to find a lower bound on Hilbert-Kunz multiplicities. Actually, it is a refinement of the argument which appeared in [22, Section 5] or in [23, Section 2]. The point is to use the tight closure \( J^* \) instead of “a parameter ideal \( J \) itself”. This enables us to investigate Hilbert-Kunz multiplicities of non-Cohen-Macaulay local rings. In Sections 3, 4, we will apply our method to unmixed local rings with \( \text{dim} A = 3 \), 4.

Before stating our theorem, we introduce the following notation: Fix \( d > 0 \). For any positive real number \( s \), we put

\[
v_s := \text{vol}\left(\left\{(x_1, \ldots, x_d) \in [0, 1]^d : \sum_{i=1}^d x_i \leq s\right\}\right), \quad v'_s := 1 - v_s,
\]

where \( \text{vol}(W) \) denotes the volume of \( W \subseteq \mathbb{R}^d \). Then it is easy to see the following fact.

**Fact 2.1.** Let \( s \) be a positive real number. Using the same notation as above, we have

1. \( v_s + v'_s = 1 \).
2. \( v'_{d-s} = v_s \).
3. \( v_{d/2} = v'_{d/2} = \frac{1}{2} \).
4. If \( 0 \leq s \leq 1 \), then \( v_s = \frac{s^d}{d!} \).

Using the above notation, the key result in this paper can be written as follows:
THEOREM 2.2. Let \((A, m, k)\) be an unmixed local ring of characteristic \(p > 0\). Put \(d = \dim A \geq 1\). Let \(J\) be a minimal reduction of \(m\), and let \(r\) be an integer with \(r \geq \mu_A(m/J^*)\), where \(J^*\) denotes the tight closure of \(J\). Also, let \(s \geq 1\) be a rational number. Then we have

\[
e_{HK}(A) \geq e(A) \left( v_s - r \cdot \frac{(s - 1)^d}{d!} \right).
\]

Remark 3. When \(1 \leq s \leq 2\), the right-hand side in Equation (2.1) is equal to \(e(A)(v_s - r \cdot v_{s-1})\).

Before proving the theorem, we need the following lemma. In what follows, for any positive real number \(\alpha\), we define \(I^\alpha := I^n\), where \(n\) is the minimum integer which does not exceed \(\alpha\).

LEMMA 2.3. Let \((A, m, k)\) be an unmixed local ring of characteristic \(p > 0\) with \(\dim A = d \geq 1\). Let \(J\) be a parameter ideal of \(A\). Using the same notation as above, we have

\[
\lim_{q \to \infty} \frac{l_A(A/J^{sq})}{q^d} = \frac{e(J)s^d}{d!}, \quad \lim_{q \to \infty} l_A \left( \frac{J^{sq} + J^{[q]}}{J^{[q]}} \right) = e(J) \cdot v'_s.
\]

Proof. First, note that our assertion holds if \(A\) is regular and \(J = m\). We may assume that \(A\) is complete. Let \(x_1, \ldots, x_d\) be a system of parameters which generates \(J\), and put \(R := k[[x_1, \ldots, x_d]]\), \(n = (x_1, \ldots, x_d)R\). Then \(R\) is a complete regular local ring and \(A\) is a finitely generated \(R\)-module with \(A/m = R/n\). Since the assertion is clear in case of regular local rings, it suffices to show the following claim.

CLAIM. Let \(I = \{I_q\}_{q=p^e}\) be a set of ideals of \(A\) which satisfies the following conditions:

(1) For each \(q = p^e\), \(I_q = J_q A\) holds for some ideal \(J_q \subseteq R\).

(2) There exists a positive integer \(t\) such that \(n^{tq} \subseteq J_q\) for all \(q = p^e\).

(3) \(\lim_{q \to \infty} l_R(R/J_q)/q^d\) exists.

Then

\[
\lim_{q \to \infty} \frac{l_A(A/I_q)}{q^d} = e(J) \cdot \lim_{q \to \infty} \frac{l_R(R/J_q)}{q^d}.
\]
In fact, since $A$ is unmixed, it is a torsion-free $R$-module of rank $e := e(J)$. Take a free $R$-module $F$ of rank $e$ such that $A_W \cong F_W$, where $W = R \setminus \{0\}$. Since $F$ and $A$ are both torsion-free, there exist the following short exact sequences of finitely generated $R$-modules:

$$0 \to F \to A \to C_1 \to 0, \quad 0 \to A \to F \to C_2 \to 0,$$

where $(C_1)_W = (C_2)_W = 0$. In particular, $\dim C_1 < d$ and $\dim C_2 < d$.

Applying the tensor product $- \otimes_R R/J_q$ to the above two exact sequences, respectively, we get

$$l_A(A/I_q) \leq l_R(F/J_qF) + l_R(C_1/J_qC_1),$$
$$l_R(F/J_qF) \leq l_A(A/I_q) + l_R(C_2/J_qC_2).$$

In general, if $\dim_R C < d$, then

$$\frac{l_R(C/J_qC)}{q^d} \leq \frac{l_R(C/n^{tq}C)}{q^d} \to 0 \quad (q \to \infty).$$

Thus the required assertion easily follows from the above observation.

Proof of Theorem 2.2. For simplicity, we put $L = J^s$ and $e = e(A)$. We will give an upper bound of $l_A(m^{[q]}/J^{[q]})$. First, we have the following inequality:

$$l_A(m^{[q]}/J^{[q]}) \leq l_A\left(\frac{m^{[q]} + m^{sq}}{J^{[q]}}\right)$$
$$= l_A\left(\frac{m^{[q]} + m^{sq}}{L^{[q]} + m^{sq}}\right) + l_A\left(\frac{L^{[q]} + m^{sq}}{L^{[q]} + J^{sq}}\right)$$
$$+ l_A\left(\frac{L^{[q]} + J^{sq}}{J^{[q]} + J^{sq}}\right) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right)$$
$$= \ell_1 + \ell_2 + \ell_3 + \ell_4.$$

Next, we see that $\ell_1 \leq r \cdot l_A(A/J^{(s-1)q}) + O(q^{d-1})$. By our assumption, we can write $m = L + Aa_1 + \cdots + Aa_r$. Since $m^{(s-1)q}a_i^q \subseteq m^{sq} \subseteq m^{sq} + L^{[q]}$, we have

$$\ell_1 = l_A\left(\frac{m^{[q]} + m^{sq}}{L^{[q]} + m^{sq}}\right) \leq \sum_{i=1}^r l_A\left(\frac{Aa_i^q + L^{[q]} + m^{sq}}{L^{[q]} + m^{sq}}\right)$$
$$= \sum_{i=1}^r l_A\left(\frac{A/(L^{[q]} + m^{sq}) : a_i^q}{L^{[q]} + m^{sq}}\right)$$
$$\leq r \cdot l_A(A/m^{(s-1)q}).$$
Since $J$ is a minimal reduction of $m$, we have $l_A(m^{(s-1)q}/J^{(s-1)q}) = O(q^{d-1})$. Thus we have the required inequality. Similarly, we get

$$
\ell_2 = l_A\left(\frac{L[q] + msq}{L[q] + Jsq}\right) \leq l_A(m^{sq}/J^{sq}) = O(q^{d-1}).
$$

Also, we have $l_A(L[q]/J[q]) = O(q^{d-1})$ by Length Criterion for Tight Closure. Hence $\ell_3 = O(q^{d-1})$ and thus

$$
l_A(m[q]/J[q]) \leq r \cdot l_A(A/J^{(s-1)q}) + l_A\left(\frac{J[q] + J^{sq}}{J[q]}\right) + O(q^{d-1}).
$$

It follows from the above argument that

$$
e_{HK}(J) - e_{HK}(m) \leq r \cdot \lim_{q \to \infty} \frac{l_A(A/J^{(s-1)q})}{q^d} + \lim_{q \to \infty} \frac{1}{q^d} l_A\left(\frac{J[q] + J^{sq}}{J[q]}\right)
\leq r \cdot e \cdot \frac{(s-1)^d}{d!} + e \cdot v'_s.
$$

Since $e_{HK}(J) = e(J) = e$, $e_{HK}(A) = e_{HK}(m)$ and $v'_s = 1 - v_s$, we get the required inequality.

The following fact is known, which gives a lower bound on Hilbert-Kunz multiplicities for hypersurface local rings.

**Fact 2.4.** (cf. [1], [2], [22]) Let $(A, m, k)$ be a hypersurface local ring of characteristic $p > 0$ with $d = \dim A \geq 1$. Then

$$
e_{HK}(A) \geq \beta_{d+1} \cdot e(A),
$$

where $\beta_{d+1}$ is given by the following equivalent formulas:

(a) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \theta}{\theta}\right)^{d+1} d\theta;

(b) \quad \frac{1}{2^d d!} \sum_{\ell=0}^{[d/2]} (-1)^\ell (d + 1 - 2\ell)^d \left(\frac{d + 1}{\ell}\right);

(c) \quad \text{vol}\{x \in [0, 1]^d : \frac{d - 1}{2} \leq \sum x_i \leq \frac{d + 1}{2}\} = 1 - v_{d-1/2} - v'_{d+1/2}.$
Table 1.

<table>
<thead>
<tr>
<th>$d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{d+1}$</td>
<td>1</td>
<td>1</td>
<td>3/4</td>
<td>2/3</td>
<td>115</td>
<td>11</td>
<td>5633</td>
</tr>
<tr>
<td></td>
<td>192</td>
<td>11</td>
<td>20</td>
<td>5633</td>
<td>115</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

Remark 4. The above inequality is not best possible in general. In case of $d \geq 4$, one cannot prove the formula in the above fact as a corollary of our theorem. See also Proposition 3.9 and Theorem 4.3.

The following lemma is an analogy of Sally’s theorem: If $A$ is a Cohen-Macaulay local ring, then $\mu_A(m/J) = \mu_A(m) - \dim A \leq e(A) - 1$.

**Lemma 2.5.** Let $(A, m, k)$ be an unmixed local ring of positive characteristic, and let $J$ be a minimal reduction of $m$.

1. $\mu_A(m/J^*) \leq e(A) - 1$.
2. If $A$ is not $F$-rational, then $\mu_A(m/J^*) \leq e(A) - 2$.

**Proof.** We may assume that $A$ is complete and thus is a homomorphic image of a Cohen-Macaulay local ring.

1. By Goto-Nakamura’s Theorem, we have that $\mu_A(m/J^*) \leq l_A(m/J^*) \leq e(J) - 1 = e - 1$.
2. If $A$ is not $F$-rational, then $l_A(A/J^*) \leq e(J) - 1 = e - 1$. Thus $\mu_A(m/J^*) \leq e - 2$, as required.

Using Theorem 2.2 and Lemma 2.5, one can prove the following corollary, which has been already proved in [23] in the case of Cohen-Macaulay local rings.

**Corollary 2.6.** (cf. [23]) Let $(A, m, k)$ be a two-dimensional unmixed local ring of characteristic $p > 0$. Put $e = e(A)$. Then

\begin{equation}
\tag{2.2}
e_{HK}(A) \geq \frac{e + 1}{2}.
\end{equation}

Also, suppose $k = \overline{k}$. Then the equality holds if and only if $\text{gr}_m(A)$ is isomorphic to the Veronese subring $k[X, Y]^{(e)} = k[X^e, X^{e-1}Y, \ldots, XY^{e-1}, Y^e]$.

Moreover, if $A$ is not $F$-rational, then we have

$$e_{HK}(A) \geq \frac{e^2}{2(e - 1)}.$$
Example 2.7. (Fakhruddin-Trivedi [7, Corollary 3.19]) Let $E$ be an elliptic curve over a field $k = \overline{k}$ of characteristic $p > 0$, and let $\mathcal{L}$ be a very ample line bundle on $E$ of degree $e \geq 2$. Let $R$ be the homogeneous coordinate ring (the section ring of $\mathcal{L}$) defined by

$$ R = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}^n). $$

Also, put $A = R_{\mathfrak{m}}$, where $\mathfrak{m}$ be the unique homogeneous maximal ideal of $R$. Then we have $e_{HK}(A) = \frac{e^3}{2(e-1)}$.

§3. Lower bounds in the case of three-dimensional local rings

In this section, we prove the main theorem in this paper, which gives the lower bound of Hilbert-Kunz multiplicities for non-regular unmixed local rings of dimension 3.

Theorem 3.1. Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. Then

1. If $A$ is not regular, then $e_{HK}(A) \geq \frac{4}{3}$.

2. Suppose that $k = \overline{k}$ and char $k \neq 2$. Then the following conditions are equivalent:
   (a) $e_{HK}(A) = \frac{4}{3}$.
   (b) $\tilde{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.
   (c) $\text{gr}_\mathfrak{m}(A) \cong k[X, Y, Z, W]/(X^2 + Y^2 + Z^2 + W^2)$. That is, $\text{gr}_\mathfrak{m}(A) \cong k[X, Y, Z, W]/(XY - ZW)$.

Proposition 3.2. Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. If $e_{HK}(A) < 2$, then $A$ is $F$-rational.

From now on, we divide the proof of Theorem 3.1 and Proposition 3.2 into several steps. In the following, we assume the following condition.

(#) Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$, and $e(A) = e$, the multiplicity of $A$. Also, suppose that $\mathfrak{m}$ has a minimal reduction $J$.

Suppose that $A$ is not regular under the assumption (#). Then $e = e(A)$ is an integer with $e \geq 2$. Thus the first assertion of Theorem 3.1 follows from the following lemma. Also, this implies that if $e_{HK}(A) = \frac{4}{3}$ then $e(A) = 2$ without extra assumptions.
Lemma 3.3. Under the assumption (#), we have

1. If \( e \geq 5 \), then \( \ell(A) > 2 \).
2. If \( e = 4 \), then \( \ell(A) \geq \frac{7}{4} > \frac{4}{3} \).
3. If \( e = 3 \), then \( \ell(A) \geq \frac{13}{8} > \frac{4}{3} \).
4. If \( e = 2 \), then \( \ell(A) \geq \frac{4}{3} \).

Remark 5. The lower bounds of \( \ell(A) \) in Lemma 3.3 are not best possible.

Proof. We may assume that \( A \) is complete. By Lemma 2.5(1), we can apply Theorem 2.2 with \( r = e - 1 \). Namely, if \( 1 \leq s \leq 2 \), then

\[
\ell(A) \geq e(v_s - (e-1)v_{s-1}) = e \left( \frac{s^3}{6} - (e+2) \frac{(s-1)^3}{6} \right).
\]

Define the real-valued function \( f_e(s) \) by the right-hand side of Eq. (3.1). Then one can easily calculate \( \max_{1 \leq s \leq 2} f_e(s) \). In fact, if \( e \geq 2 \), then

\[
\max_{1 \leq s \leq 2} f_e(s) = f \left( \frac{e + 2 + \sqrt{e+2}}{e + 1} \right) = \frac{e}{6} \left( \frac{e + 2 + \sqrt{e+2}}{e + 1} \right)^2.
\]

But, in order to prove the lemma, it is enough to use the following values only:

<table>
<thead>
<tr>
<th>( e )</th>
<th>( \frac{3}{2} )</th>
<th>( \frac{7}{4} )</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_e(s) )</td>
<td>( \frac{e(25-e)}{48} )</td>
<td>( \frac{e(289-27e)}{384} )</td>
<td>( \frac{e(6-e)}{6} )</td>
</tr>
</tbody>
</table>

(1) We show that \( \ell(A) > 2 \) if \( e \geq 5 \). If \( e \geq 13 \), then by Proposition 1.4,

\[
\ell(A) \geq \frac{e}{3!} \geq \frac{13}{6} > 2.
\]

So we may assume that \( 5 \leq e \leq 12 \). Applying Eq. (3.1) for \( s = \frac{3}{2} \), we get

\[
\ell(A) \geq \frac{e(25-e)}{48} \geq \frac{5(25-5)}{48} = \frac{25}{12} > 2.
\]

(2) Suppose that \( e = 4 \). Applying Eq. (3.1) for \( s = \frac{3}{2} \), we get

\[
\ell(A) \geq \frac{e(25-e)}{48} = \frac{7}{4}.
\]
(3) Suppose that $e = 3$. Applying Eq. (3.1) for $s = \frac{7}{4}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(289 - 27e)}{384} = \frac{13}{8}.$$ 

(4) Suppose that $e = 2$. Applying Eq. (3.1) for $s = 7/4$, we get

$$e_{\text{HK}}(A) \geq \frac{e(289 - 27e)}{384} = \frac{13}{8},$$

as required.

Before proving the second assertion of Theorem 3.1, we prove Proposition 3.2. For that purpose, we now focus non-$F$-rational local rings.

Now suppose that $A$ is not $F$-rational. If $e = 2$, then $e_{\text{HK}}(A) = 2$ by Lemma 1.10. On the other hand, if $e \geq 5$, then $e_{\text{HK}}(A) > 2$ by Lemma 3.3. Thus in order to prove Proposition 3.2, it is enough to investigate the cases of $e = 3, 4$. Namely, Proposition 3.2 follows from the following lemma.

**Lemma 3.4.** Suppose that $A$ is not $F$-rational under the assumption $(\#)$. Then

(1) If $e = 3$, then $e_{\text{HK}}(A) \geq 2$.

(2) If $e = 4$, then $e_{\text{HK}}(A) > 2$.

**Proof.** By Lemma 2.5(2), we can apply Theorem 2.2 for $r = e - 2$. Thus if $1 \leq s \leq 2$, then

(3.2) $$e_{\text{HK}}(A) \geq e\left(\frac{s^3}{6} - (e + 1)\frac{(s - 1)^3}{6}\right).$$

(1) Suppose that $e = 3$. Applying Eq. (3.2) for $s = 2$, we get

$$e_{\text{HK}}(A) \geq \frac{e(7 - e)}{6} = 2.$$ 

(2) Suppose that $e = 4$. Applying Eq. (3.2) for $s = \frac{7}{4}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(316 - 27e)}{384} = \frac{13}{6} > 2,$$

as required. □
Example 3.5. Let $R = k[T, xT, xyT, yT, x^{-1}yT, x^{-2}yT, \ldots, x^{-n}yT]$ be a rational normal scroll and put $m = (T, xT, xyT, yT, x^{-1}yT, \ldots, x^{-n}yT)$. Then $A = R_m$ is a three-dimensional Cohen-Macaulay $F$-rational local domain with $e(A) = n + 2$, and

$$e_{HK}(A) = \frac{e(A)}{2} + \frac{e(A)}{6(n+1)}.$$  

Proof. Let $P \subseteq \mathbb{R}$ be a convex polytope with vertex set

$$\Gamma = \{(0,0), (1,0), (1,1), (0,1), (-1,1), \ldots, (-n,1)\},$$

and put $\tilde{P} := \{(\alpha, 1) \in \mathbb{R}^3 : \alpha \in P\}$ for every integer $d \geq 0$. Also, if we define a cone $C = C(\tilde{P}) := \{r\beta : \beta \in \tilde{P}, 0 \leq r \in \mathbb{Q}\}$ and regard $R$ as a homogeneous $k$-algebra with $\deg x = \deg y = 0$ and $\deg T = 1$, then the basis of $R_d$ corresponds to the set $\{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2 \cap dP\} = \{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2\} \cap C$.

If we put $\Gamma_q = \{(0,0), (q,0), (q,q), (0,q), (-q,q), \ldots, (-nq,q)\}$, then $m[q] = (x^ay^bt^q : (a,b) \in \Gamma_q)$. Since $[m[q]]_d = \sum_{(a,b) \in \Gamma_q} R_{d-q} x^ay^bt^q$, we have

$$e_{HK}(A) = \lim_{q \to \infty} \frac{1}{q^3} I_A(A/m[q])$$

$$= \lim_{q \to \infty} \frac{1}{q^3} \# \left\{ \mathbb{Z}^3 \cap \left( C \setminus \bigcup_{(a,b) \in \Gamma_q} (a, b, q) + C \right) \right\},$$

that is,

$$e_{HK}(A) = \lim_{q \to \infty} \frac{1}{q^3} \left[ \sum_{d=0}^{\infty} \# \left\{ \mathbb{Z}^2 \cap \left( dP \setminus \bigcup_{(a,b) \in \Gamma_q} (a, b) + \max\{0, d-q\} P \right) \right\} \right].$$

Also, if we define a real continuous function $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = \text{the volume of } tP \setminus \bigcup_{(a,b) \in \Gamma} (a, b) + \max\{0, t-1\} P \text{ in } \mathbb{R}^2,$$
then $e_{HK}(A) = \int_0^\infty f(t) \, dt$. Let us denote the volume of $M \subseteq \mathbb{R}^2$ by vol($M$). To calculate $e_{HK}(A)$, we need to determine $f(t)$. Namely, we need to show the following claim.

**Claim.**

$$
f(t) = \begin{cases} 
\text{vol}(t\mathcal{P}), & 0 \leq t < 1; \\
\text{vol}(t\mathcal{P}) - (n+4)\text{vol}((t-1)\mathcal{P}), & 1 \leq t < \frac{n+2}{n+1}; \\
\frac{(n+2)(2-t^2)}{2} + (n+2)\frac{(2-t)^2}{2n}, & \frac{n+2}{n+1} \leq t < 2; \\
0, & t \geq 2.
\end{cases}
$$

To prove the claim, we may assume that $t \geq 1$. For simplicity, we put $M_{a,b} = (a, b) + (t-1)\mathcal{P}$ for every $(a, b) \in \Gamma$. First suppose that $1 \leq t < \frac{n+2}{n+1}$. Then since $1 - n(t-1) > t - 1$, $M_{0,0} \cap M_{1,0} = \emptyset$. Similarly, one can easily see that any two $M_{a,b}$ do not intersect each other; see Figure 1. Thus $f(t) = \text{vol}(t\mathcal{P}) - (n+4)\text{vol}((t-1)\mathcal{P})$.

Next suppose that $\frac{n+2}{n+1} \leq t < 2$. Then $\mathcal{P} \cap \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq t-1\} = M_{0,0} \cup M_{1,0} \cup T_0$, where $T_0$ is a triangle with vertex $(t-1, 0)$, $(1, 0)$ and $(t-1, \frac{2-t}{n})$. Similarly, there exist $(n+1)$-triangles $T_1, \ldots, T_{n+1}$ having the same volumes as $T_0$ such that $\mathcal{P} \cap \{(x, y) \in \mathbb{R}^2 : 1 \leq y \leq t\} = M - n_1 \cup \cdots \cup M_{n,1} \cup M_{0,1} \cup M_{1,1} \cup T_1 \cup \cdots \cup T_{n+1}$ and any two $T_i$’s do not intersect each other; see Figure 2. Thus

$$f(t) = \text{vol}(\mathcal{P} \cap \{(x, y) \in \mathbb{R}^2 : t-1 \leq y \leq 1\}) + (n+2)\text{vol}(T_0) = \frac{(n+2)t(2-t)}{2} + (n+2)\frac{(2-t)^2}{2n}.
$$

Finally, suppose that $t \geq 2$. Then since $\mathcal{P}$ is covered by $M_{a,b}$’s, we have $f(t) = 0$, as required.

Using the above claim, let us calculate $e_{HK}(A)$. Note that $\text{vol}(t\mathcal{P}) = \frac{(n+2)t^2}{2}$.

$$e_{HK}(A) = \int_0^{\infty} \frac{(n+2)t^2}{2} \, dt - (n+4)\int_1^{\frac{n+2}{n+1}} \frac{(n+2)(t-1)^2}{2} \, dt \\
\quad + \int_{\frac{n+2}{n+1}}^{2} \frac{(n+2)t(2-t)}{2} \, dt + (n+2)\int_{\frac{n+2}{n+1}}^{2} \frac{(2-t)^2}{2n} \, dt \\
= (n+2) \left[ \frac{1}{2} + \frac{1}{6(n+1)} \right].$$
Discussion 3.6. Let $A$ be a complete local ring which satisfies (#). Also, suppose that $e = 3$. What is the smallest value of $e_{HK}(A)$ among such rings?

The function $f_e(s) = 3\left(\frac{s^3}{6} - 5\left(\frac{s-1)^3}{6}\right)\right)$, which appeared in Eq. (3.1), takes the maximal value

$$f\left(\frac{5 + \sqrt{5}}{4}\right) = \frac{15 + 5\sqrt{5}}{16} = 1.636\cdots$$

in $s \in [1, 2]$. Hence $e_{HK}(A) \geq 1.636\cdots$. But we believe that this is not best possible.

Suppose that $e_{HK}(A) < 2$. Then $A$ is $F$-rational by Lemma 3.4. Thus it is Cohen-Macaulay and $3 + 1 \leq v = \text{emb}(A) \leq d + e - 1 = 3 + 3 - 1 = 5$. If $v \neq 5$, then $A$ is a hypersurface and $e_{HK}(A) \geq \frac{2}{3} \cdot e = 2$ by Fact 2.4. Hence we may assume that $v = 5$, that is, $A$ has maximal embedding dimension. If we write as $A = R/I$, where $R$ is a complete regular local ring with $\dim R = 5$, then height $I = 2$. By Hilbert-Burch’s theorem, there exists a $2 \times 3$-matrix $M$ such that $I = I_2(M)$, the ideal generated by all 2-minors of $M$. In particular, $A$ can be written as $A = B/aB$, where $B = k[X]/I_2(X)$,
$X$ is a generic $2 \times 3$-matrix and $a$ is a prime element of $B$. This implies that

$$e_{HK}(A) = e_{HK}(B/aB) \geq e_{HK}(B) = 3 \left\{ \frac{1}{2} + \frac{1}{4!} \right\} = \frac{13}{8} = 1.625;$$

see [5, Section 3].

For example, if $A = k[[T, xT, yT, yT, x^{-1}yT]]$ is a rational normal scroll, then $e_{HK}(A) = \frac{7}{4} = 1.75$ by Example 3.5. Is this the smallest value?

**Discussion 3.7.** Let $A$ be a complete local ring which satisfies (\#). Also, suppose that $e = 4$. What is the smallest value of $e_{HK}(A)$ among such rings?

As in Discussion 3.6, it suffices to consider $F$-rational local rings only. For example, let $A = k[[x, y, z]]^{(2)}$ be the Veronese subring. Then $A$ is an $F$-rational local domain with $e(A) = 4$ and $e_{HK}(A) = 2$. Also, let $A$ be the completion of the Rees algebra $R(n)$ over an $F$-rational double point $(R, n)$ of dimension 2. Then $A$ is an $F$-rational local domain with $e(A) = 4$ and $e_{HK}(A) \geq 2$ (we believe that this inequality is strict).

On the other hand, the function $f_e(s)$ which appeared in Eq. (3.1), takes the maximal value

$$\frac{28 + 8\sqrt{6}}{25} = 1.903 \ldots$$

in $s \in [1, 2]$. Hence the fact that we can prove now is “$e_{HK}(A) \geq 1.903 \ldots$” only.

Based on Corollary 2.6 and Discussion 3.7, we pose the following conjecture.

**Conjecture 3.8.** Let $A$ be a complete local ring which satisfies (\#), and let $r \geq 2$ be an integer. If $e(A) = r^2$, then

$$e_{HK}(A) \geq \frac{(r + 1)(r + 2)}{6}.$$ 

Also, the equality holds if and only if $A$ is isomorphic to $k[[x, y, z]]^{(r)}$.

In the rest of this section, we prove the second statement of Theorem 3.1. Let $(A, m, k)$ be a complete local ring which satisfies (\#). If
\[ e_{HK}(A) = \frac{4}{3}, \] then \( A \) is an \( F \)-rational hypersurface with \( e(A) = 2 \) by the above observation. Furthermore, suppose that \( k = \mathbb{K} \) and \( \text{char } k \neq 2 \). Then we may assume that \( A \) can be written as the form \( k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W)) \). To study Hilbert-Kunz multiplicities for these rings, we prove the improved version of Theorem 2.2.

**Proposition 3.9.** Let \( k \) be an algebraically closed field of \( \text{char } k \neq 2 \), and let \( A = k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W)) \) be an \( F \)-rational hypersurface local ring. Let \( a, b, c \) be integers with \( 2 \leq a \leq b \leq c \).

Suppose that there exists a function \( \text{ord} : A \rightarrow \mathbb{Q} \cup \{\infty\} \) which satisfies the following conditions:

1. \( \text{ord}(\alpha) \geq 0; \text{and } \text{ord}(\alpha) = \infty \iff \alpha = 0. \)
2. \( \text{ord}(x) = 1/2, \text{ord } y = 1/a, \text{ord } z = 1/b, \text{and } \text{ord } w = 1/c. \)
3. \( \text{ord}(\varphi) \geq 1. \)
4. \( \text{ord}(\alpha + \beta) \geq \min\{\text{ord}(\alpha), \text{ord}(\beta)\}. \)
5. \( \text{ord}(\alpha \beta) \geq \text{ord}(\alpha) + \text{ord}(\beta). \)

Then we have
\[
e_{HK}(A) \geq 2 - \frac{abc}{12}(N^3 - n^3),
\]
where
\[
N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}, \quad n = \max\left\{0, N - \frac{2}{c}\right\}.
\]
In particular, if \((a, b, c) \neq (2, 2, 2)\), then \( e_{HK}(A) > \frac{4}{3} \).

**Remark 6.** The third condition \( \text{ord}(\varphi) \geq 1 \) is important. For example, if \( \varphi \equiv y^2 \mod (z, w)^3 \), then one can take \((a, b, c) = (2, 3, 3)\), but \((a, b, c) = (2, 3, 4)\).

**Proof.** First, we define a filtration \( \{F_n\}_{n \in \mathbb{Q}} \) as follows:
\[
F_n := \{\alpha \in A : \text{ord}(\alpha) \geq n\}.
\]
Then every \( F_n \) is an ideal and \( F_m F_n \subseteq F_{m+n} \) holds for all \( m, n \in \mathbb{Q} \). Using \( F_n \) instead of \( m^n \), we shall estimate \( l_A(m^{[n]}/J^{[n]}) \).

Set \( J = (y, z, w)A \) and fix a sufficiently large power \( q = p^e \). Put
\[
s = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}.
\]
Since $J$ is a minimal reduction of $\mathfrak{m}$ and $x^{q-1}y^{q-1}z^{q-1}w^{q-1}$ generates the socle of $A/J[q]$, we have that $F_{sq} \subseteq J[q]$. Also, since $B = A/J[q]$ is an Artinian Gorenstein local ring, we get

$$F_{\frac{(N+1)q}{2}} B \subseteq 0 :_B F_{\frac{Nq}{2}} B \cong K_B/F_{\frac{Nq}{2}} B,$$

where $K_C$ denotes a canonical module of a local ring $C$. Hence, by the Matlis duality theorem, we get

$$l_A \left( \frac{F_{\frac{(N+1)q}{2}} + J[q]}{J[q]} \right) \leq l_B \left( \frac{F_{\frac{(N+1)q}{2}} + J[q]}{J[q]} \right) \leq l_B (K_B/F_{\frac{Nq}{2}} B) = l_B (B/F_{\frac{Nq}{2}} B) = 0.$$

On the other hand, since $x^q \in F_{\frac{q}{2}}$ by the assumption, we have

$$x^q F_{\frac{Nq}{2}} \subseteq F_{\frac{(N+1)q}{2}}.$$

Therefore by a similar argument as in the proof of Theorem 2.2, we get

$$l_A (m[q]/J[q]) \leq l_A \left( \frac{Ax^q + J[q] + F_{\frac{(N+1)q}{2}}}{J[q]} \right) + l_A \left( \frac{F_{\frac{(N+1)q}{2}} + J[q]}{J[q]} \right)
\leq l_A \left( A/(J[q] + F_{\frac{(N+1)q}{2}}) : x^q \right) + l_B (B/F_{\frac{Nq}{2}} B)
\leq 2 \cdot l_A \left( A/J[q] + F_{\frac{Nq}{2}} \right).$$

In fact, since

$$\lim_{q \to \infty} \frac{1}{q^3} l_A \left( A/J[q] + F_{\frac{Nq}{2}} \right)
= e(A) \cdot \lim_{q \to \infty} \frac{1}{q^3} \text{vol} \left\{ (x, y, z) \in [0, q]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{Nq}{2} \right\}
= 2 \cdot \text{vol} \left\{ (x, y, z) \in [0, 1]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{N}{2} \right\}
= \frac{abc}{24} (N^3 - n^3),$$

we get

$$e_{\text{HK}}(A) \geq 2 - 2 \cdot \frac{abc}{24} (N^3 - n^3) = 2 - \frac{abc}{12} (N^3 - n^3),$$

as required.
Example 3.10. Let $k$ be an algebraically closed field of char $k \neq 2$, and let $(A, m, k)$ be a hypersurface. Put $\text{gr}_m(A) = k[X, Y, Z, W]/(g(X, Y, Z, W))$.

\[
g(X, Y, Z, W) = X^2 + Y^3 + Z^3 + W^3 \implies e_{HK}(A) \geq \frac{55}{32};
\]
\[
g(X, Y, Z, W) = X^2 + Y^2 + Z^3 + W^3 \implies e_{HK}(A) \geq \frac{14}{9};
\]
\[
g(X, Y, Z, W) = X^2 + Y^2 + Z^2 + W^c \implies e_{HK}(A) \geq \frac{3}{2} - \frac{2}{3c^2}.
\]

Proof of Theorem 3.1(2). Put $G = \text{gr}_m(A)$ and $\mathfrak{M} = \text{gr}_m(A)_+$. The implication $(a) \Rightarrow (b)$ follows from Proposition 3.9. $(b) \Rightarrow (c)$ is clear. Suppose $(c)$. Then $e_{HK}(G_{\mathfrak{M}}) = \frac{4}{3}$. Also, by Proposition 1.3 and Theorem 3.1(1), we have that $\frac{4}{3} \leq e_{HK}(A) \leq e_{HK}(G_{\mathfrak{M}}) = \frac{4}{3}$. Thus $e_{HK}(A) = \frac{4}{3}$, as required.

Also, the following corollary follows from the proof of Proposition 3.9 and Example 3.10.

Corollary 3.11. Let $A$ be a local ring which satisfies $(\#)$. Also, assume that $k = \overline{k}$ and $p \neq 2$. Then the following conditions are equivalent:

1. $\frac{4}{3} < e_{HK}(A) \leq \frac{3}{2}$.
2. $\text{gr}_m(A) \cong k[X, Y, Z]/(X^2 + Y^2 + Z^2)$.
3. $A$ is isomorphic to a hypersurface $k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^c)$ for some integer $c \geq 3$.

When this is the case, $e_{HK}(A) \geq \frac{4}{3} - \frac{2}{3c^2}$.

§4. A generalization of the main result to higher dimensional case

In this section, we want to generalize Theorem 3.1 to the case of $\dim A \geq 4$. Let $d \geq 1$ be an integer and $p > 2$ a prime number. If we put

\[A_{p,d} := \overline{\mathbb{F}}_p[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2),\]

then we can guess that $e_{HK}(A_{p,d}) = s_{HK}(p, d)$ holds according to the observations until the previous section. In the following, let us formulate this as a conjecture and prove that it is also true in case of $\dim A = 4$. 
In [10], Han and Monsky gave an algorism to calculate $e_{HK}(A_{p,d})$, but it is not so easy to represent $e_{HK}(A_{p,d})$ as a quotient of two polynomials of $p$ for any fixed $d \geq 1$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{HK}(A_{p,d})$</td>
<td>$2$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{29p^2 + 15}{24p^2 + 12}$</td>
</tr>
</tbody>
</table>

On the other hand, surprisingly, Monsky proved the following theorem:

**Theorem 4.1.** (Monsky [19]) Under the above notation, we have

\[
\lim_{p \to \infty} e_{HK}(A_{p,d}) = 1 + \frac{c_d}{d!},
\]

where

\[
\sec x + \tan x = \sum_{d=0}^{\infty} \frac{c_d}{d!} x^d \quad (|x| < \frac{\pi}{2}).
\]

**Remark 7.** It is known that the Taylor expansion of $\sec x$ (resp. $\tan x$) at origin can be written as follows:

\[
\sec x = \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i},
\]

\[
\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i} - 1)B_{2i}}{(2i)!} x^{2i-1},
\]

where $E_{2i}$ (resp. $B_{2i}$) is said to be Euler number (resp. Bernoulli number).

Also, $c_d$ appeared in Eq. (4.1) is a positive integer since $\cos t$ is an unit element in a ring $\mathcal{H} = \{ \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} : a_n \in \mathbb{Z} \text{ for all } n \geq 0 \}$.

Based on the above observation, we pose the following conjecture.

**Conjecture 4.2.** Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put

\[ A_{p,d} := \mathbb{F}_p[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2). \]

Let $(A, m, k)$ be a $d$-dimensional unmixed local ring with $k = \mathbb{F}_p$. Then the following statements hold.

1. If $A$ is not regular, then $e_{HK}(A) \geq e_{HK}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$. In particular, $s_{HK}(p, d) = e_{HK}(A_{p,d})$. 


(2) If $e_{HK}(A) = e_{HK}(A_{p,d})$, then $\hat{A} \cong A_{p,d}$ as local rings.

In the following, we prove that this is true in case of $\dim A = 4$. Note that
\[
\lim_{p \to \infty} e_{HK}(A_{p,4}) = \lim_{p \to \infty} \frac{29p^2 + 15}{24p^2 + 12} = \frac{29}{24} = 1 + \frac{c_4}{4!}.
\]

**Theorem 4.3.** Let $(A, m, k)$ be an unmixed local ring of characteristic $p > 0$ with $\dim A = 4$. If $e(A) \geq 3$, then $e_{HK}(A) \geq \frac{5}{4} = \frac{30}{24}$.

Suppose that $k = \overline{k}$ and $\text{char } k \neq 2$. Put
\[A_{p,4} = \mathbb{F}_p[[X_0, X_1, \ldots, X_4]]/(X_0^2 + \cdots + X_4^2).
\]
Then the following statement holds.

(1) If $A$ is not regular, then
\[e_{HK}(A) \geq e_{HK}(A_{p,4}) = \frac{29p^2 + 15}{24p^2 + 12}.
\]

(2) The following conditions are equivalent:

(a) Equality holds in (1).

(b) $e_{HK}(A) < \frac{5}{4}$.

(c) The completion of $A$ is isomorphic to $A_{p,4}$.

**Proof.** Put $e = e(A)$, the multiplicity of $A$. We may assume that $A$ is complete with $e \geq 2$ and $k$ is infinite. In particular, $A$ is a homomorphic image of a Cohen-Macaulay local ring, and there exists a minimal reduction $J$ of $m$. Then $\mu_A(m/J^*) \leq e - 1$ by Lemma 2.5. We first show that $e_{HK}(A) \geq \frac{5}{4}$ if $e \geq 3$.

**Claim 1.** If $3 \leq e \leq 10$, then $e_{HK}(A) \geq \frac{5}{4}$.

Putting $r = e - 1$ and $s = 2$ in Theorem 2.2, since $v_2 = \frac{1}{2}$, we have
\[e_{HK}(A) \geq e \left\{ v_2 - \frac{(e - 1)^4}{4!} \right\} = \frac{(13 - e)e}{24} \geq \frac{(13 - 3) \cdot 3}{24} = \frac{30}{24},
\]
as required.

**Claim 2.** If $11 \leq e \leq 29$, then $e_{HK}(A) \geq \frac{737}{384} (> \frac{5}{4})$. 


By Fact 2.4, we have $v_{3/2} = \frac{1-\beta_1+1}{2} = \frac{77}{384}$. Putting $r = e - 1$ and $s = \frac{3}{2}$ in Theorem 2.2, we have

$$e_{HK}(A) \geq e\left(v_{3/2} - \frac{e-1}{24} \cdot \left(\frac{1}{2}\right)^4\right) = \frac{(78-e)e}{384} \geq \frac{11(78-11)}{384} = \frac{737}{384},$$

as required.

**Claim 3.** If $e \geq 30$, then $e_{HK}(A) \geq \frac{5}{4}$.

By Proposition 1.4, we have $e_{HK}(A) \geq \frac{e}{4} \geq \frac{30}{24}$.

In the following, we assume that $k = \bar{k}$, char $k \neq 2$ and $e \geq 2$. To see (1), (2), we may assume that $e = 2$ by the above argument. Then since $e_{HK}(A) = 2$ if $A$ is not $F$-rational, we may also assume that $A$ is $F$-rational and thus is a hypersurface. Thus $A$ can be written as the following form:

$$A = k[[X_0, X_1, \ldots, X_4]]/(X_0^2 - \varphi(X_1, X_2, X_3, X_4)).$$

If $A$ is isomorphic to $A_{p,4}$, then by [10], it is known that $e_{HK}(A) = \frac{29p^2 + 15}{24p^2 + 12}$. Suppose that $A$ is not isomorphic to $A_{p,4}$. Then one can take a minimal numbers of generators $x, y, z, w, u$ of $m$ and one can define a function $\text{ord} : A \to \mathbb{Q} \cup \{\infty\}$ such that

$$\text{ord}(x) = \text{ord}(y) = \text{ord}(z) = \text{ord}(w) = \frac{1}{2}, \quad \text{ord}(u) = \frac{1}{3}.$$

If we put $J = (y, z, w, u)A$ and $F_n = \{\alpha \in A : \text{ord}(\alpha) \geq n\}$, then by a similar argument as in the proof of Proposition 3.9, we have

$$l_A(m^{[q]}/J^{[q]}) \leq 2 \cdot l_A(A/J^{[q]} + F_{2q/3}).$$

Divided the both-side by $q^d$ and taking a limit $q \to \infty$, we get

$$e(A) - e_{HK}(A) \leq 2 \cdot e(A) \cdot \text{vol}\left\{(y, z, w, u) \in [0, 1]^4 : \frac{y}{2} + \frac{z}{2} + \frac{w}{2} + \frac{u}{3} \leq \frac{2}{3}\right\}.$$

To calculate the volume in the right-hand side, we put

$$F_u = \begin{cases} \frac{1}{6}(\frac{4}{3} - \frac{2}{3}u)^3 - 3 \cdot \frac{1}{6}(\frac{1}{3} - \frac{2}{3}u)^3 & (0 \leq u \leq \frac{1}{2}) \\ \frac{1}{6}(\frac{4}{3} - \frac{2}{3}u)^3 & (\frac{1}{2} \leq u \leq 1) \end{cases}$$
Then one can easily calculate

the above volume = \( \int_0^1 F_u \, du = \frac{237}{2434} \).

It follows that

\[ e_{HK}(A) \geq 2 - 4 \times \frac{237}{2434} = \frac{411}{324} > \frac{5}{4}. \]

The following conjecture also holds if \( \dim A \leq 4 \).

**Conjecture 4.4.** Under the same notation as in Conjecture 4.2, if \( e(A) \geq 3 \), then

\[ e_{HK}(A) \geq 1 + \frac{c_d + 1}{d!}. \]

**Discussion 4.5.** Let \( d \geq 2 \) be an integer and fix a prime number \( p \gg d \). Assume that Conjectures 4.2 and 4.4 are true. Also, assume that \( s_{HK}(p,d) < s_{HK}(p,d - 1) \) for all \( d \geq 3 \). Let \( A = k[X_0, \ldots, X_v]/I \) be a \( d \)-dimensional homogeneous unmixed \( k \)-algebra with \( \deg X_i = 1 \), and let \( m \) be the unique homogeneous maximal ideal of \( A \). Suppose that \( k \) is an algebraically closed field of characteristic \( p > 0 \). Then \( e_{HK}(A) = s_{HK}(p,d) \) implies that \( \widehat{A_m} \cong A_{p,d} \).

In fact, if \( e_{HK}(A) = s_{HK}(p,d) \), then we may assume that \( e_{HK}(A) < 1 + \frac{c_d + 1}{d!} \). Thus \( e(A_m) = 2 \) if Conjecture 4.4 is true. For any prime ideal \( P A_m \) of \( A_m \) such that \( P \neq m \), we have \( e_{HK}(A_P) \leq e_{HK}(A_m) = s_{HK}(p,d) < s_{HK}(p,n) \), where \( n = \dim A_P < d \). Since \( A_P \) is also unmixed, it is regular. Thus \( A_m \) has an isolated singularity. Hence \( A \) is a non-degenerate quadric hypersurface. In other words, \( \widehat{A_m} \) is isomorphic to \( A_{p,d} \).

**References**


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