SYMMETRIES AND PSEUDO-SYMMETRIES OF HYPERELLIPTIC SURFACES

by DAVID SINGERMAN

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1. Let X be a closed Riemann surface of genus $g \geq 2$ and let Aut X denote the group of automorphisms of X where, in this paper, an automorphism means a conformal or anticonformal self-homeomorphism. X is called *hyperelliptic* if it admits a conformal automorphism J of order 2 such that X/H has genus 0, where $H = \langle J \rangle$ is the group of order 2 generated by J. Thus X is a two-sheeted covering of the sphere which is branched over $2g+2$ points and J is the sheet-interchange map. J is the unique conformal automorphism of order 2 such that $X/\langle J \rangle$ has genus 0 and it follows that if $U \in \text{Aut } X$, then $UJU^{-1} = J$. Thus J is central in Aut X and $H \trianglelefteq \text{Aut } X$. (Cf. [**8**])

A *symmetry* of X is an anticonformal automorphism of order 2. Another type of automorphism which occurs naturally is an anticonformal automorphism Q such that $Q^2 = J$. Following Zarrow [**13**] we shall call such automorphisms *pseudo-symmetries*. If T is a symmetry of X then so is JT. We shall call $\{T, JT\}$ a *pair* of symmetries. Similarly, if Q is a pseudo-symmetry $\{Q, JQ\}$ is called a *pair* of pseudo-symmetries.

In this paper we shall be concerned with the number of conjugacy classes of pairs of symmetries of a hyperelliptic surface and show that this is at most 3 if g is even, and at most 4 if g is odd. We shall also find a sharp upper bound for the number of symmetries of a hyperelliptic surface and show that there is at most one pair of pseudo-symmetries. We also find conditions on the uniformizing Fuchsian group for a Riemann surface to be pseudo-symmetric and describe the subspace of Teichmüller space corresponding to pseudo-symmetric surfaces.

The study of symmetric Riemann surfaces was begun by Klein (see e.g. [**6**]) and their importance is partly due to their relationship to real algebraic curves. For some modern papers on the subject see [**2**], [4], [**11**], [**13**]. In further work we shall study a symmetric hyperelliptic surface according to the topological character of its symmetries, and this motivated the study of conjugacy, as conjugate symmetries have the same topological character.

2. Let G be the group of automorphisms of the hyperelliptic surface X. Then G/H is a finite group of automorphisms of X/H, which being a Riemann surface of genus 0 may be regarded as the Riemann sphere Σ. We now consider the symmetries and finite automorphism groups of Σ. Every symmetry of Σ is conjugate in Aut Σ either to the reflection $z \to \bar{z}$ or to the antipodal map $z \to -1/\bar{z}$ [**2**]. These are distinguished by the fact that only the first of these has fixed points. In the following table we list the finite groups of conformal automorphisms of Σ, their possible extensions by adjoining anticonformal automorphisms and information about the symmetries.

The information about the groups can be found in [**3**], §4.3, 4.4 and Table 2. The symmetries are the involutions which lie in the extended group, but outside the original group.

Group	Extended group	Number of symmetries	Number of conjugacy classes of symmetries
C_q	$C_q \times C_2$	$1(q$ odd), $2(q$ even)	$1(q$ odd), $2(q$ even)
	D_q	q	$1(q$ odd), $2(q$ even)
D_q	$C_2 \times D_q$	$q+1(q$ odd), $q+2(q$ even)	$2(q$ odd), $4(q$ even)
A_4	$C_2 \times A_4$	4	2
	S_4	6	1
S_4	$C_2 \times S_4$	10	3
A_5	$C_2 \times A_5$	16	2

As an example consider the group D_q, where q is even, acting as a group o conformal automorphisms. This is generated by $P: z \to 1/z$ and $R: z \to \varepsilon z$, where $\varepsilon =$ $\exp(2\pi i/q)$. The extended group is found by adjoining the antipodal map $A: z \to -1/\bar{z}$ The anticonformal automorphisms have the form AP^uR^v, where $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant$ $q-1$. The symmetries are A, $AR^{q/2}$, APR^v, $(0 \leqslant v \leqslant q-1)$. The 4 conjugacy classes o symmetries are $\{A\}$, $\{AR^{q/2}\}$, $\{AP, APR^2, \ldots, APR^{q-2}\}$, $\{APR, APR^3, \ldots, APR^{q-1}\}$.

We are using the result that every finite group of conformal automorphisms of Σ is group of rotations of Σ. Using the facts that a conformal automorphism of Σ is a linea fractional transformation and that a rotation must commute with A we see that ever rotation has the form

$$z \to \frac{az - \bar{c}}{cz + \bar{a}}, \quad \text{where} \quad a\bar{a} + c\bar{c} = 1.$$

Thus the rotation group of Σ is isomorphic to $\mathrm{PSU}(2, \mathbb{C})$. The other elements of th extended groups are either reflections in planes passing through the origin or th antipodal map A. As all these elements commute with A we see that the extended group are all naturally isomorphic to subgroups of $\mathrm{PSU}(2, \mathbb{C}) \times C_2$, where the C_2 is generated b A. The only symmetry in this group which acts without fixed points is A and hence ther is at most one fixed point free symmetry in these extended groups, namely A itself. (O the extended groups only D_q and S_4 do not contain A).

3. Let G be the group of automorphisms of the hyperelliptic surface X. Then by § there is a homomorphism $\Phi: G \to PSU(2, \mathbb{C}) \times C_2$.

LEMMA 1. *If $T \in G$ and if $\Phi(T)$ is a symmetry of Σ then T is either a symmetry or pseudo-symmetry of X.*

Proof. The kernel of Φ is $\{E, J\}$, where E is the identity, and so $T^2 = E$ or $T^2 = J$. A $\ker \Phi$ consists of conformal automorphisms, Φ must preserve the sense of each element o G. Thus T is sense reversing and hence is either a symmetry or a pseudo-symmetry.

If T_1, T_2 are symmetries or pseudo-symmetries then the pairs $\{T_1, JT_1\}$, $\{T_2, JT_2\}$ ar called *conjugate* if T_1 is conjugate in Aut X to either T_2 or JT_2 (and then JT_1 is conjugat to JT_2 or T_2).

LEMMA 2. *$\{T_1, JT_1\}$ is conjugate to $\{T_2, JT_2\}$* if and only if *$\Phi(T_1)$ is conjugate to $\Phi(T_2)$ in $\Phi(G)$.*

Proof. Clearly Φ maps conjugate pairs to conjugate symmetries. Now suppose that T_1 and T_2 are symmetries or pseudo-symmetries in G and that $\Phi(T_1)$ is conjugate to $\Phi(T_2)$. Then there exists $S \in G$ such that $\Phi(ST_1S^{-1}) = \Phi(T_2)$. Hence $ST_1S^{-1}T_2^{-1} = E$ or J and therefore $\{T_1, JT_1\}$ is conjugate to $\{T_2, JT_2\}$.

It follows from the table in §2 that the number of pairs of symmetries and pairs of pseudo-symmetries is at most 4. To calculate the number of conjugacy classes of symmetries we have to distinguish them algebraically from pseudo-symmetries in relation to Φ. This is done in Theorem 2. For a proof it is convenient to introduce non-Euclidean crystallographic (NEC) groups.

An NEC group is a discrete subgroup of $\mathscr{L}$, the group of all automorphisms of D, the unit disc. We let $\mathscr{L}^+$ denote the subgroup of $\mathscr{L}$ consisting of conformal automorphisms so that if Γ is a subgroup of $\mathscr{L}^+$, then Γ is a Fuchsian group. The structure of NEC groups with reflections is rather more complicated than the structure of Fuchsian groups, but in this paper we shall only need to use NEC groups without reflections. (See [7] for the structure of NEC groups). The reason for the importance of NEC groups in the study of symmetries is quite clear. If X is a Riemann surface of genus $g \geq 2$, then there is a torsion free Fuchsian group K such that X is conformally equivalent to D/K. If X admits a symmetry T, then T can be lifted to an element $t \in \mathscr{L} - \mathscr{L}^+$ such that $tKt^{-1} = K$ and $t^2 \in K$. Now $\Gamma = K + Kt$ is an NEC group and t is either a reflection or a glide-reflection. A glide-reflection acts without fixed points on D, a reflection has a whole non-Euclidean line of fixed points. It follows that T acts without fixed points if and only if Γ has no reflections. (If T has fixed points then it leaves pointwise fixed $k \leq g+1$ disjoint Jordan curves) (see [6]).

NEC groups without reflections can be divided into two families depending on whether their quotient space is orientable or not. If D/Γ is orientable of genus g then Γ is a Fuchsian group. If there are r points on D/Γ over which the projection map from D to D/Γ is branched with orders of ramification $m_1, \ldots, m_r$, then Γ has presentation

$$\left\{a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r \,\middle|\, \prod_{i=1}^{r} x_i \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = x_1^{m_1} = \ldots = x_r^{m_r} = 1\right\}.$$

Here a_i, b_i are hyperbolic and the x_i elliptic. We shall say that Γ has signature

$$(g; +, m_1, \ldots, m_r).$$

$m_i \geq 2$ for $i = 1, \ldots, r$ and are called the periods of Γ. If D/Γ is non-orientable with g cross-caps then Γ has a presentation of the form

$$\left\{a_1, \ldots, a_g, x_1, \ldots, x_r \,\middle|\, \prod_{i=1}^{r} x_i \prod_{i=1}^{g} a_i^2 = x_1^{m_1} = \ldots x_r^{m_r} = 1\right\}.$$

Here a_i are glide-reflections and the x_i elliptics, $m_1, \ldots, m_r$ are again the orders of ramification. We shall say that Γ has signature

$$(g: - m_1, \ldots, m_r).$$

If there is no branching we shall write the signature as $(g; +)$ or $(g; -)$.

The measure of a fundamental region for Γ is

$$\mu(\Gamma) = 2\pi\left(\eta g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right),$$

where $\eta = 2$ if D/Γ is orientable and $\eta = 1$ if D/Γ is non-orientable. If $\Gamma_1 \subseteq \Gamma_2$, then the index is given by

$$|\Gamma_2 : \Gamma_1| = \mu(\Gamma_1)/\mu(\Gamma_2). \tag{1}$$

If Γ has signature $(g; -, m_1, \ldots, m_r)$, then the subgroup $\Gamma^+ = \Gamma \cap \mathcal{L}^+$ has signature $(g-1; +, m_1, m_1, \ldots, m_r, m_r)$ by [**10**]. In particular if Γ has signature $(1; -, 2^{(g+1)})$, then Γ^+ has signature $(0; +, 2^{(2g+2)})$, where $2^{(k)}$ means the period 2 is repeated k times.

If K has signature $(g; +)$ then as Maclachlan ([**8**]) observed, D/K is hyperelliptic if and only if K is a subgroup of index 2 in a group of signature $(0; +, 2^{(2g+2)})$, and the induced automorphism is the hyperelliptic involution. Our main results on symmetries and pseudo-symmetries follow from the next lemma.

Lemma 3. *Let* Γ *have signature* $(1; -; 2^{(g+1)})$. *Then* Γ *has a unique normal subgroup* K *of index* 4 *with signature* $(g; +)$. *If* g *is even, then* $\Gamma/K \cong C_4$, *an if* g *is odd, then* $\Gamma/K \cong C_2 \times C_2$.

Proof. Let Γ have presentation

$$\{a, x_1, \ldots, x_{g+1} \mid a^2 x_1 \ldots x_{g+1} = x_1^2 = \ldots = x_{g+1}^2 = 1\}.$$

Let g be even. We construct an epimorphism $\theta : \Gamma \to C_4$ whose kernel is torsion free. For this we need θ to preserve the orders of elements of finite order. Let $C_4 = \{u \mid u^4 = 1\}$; then we must have $\theta(x_i) = u^2$ $(i = 1, \ldots, g+1)$ and as θ is onto $\theta(a) = u^{\pm 1}$. As g is even the relation $a^2 x_1 \ldots x_{g+1} = 1$ is then preserved and so θ is an epimorphism. The kernel of θ is the normal subgroup generated by a^2 and hence is a Fuchsian group. As it is torsion free it follows from (1) that it has signature $(g; +)$. There are two epimorphisms from Γ to C_4 but as they differ by an automorphism of C_4 (namely $u \to u^{-1}$) they both have the same kernel. Also, there is no epimorphism from Γ to C_4 whose kernel is torsion free when g is odd.

On the other hand, suppose there is an epimorphism $\theta : \Gamma \to C_2 \times C_2$ whose kernel has signature $(g; +)$, where g is even. Let $C_2 \times C_2 = \{u, v \mid u^2 = v^2 = (u\,v)^2 = 1\}$. As the kernel is a Fuchsian group, it does not contain the glide reflection a. Assume without loss of generality that $\theta(a) = u$; then $\theta(x_1) \neq u$, for otherwise ax_1 would lie in the kernel, which is impossible as ax_1 is anticonformal. Hence we can assume that $\theta(x_1) = v$. If $1 < i \leq g+1$

then, as above, $\theta(x_i) \neq u$, but also $\theta(x_i) \neq uv$, for otherwise ax_1x_i would lie in the kernel which is impossible. Hence $\theta(a) = u$, $\theta(x_i) = v$, $1 \leq i \leq g+1$. From the relation $a^2x_1x_2 \ldots x_{g+1} = 1$ we obtain $u^2v^{g+1} = 1$, implying that g is odd, a contradiction.

However, if g is odd there is an epimorphism from Γ to $C_2 \times C_2$ whose kernel has signature $(g; +)$, and using the fact that the automorphism group of $C_2 \times C_2$ is the symmetric group on the 3 nonidentity elements, we deduce that the kernel is unique.

LEMMA 4. *If Q is a pseudo-symmetry of X, then Q acts without fixed points on X and $X/\langle Q \rangle = \Pi$, the projective plane.*

Proof. Suppose $p \in X$ and $Q(p) = p$. Let $f : X \to \Sigma$ be the natural projection and let Q induce a map $B : \Sigma \to \Sigma$. Thus $f \circ Q = B \circ f$ and B must be a symmetry of Σ fixing $f(p)$. It follows that B must fix pointwise a whole Jordan curve χ. If $f(p') \in \chi$, then $f \circ Q(p') = f(p')$. Hence $Q(p') = p'$ or $Q(p') = J(p')$; but $J = Q^2$ and so we must have $Q(p') = p'$. Therefore Q and hence J fix an infinite number of points, which is a contradiction as J fixes only $(2g+2)$ points.

It follows that the symmetry B must act without fixed points and so $X/\langle Q \rangle = \Sigma/\langle B \rangle = \Pi$.

NOTE. The fact that Q is fixed point free does not mean that X is a smooth covering of Π (which of course it cannot be) for $Q^2 = J$ does have fixed points.

Now let $X = D/K$ where K has signature $(g; +)$. Then K is uniquely determined up to conjugacy in $\mathscr{L}$.

THEOREM 1. *X is pseudo-symmetric if and only if X has even genus and there is an NEC group Γ of signature $(1; -, 2^{(g+1)})$ such that $K \lhd \Gamma$.*

Proof. Suppose that X has even genus and that such an NEC group exists. By Lemma 3, $\Gamma/K \cong C_4$ which we will suppose is generated by Q. As Q is the image in Γ of a glide reflection and as the kernel is a Fuchsian group, Q is anticonformal. Γ^+ has signature $(0; +, 2^{(2g+2)})$ and Γ^+/K is the subgroup of Γ/K generated by Q^2. By the remark preceding Lemma 3, X is hyperelliptic and Q is the hyperelliptic involution. Thus X is pseudo-symmetric.

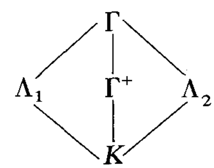
Conversely, if X is pseudo-symmetric, then K is normal in an NEC group Γ of index 4 with the properties that Γ^+ has signature $(0; +, 2^{(2g+2)})$ and $D/\Gamma = D/K/\Gamma/K = \Pi$ (by Lemma 4). It follows that Γ has signature $(1; -, 2^{(g+1)})$ and as $\Gamma/K \cong C_4$, g is even.

The fact that a pseudo-symmetric surface has even genus also follows from the work of Zarrow [**13**].

THEOREM 2. *Let X be a hyperelliptic surface of genus g. Then Q is a pseudo-symmetry of X if and ony if g is even and $\Phi(Q) = A$, the antipodal map.*

Proof. If Q is a pseudo-symmetry then X has even genus by Theorem 1 and $\Phi(Q)=A$ by Lemma 4.

Now suppose that g is even and $\Phi(Q)=A$. By Lemma 1, Q is a symmetry or a pseudo-symmetry. Suppose that Q is a symmetry. Then as $\Phi(Q)=\Phi(JQ)=A$ both Q and JQ would act without fixed curves. $\{E, J, Q, JQ\}$ would be a group of automorphisms of X isomorphic to $C_2\times C_2$ and we would be able to construct the following lattice of subgroups



where K has signature $(g;+)$, Λ_1 and Λ_2 have signatures $(g+1;-)$ and Γ^+ has signature $(0;+,2^{(2g+2)})$. Here Λ_1 and Λ_2 are the inverse images of the groups $\langle Q\rangle$, $\langle JQ\rangle$ under the epimorphism $\theta:\Gamma\to\Gamma/K$. If $c\in\Gamma$ is a reflection, then $\theta(c)=Q$ or JQ; but then Λ_1 or Λ_2 would contain reflections. Hence Γ has no reflections and therefore it must have signature $(1;-,2^{(g+1)})$. By Lemma 3, g is odd which is a contradiction. Therefore Q is a pseudo-symmetry.

COROLLARY. *X admits at most one pair of pseudo-symmetries.*

Proof. If $\{T_1, JT_1\}$, $\{T_2, JT_2\}$ are two pairs of pseudo-symmetries then as $\Phi(T_1)=\Phi(T_2)=A$, $T_1=T_2$, $JT_1=JT_2$ or $T_1=JT_2$, $JT_1=T_2$.

Using Lemma 2, Theorem 2 and its corollary it is now possible to determine the number of conjugacy classes of pairs of symmetries in G, a group of automorphisms containing anticonformal elements. If g is odd then it is equal to the number of conjugacy classes of symmetries in $\Phi(G)$, while if g is even, and if $\Phi(G)$ contains A it is one less than the number of symmetries in $\Phi(G)$.

$\Phi(G)$	Number of conjugacy classes of pairs of symmetries when g is even	Number of conjugacy classes of pairs of symmetries when g is odd
$C_2\times C_q$	0 (q odd), 1 (q even)	1 (q odd), 2 (q even)
D_q	1 (q odd), 2 (q even)	1 (q odd), 2 (q even)
$C_2\times D_q$	1 (q odd), 3 (q even)	2 (q odd), 4 (q even)
$C_2\times A_4$	1	2
S_4	1	1
$C_2\times S_4$	2	3
$C_2\times A_5$	1	2

From this table we deduce the following result.

THEOREM 3. *Let X be a hyperelliptic surface of genus g. If g is odd then X admits at most 4 conjugacy classes of pairs of symmetries. If g is even then X admits at most 3 conjugacy classes of pairs of symmetries.*

4. In this paragraph we present a family of surfaces for which the bound in Theorem 3 is attained. For each g there are infinitely many examples but the surfaces presented here are the most interesting in that, except for 3 values of g, they have the greatest number of symmetries of all hyperelliptic surfaces of genus g. For each $g \geqslant 2$, they are the unique hyperelliptic surface of genus g admitting a group of $8(g+1)$ conformal automorphisms and for infinitely many values of g this is the largest group of conformal automorphisms for any Riemann surface of genus g. In this context they were discovered independently by Accola [**1**] and Maclachlan [**9**]. Let

$$L^+(g) = \{R, S \mid R^4 = S^{2g+2} = (RS)^2 = (R^{-1}S)^2 = E\}.$$

(In the notation of [**3**] this is the group $(4, 2g+2 \mid 2, 2)$.)

$L^+(g)$ contains the central involution R^2 and $L^+(g)/\langle R^2\rangle \cong D_{2(g+1)}$. Hence $L^+(g)$ has order $8(g+1)$. It follows that $\langle R\rangle \cap \langle S\rangle = \{E\}$, and hence every element of the group has the form R^iS^j, $0 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 2g+1$.

Let Γ be a Fuchsian group with signature $(0; +, 4, 2g+2, 2)$. There is an obvious epimorphism $\Psi: \Gamma \to L^+(g)$ whose kernel $K(g)$ is a Fuchsian group with signature $(g; +)$. Let $D/K(g) = Y(g)$. $\psi^{-1}\langle R^2\rangle$ has signature $(0; +, 2^{(2g+2)})$ (see e.g. Proposition 4 of [**12**]) and so $Y(g)$ is hyperelliptic with hyperelliptic involution R^2. (Alternatively, we could follow Accola and construct $Y(g)$ directly as a two-sheeted covering of the sphere branched at the vertices of a regular $(2g+2)$-gon on the equatorial circle).

As $R \to R^{-1}$, $S \to S^{-1}$ extends to an automorphism of $L^+(g)$ it follows by Theorem 2 of [**11**] that $Y(g)$ is symmetric and has $L(G) = \{R, S, T \mid T^2 = (TR)^2 = (TS)^2 = R^4 = S^{2g+2} = (RS)^2 = (R^{-1}S)^2 = E\}$ as a group of automorphisms. Here T is a symmetry and $L(g)$ has order $16(g+1)$. We now find all the symmetries in $L(g)$. The anticonformal automorphisms all have the form TR^iS^j, and clearly TS^j, TR^2S^j are a pair of symmetries. Now

$$(TRS^j)^2 = TRS^jT^{-1}RS^j = R^{-1}S^{-j}RS^j = (S^{-1}R^2)^{-j}S^j = R^{-2j}S^{2j}.$$

(We are using $R^{-1}SR = S^{-1}R^2$ which is easily derivable from the relations for $L^+(G)$)

Thus TRS^j is a symmetry if and only if $R^{2j} = S^{2j} = E$. If g is odd this occurs when $j = 0$ or $j = g+1$, but for g even this only occurs when $j = 0$. Thus the only symmetries of the form TRS^j are TR and TRS^{g+1} when g is odd and TR when g is even. However, note that when g is even, TRS^{g+1} is a pseudo-symmetry. Now

$$S^{-1}(TS^j)S = TS^{j+2},$$

$$T^{-1}(TS^j)T = TS^{-j},$$

$$R^{-1}(TS^j)R = \begin{cases} TS^{-j} & j \text{ odd} \\ TS^{-j}R^2 & j \text{ even} \end{cases}$$

We thus see that the 4 conjugacy classes of pair of symmetries and pseudo-symmetries are

1. $\{T, TR^2\}, \{TS^2, TR^2S^2\}, \ldots, \{TS^{2g}, TR^2S^{2g}\}$,
2. $\{TS, TSR^2\}, \{TS^3, TR^2S^3\}, \ldots, \{TS^{2g+1}, TR^2S^{2g+1}\}$,
3. $\{TR, TR^3\}$,
4. $\{TRS^{g+1}, TR^3S^{g+1}\}$.

Classes 1, 2, 3 always consists of symmetries, while class 4 consists of symmetries if g is odd and pseudo-symmetries if g is even. (Note that in these groups we get examples where symmetries of a pair are themselves conjugate; for example $R^{-1}TR = TR^2$.)

$Y(g)$ admits $4g+4$ symmetries if g is odd and $4g+2$ symmetries if g is even. Now D_{2g+2} is the largest group of conformal homeomorphisms of Σ which lifts to a group on a hyperelliptic surface of genus g. For D_q has orbit lengths 2 and q in its action on Σ and, for it to lift, the $2g+2$ branch points must be an orbit implying that $q = 2g+2$. It follows by examining the table in §2 that $Y(g)$ admits the greatest number of symmetries for a hyperelliptic surface of genus g except possibly when $\Phi(G) \cong C_2 \times A_4$, $C_2 \times S_4$, S_4 or $C_2 \times A_5$, for low values of g.

As an example suppose that $\Phi(G) \cong C_2 \times S_4$. If G^+ is the subgroup of G consisting of conformal homeomorphisms then $\Phi(G^+) \cong S_4$ and the action of S_4 must lift up to the hyperelliptic surface. The orbits of S_4 have lengths 6, 8 or 12 which can be considered as the vertices of a regular cube, octahedron or the mid-points of an edge. Thus, if g is the genus of the hyperelliptic surface $2g+2 = 6$, 8 or 12 and hence $g = 2$, 3, or 5. For $g = 2$, 3 we get more symmetries (namely 20) than occur in $Y(2)$, $Y(3)$.

By comparing with §8.8 of **[3]** we can get presentations for these groups as follows:

$$(g=2)\{R, S \mid R^8 = S^3 = (RS)^2 = (R^{-3}S)^2 = E\};$$

$$(g=3)\{R, S \mid R^6 = S^4 = (RS)^2 = (R^{-2}S)^2 = E\}.$$

The first group has centre $\langle R^4 \rangle$ and the second group has centre $\langle R^3 \rangle$, the quotient in both cases being S_4. Thus, both groups have order 48 and we can prove that they act on symmetric surfaces as before. Therefore, for $g = 2$ and $g = 3$ we get hyperelliptic surfaces with 18 and 20 symmetries respectively. (For $g = 2$ the surface is the one which Coxeter and Moser remark has been beautifully drawn by Burnside.)

In a similar way we can get a surface of genus 5 with 32 symmetries. (The group here is $\{R, S \mid R^{10} = S^3 = (RS)^2 = (R^{-4}S)^2 = E\}$, centre $\langle R^5 \rangle$ and quotient A_5.) The next smallest genus which A_5 lifts up to is $g = 9$. We thus get the following result.

THEOREM 4. *Let $M(g)$ be the maximum number of symmetries for a hyperelliptic surface of genus g. Then*

$$M(2) = 18,$$
$$M(3) = 20,$$
$$M(5) = 32,$$
$$M(g) = 4g+2 \qquad (g \text{ even}, g \geq 4),$$
$$M(g) = 4g+4 \qquad (g \text{ odd}, g \geq 7).$$

Thus, the Accola–Maclachlan surfaces $Y(g)$ can be considered, (for $g \neq 2, 3, 5$) as the most symmetric of the hyperelliptic surfaces. It is interesting to compare with the elliptic case $(g=1)$.

Every complex torus is a two-sheeted covering of the sphere and again we have a central involution in the automorphism group so we can speak of paired symmetries. If we regard the torus as $\mathbb{C}$ modulo a lattice then this involution is induced by $z \to -z$.

The symmetries of complex tori are catalogued on page 65 of [**2**] and we see that the torus obtained from the square lattice has 3 conjugacy classes of pairs of symmetries, that obtained from a rectangular non-square lattice has 2 conjugacy classes of pairs and every other symmetric complex torus has one conjugacy class of pairs of symmetries. Thus, the square lattice gives the most symmetric torus. Now tori differ from Riemann surfaces of higher genus in that they admit continuous groups of automorphisms, (induced by $z \to z+\lambda$) and the torus obtained from the square lattice admits further automorphisms. In particular it admits a group of 16 automorphisms induced by $z \to \varepsilon z + \delta$, where $\varepsilon = \pm 1$, $\pm i$, $\delta = 0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2}+\frac{1}{2}i$. For simplicity of notation we regard these as being the automorphisms, so we consider δ as being a residue class in $\mathbb{C}/Z[i]$. Now let $R(z) = iz$, $S(z) = iz + \frac{1}{2}$. Then

$$R^4 = S^4 = (RS)^2 = (R^{-1}S)^2 = E,$$

so that this group is $L^+(1)$.

Now let $T(z) = i\bar{z}$. Then T gives a symmetry of the torus and $T^2 = (TR)^2 = (TS)^2 = E$. The 3 conjugacy classes of symmetries given in [**2**] have representatives $z \to \bar{z}$, $z \to i\bar{z}$, $z \to \bar{z} + \frac{1}{2}$, i.e. TR, T, TS^3R^2. (In this torus every symmetry is conjugate to its paired symmetry.)

If we work in analogy to the cases where $g \geq 2$, we would expect another conjugacy class, for example represented by $TR^3S^2(z) = \bar{z} + \frac{1}{2} + \frac{1}{2}i$. However, TS is conjugate to TR in the automorphism group of the torus but not in $L(1)$. For if $U(z) = z - \frac{1}{4}i$, $U^{-1}(TS^3R^2)U = TR^3S^2$.

5. Finally, we consider briefly the subspace of Teichmüller space corresponding to the pseudo-symmetric surfaces. The ideas and definitions can be found in [**8**]. See also [**4**], [**11**].

Let K be a Fuchsian group with signature $(g; +)$, such that D/K is pseudo-symmetric. Then by Lemma 3, there exists a unique NEC group Γ of signature $(1; -; 2^{(g+1)})$ such that $K \lhd \Gamma$. The normal inclusion of K in Γ gives rise to an embedding $T(\Gamma) \to T(K)$ of the corresponding Teichmüller spaces. The image of this inclusion corresponds to the set of pseudo-symmetric surfaces $Tp(K)$ in $T(K)$. As the NEC group Γ is the unique normal extension of Γ such that Γ/K has order 4 (by the corollary to Theorem 2) we can argue as in Lemma 3 of [**8**] that $Tp(K)$ is a submanifold of $T(\Gamma)$ with infinitely many components all equivalent under the Teichmüller modular group. Now the dimension of $T(\Gamma)$ is half the dimension of $T(\Gamma^+)$ ([**11**]) and we obtain

$$\dim Tp(K) = \dim T(\Gamma) = \tfrac{1}{2} \dim T(\Gamma^+) = 2g-1.$$

We can now apply these ideas to a problem considered by Earle in [**4**]. If there is an anticonformal homeomorphism between Riemann surfaces X and Y then X and Y are called *conjugate* surfaces. Symmetric Riemann surfaces are conformally equivalent to their conjugate surfaces and in [**4**], Earle found examples of non-symmetric surfaces of genera 2 and 5 which are conformally equivalent to their conjugates, by constructing surfaces which admit anticonformal automorphisms of order 4 but none of order 2.

Let Γ have signature $(1; -, 2^{(g+1)})$. Then Γ^+ has signature $(0; +, 2^{(2g+2)})$. Hence most Fuchsian groups isomorphic to Γ^+ are maximal and hence most NEC groups isomorphic to Γ are maximal ([**12**]). The points in the image of the embedding of $T(\Gamma)$ in $T(K)$ corresponding to these maximal groups will give pseudo-symmetric surfaces with no further automorphisms and, in particular, no symmetries. They are therefore non-symmetric surfaces conformally equivalent to their conjugates. These surfaces will correspond to a dense subset of $Tp(K)$.

To construct such surfaces for odd g, we cannot use pseudo-symmetric surfaces, but similar ideas will apply. For example, let Γ_1 be a maximal group with signature $(2; -, 2^{(g-1)})$. Define an epimorphism $\theta : \Gamma_1 \to C_4 = \{u \mid u^4 = 1\}$, by

$$\theta(a_1) = \theta(a_2) = u$$

$$\theta(x_i) = u^2 \quad (i = 1, \ldots, g-1).$$

If K is the kernel of θ, D/K_1 is a non-symmetric surface conformally equivalent to its conjugate and such surfaces correspond to a dense subset of $T(T_1)$ embedded in $T(K_1)$. Thus an infinite number of examples exist for each $g \geq 2$.

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