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ON THE GORENSTEINNESS OF REES AND FORM RINGS OF ALMOST COMPLETE INTERSECTIONS

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§1. Introduction

Let A be a Noetherian local ring and p a prime ideal in A. Let

$$R(p) = \bigoplus_{i \geqq 0} p^i \quad \text{and} \quad G(p) = \bigoplus_{i \geqq 0} p^i/p^{i+1}$$

and call them, respectively, the Rees ring and the form ring of p. The purpose of this paper is to clarify, provided p is an almost complete intersection in A (cf. (2.1) for definition), when the rings $R(p)$ and $G(p)$ are Gorenstein. The main results are contained in the following two theorems (see also Theorem (3.4)):

THEOREM (1.1). *The following conditions are equivalent.*

(1) *$G(p)$ is a Gorenstein ring.*

(2) *A is a Gorenstein ring and A/p is a Cohen-Macaulay ring.*

THEOREM (1.2). *Let $r = \dim A_p$ and assume that A is a Cohen-Macaulay ring. Then*

(1) *$R(p)$ is not a Gorenstein ring, if $r \geqq 3$.*

(2) *Suppose that $r = 2$. Then $R(p)$ is a Gorenstein ring if and only if $G(p)$ is a Gorenstein ring.*

Similar results for $r \leqq 1$ on the Gorensteinness of $R(p)$ may be found in Section 4 (cf. Lemma (4.1) and Theorem (4.4)).

The proof of Theorem (1.1) (resp. Theorem (1.2)) shall be given in Section 3 (resp. Section 4). We will summarize in Section 2 some basic results on almost complete intersections, which we frequently need to prove the above theorems.

Throughout this paper, let A be a Noetherian local ring of $\dim A = d$ and m the maximal ideal in A. Let $H_m^i(\cdot)$ denote, for each i, the i^{th} local cohomology functor relative to m.

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§ 2. Preliminaries

First of all, let us recall the definition of almost complete intersections.

DEFINITION (2.1). Let p be a prime ideal of A with $\dim A_p = r$. Then p is said to be an almost complete intersection in A, if p is minimally generated by $r+1$ elements and if the local ring A_p is regular.

For instance, let $d > r \geqq 0$ be integers and let R be a regular local ring of dimension $d+1$. Let

$$f = a_1 b + \sum_{i=2}^{r+1} a_i^2 \quad \text{and} \quad P = (a_1, a_2, \cdots, a_{r+1})R ,$$

where $a_1, a_2, \cdots, a_d, b$ is a regular system of parameters for R. We put $A = R/fR$ and $p = P/fR$. Then $\dim A_p = r$, and the ideal p is an almost complete intersection in A. Clearly A is a Gorenstein ring of dimension d and A/p is regular.

EXAMPLE (2.2). Let $k[[X, Y, Z]]$ and $k[[t]]$ be formal power series rings over a field k and let H be a numerical semigroup generated by 3 elements, say $H = \langle a, b, c \rangle$. Let $\varphi\colon k[[X, Y, Z]] \to k[[t]]$ be the k-algebra map defined by

$$\varphi(X) = t^a , \quad \varphi(Y) = t^b \quad \text{and} \quad \varphi(Z) = t^c .$$

We put $p = \operatorname{Ker} \varphi$. Then if the ring $k[[t^a, t^b, t^c]]$ is not Gorenstein, the prime ideal p is minimally generated by 3 elements and is an almost complete intersection in $k[[X, Y, Z]]$ (cf. [6]). For instance, let $H = \langle 3, 4, 5 \rangle$. Then

$$p = (X^3 - YZ,\ Y^2 - XZ,\ Z^2 - X^2 Y) ,$$

which is a typical example of almost complete intersections in $k[[X, Y, Z]]$.

Let $e(A)$ be the multiplicity of A relative to m. Given a finitely generated A-module M, let $v_A(M)$ denote the number of elements in a minimal system of generators for M. We put $v(A) = v_A(m)$, the embedding dimension of A.

PROPOSITION (2.3). *Let A be a regular local ring and p a prime ideal of A contained in m^2. Assume that A/p is a Cohen-Macaulay ring with*

$$v(A/p) = e(A/p) + \dim A/p - 1 .$$

Then the following conditions are equivalent.

(1) *p is an almost complete intersection in A.*

(2) $e(A/p) = 3$.

Proof. By [14, Theorem 1] we get $v_A(p) = \binom{e}{2}$, where $e = e(A/p)$. Notice that

$$\begin{aligned} v_A(p) - \dim A_p &= v_A(p) - (d - \dim A/p) \\ &= v_A(p) - (e - 1) \\ &= (v_A(p) - e) + 1\,, \end{aligned}$$

since $d = e + \dim A/p - 1$ by our assumption. Then p is an almost complete intersection in A if and only if $\binom{e}{2} = e$, i.e., $e = 3$.

Let $H \subset N^2$ be a finitely generated additive submonoid of N^2 with $\operatorname{rank}_Z H = 2$. Let k be a field and $k[H]$ the monoid algebra of H over k. We denote by t^h, for each $h \in H$, the canonical image of h in $k[H]$. Let $h_1, h_2, \cdots, h_v$ be a minimal system of generators for H. Let $R = k[X_1, X_2, \cdots, X_v]$ be a polynomial ring and let $\varphi: R \to k[H]$ denote the epimorphism of k-algebras defined by $\varphi(X_i) = t^{h_i}$ for all $1 \leqq i \leqq v$. We put $A = R_M$ and $p = PA$, where

$$M = (X_1, X_2, \cdots, X_v)R \quad \text{and} \quad P = \operatorname{Ker} \varphi\,.$$

COROLLARY (2.4). *Suppose that the ring $k[H]$ is normal. Then the following conditions are equivalent.*

(1) *p is an almost complete intersection in A.*

(2) $v = 4$.

Proof. Let $S = k[H]$ and $N = (t^{h_i} \mid 1 \leqq i \leqq v)S$. Clearly S is a Cohen-Macaulay ring, as it is normal and of dimension 2. Moreover by virtue of [3], we get that $v(S_N) = e(S_N) + 1$. Hence as $v = v(S_N)$, the assertion follows from (2.3).

Because the structure of 2-dimensional normal semigroups H ($\subset N^2$) is completely known (cf., e.g., [1]), one has, by (2.4), numerous examples of almost complete intersections p of height 2 in the regular local ring $A = k[X_1, X_2, X_3, X_4]_M$. For instance,

$$p = (X_3^2 - X_2X_4,\ X_1X_4 - X_2^2X_3,\ X_2^3 - X_1X_3)A \quad \text{where } H = \langle (0, 5),\ (1, 2),\ (3, 1),\ (5, 0) \rangle\,,$$

$$p = (X_2^2 - X_1X_3,\ X_1X_4 - X_2X_3,\ X_3^2 - X_2X_4)A \quad \text{where } H = \langle (0, 3),\ (1, 2),\ (2, 1),\ (3, 0) \rangle\,,$$

$$p = (X_2^3 - X_1X_3,\ X_3^2 - X_2X_4,\ X_1X_4 - X_2^2X_3^2)A$$
$$\text{where } H = \langle (0, 8),\ (1, 3),\ (3, 1),\ (8, 0)\rangle .$$

Now let us fix a prime ideal p of a Cohen-Macaulay local ring A and assume that p is an almost complete intersection. We put $r = \dim A_p$.

The next result (2.5) is already known (cf., e.g., [4] and [9]). However we need it so frequently that we shall give a proof for completeness.

LEMMA (2.5). *There exist elements* $a_1, a_2, \cdots, a_{r+1}$ *of* A *such that*

(1) $p = (a_1, a_2, \cdots, a_{r+1})$ *and* $pA_p = (a_1, a_2, \cdots, a_r)A_p$,

(2) $a_1, a_2, \cdots, a_r$ *is an* A*-regular sequence,*

and

(3) $(a_1, \cdots, a_r)\colon a_{r+1} = (a_1, \cdots, a_r)\colon a_{r+1}^2$.

Proof. Choose elements $a_1, a_2, \cdots, a_{r+1}$ of A so that $p = (a_1, a_2, \cdots, a_{r+1})$, $pA_p = (a_1, a_2, \cdots, a_r)A_p$, and $ht_A(a_1, a_2, \cdots, a_r) = r$ (cf. [12, p. 427, Corollary 2]). Then $a_1, a_2, \cdots, a_r$ is certainly an A-regular sequence, as A is Cohen-Macaulay.

Let us prove the assertion (3). Passing to the ring $A/(a_1, a_2, \cdots, a_r)$, we may assume that $r = 0$. Let $x \in A$ and suppose that $a_1x \neq 0$. We choose $P \in \operatorname{Ass} A$ so that $a_1x \neq 0$ in A_P. Then as $a_1 \neq 0$ in A_P, we get $P \not\supset p = (a_1)$ (recall that A_p is a field). Consequently $a_1^2x \neq 0$ in A_P, since a_1 is a unit and $x \neq 0$ in A_P. Thus $a_1^2x \neq 0$ in A and we get that $(0)\colon a_1 = (0)\colon a_1^2$ as required.

Let $a_1, a_2, \cdots, a_{r+1}$ be elements of A obtained by Lemma (2.5). Then by the conditions (2) and (3) we get

$$(a_1, \cdots, a_{i-1})\colon a_j = (a_1, \cdots, a_{i-1})\colon a_ia_j\ ,$$

whenever $1 \leqq i \leqq j \leqq r + 1$. Accordingly we immediately have the following

COROLLARY (2.6) ([9]). *The equality*

$$(a_1, \cdots, a_i) \cap p^n = (a_1, \cdots, a_i)p^{n-1}$$

holds for any integers $1 \leqq i \leqq r + 1$ *and* n.

COROLLARY (2.7) (cf., e.g., [8, p. 488, Remark]). *Let* $S(p)$ *denote the symmetric algebra of the* A*-module* p. *Then*

$$S(p) \cong R(p)$$

as A*-algebras.*

We put $s = \operatorname{depth} A/p$. Let $N = mR(p) + [R(p)]_+$ (resp. $M = mG(p) + [G(p)]_+$) be the unique graded maximal ideal of the Rees ring $R(p)$ (resp. the form ring $G(p)$) of p. Then as is well-known,

$$\dim [R(p)]_N = d + 1 \quad \text{and} \quad \dim [G(p)]_M = d$$

(cf., e.g., [17]). Moreover we have

PROPOSITION (2.8) ([4]). (1) $\operatorname{depth} [G(p)]_M = \min \{d, s + r + 1\}$.
(2) $\operatorname{depth} [R(p)]_N = \min \{d, s + 2\} \qquad (r = 0)$,
$= \min \{d + 1, s + r + 2\} \qquad (r > 0)$.

By virtue of this result we can easily estimate, in terms of $s = \operatorname{depth} A/p$ and $r = \dim A_p$, when the rings $R(p)$ and $G(p)$ are Cohen-Macaulay.

We close this section with one more remark. Let B be, for a moment, an arbitrary Noetherian local ring with maximal ideal n and E a Cohen-Macaulay B-module of dimension k. We put

$$r_B(E) = \dim_{B/n} \operatorname{Ext}^k_B (B/n, E)$$

and call it the type of E.

LEMMA (2.9) ([7, 1.22]). *Let x be an E-regular element of B. Then* $r_B(E) = r_B(E/xE)$.

We denote $r_B(B)$ simply by $r(B)$, in case B itself is Cohen-Macaulay. When B is not necessarily local, we put

$$r(B) = \sup_{P \in \operatorname{Spec} B} r(B_P)$$

and call it the global type of B (cf. [2]). Notice that the invariant $r(B)$ measures how the ring B differs from Gorenstein rings: B is a Gorenstein ring if and only if B is Cohen-Macaulay and $r(B) = 1$.

§ 3. The Gorensteinness of form rings $G(p)$ and Proof of Theorem (1.1)

Similarly as in the previous section, we suppose that A is a Cohen-Macaulay local ring. Let p be a prime ideal in A and assume that p is an almost complete intersection. We put

$R = R(p)$, the Rees ring of p ,
$G = G(p)$, the form ring of p ,
$N = mR + R_+$, the unique graded maximal ideal of R ,

$M = mG + G_+$, the unique graded maximal ideal of G,
$r = \dim A_p$,
$s = \operatorname{depth} A/p$.

Usually we identify R with the A-subalgebra

$$A[aT \mid a \in p]$$

of $A[T]$, where T is an indeterminate over A. Let $a_1, a_2, \cdots, a_{r-1}$ be elements of A obtained by Lemma (2.5). We put

$$I = (a_1, \cdots, a_r): a_{r+1}.$$

To begin with, we need the following

PROPOSITION (3.1). *$\dim A/I = d - r$ and*

$$\operatorname{depth} A/I = \min\{s + 1,\ d - r\}.$$

Proof. Since $I = (a_1, \cdots, a_r): a_{r+1}$ and $p = (a_1, \cdots, a_r, a_{r+1})$, we get a short exact sequence

$$(\sharp) \qquad 0 \longrightarrow A/I \longrightarrow A/(a_1, a_2, \cdots, a_r) \longrightarrow A/p \longrightarrow 0$$

of A-modules. As $s = \operatorname{depth} A/p$ and as $A/(a_1, a_2, \cdots, a_r)$ is a Cohen-Macaulay ring of dimension $d - r$, the second assertion follows from the sequence ($\sharp$).

For the first assertion, notice that the ideal I consists of zero-divisors in $A/(a_1, a_2, \cdots, a_r)$. Therefore we may choose $P \in \operatorname{Ass}_A A/(a_1, a_2, \cdots, a_r)$ so that $P \supset I$, whence $\dim A/I \geqq \dim A/P = d - r$. As $\dim A/I \leqq \dim A/(a_1, a_2, \cdots, a_r) = d - r$, the required equality $\dim A/I = d - r$ now follows.

COROLLARY (3.2). *Assume that $r = 0$ and that A/p is Cohen-Macaulay. Then A/I is a Cohen-Macaulay ring of dimension d.*

LEMMA (3.3). *$p \cap I = (a_1, a_2, \cdots, a_r)$.*

Proof. Passing to the ring $A/(a_1, a_2, \cdots, a_r)$, we may assume that $r = 0$. Let $x \in p \cap I$ and write $x = a_1 y$ with $y \in A$. Then $a_1 x = a_1^2 y = 0$, as $x \in I = (0): a_1$. Therefore $x = a_1 y = 0$, since $(0): a_1 = (0): a_1^2$ by (2.5) (3) and we get that $p \cap I = (0)$.

THEOREM (3.4). *Assume that A/p is Cohen-Macaulay. Then G is a Cohen-Macaulay ring of $r(G) = r(A)$.*

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Proof. As $s = d - r$, we see by (2.8) (1) that the local ring G_M is Cohen-Macaulay whence G is (cf. [11]). Let

$$f_i = a_i \bmod p^2$$

$(1 \leqq i \leqq r)$. Then it follows from (2.6) and [16, Theorem 1.1 and Proposition 2.6] that the forms $f_1, f_2, \cdots, f_r$ constitute a G-regular sequence and

$$G/(f_1, f_2, \cdots, f_r) \cong G(p/(a_1, a_2, \cdots, a_r))$$

as A-algebras. Consequently, as $r(G) = r(G_M)$ (cf. [2, Theorem 3.1]), by (2.9) we may assume, passing to the ring $A/(a_1, a_2, \cdots, a_r)$, without loss of generality that $r = 0$. Notice that in this situation all the A-modules A/p, A/I, and I are Cohen-Macaulay and of dimension d (cf. (3.2)).

Choose $t \in A$ so that $p = (0) : t$. (Notice that $t \in I$ but $t \notin p$.) We put $a = a_1$.

Claim 1. $a + t$ is regular in A.

Proof. We put $b_1 = a + t$. Then clearly b_1 is regular on A/p, as $b_1 \notin p$. Let $x \in A$ and assume that $b_1 x \in I$. Then $a \cdot b_1 x = a^2 x = 0$, because $at = 0$ and $I = (0) : a$. Hence $x \in I = (0) : a^2$ and b_1 is regular on A/I. Since the ring A is, by (3.3), embedded into the direct sum $A/p \oplus A/I$, we get that b_1 is regular also in A.

By this claim we find that the element $b_1 = a + t$ can be extended to a system, say $b_1, b_2, \cdots, b_d$ of parameters for A. We put

$$J = (b_2, b_3, \cdots, b_d) .$$

Claim 2. $\dim A/(p + tI + J) = 0$ and $r(G) = r(A/(p + tI + J))$.

Proof. We get $\dim A/(p + tI + J) = 0$, as $p + tI + J \supset (b_1^2) + J$ (recall that $t \in I$).

Consider the second equality. Let X be an indeterminate over A and let $\varphi \colon A[X] \to R$ be the epimorphism of A-algebras such that $\varphi(X) = aT$. Then as $I = (0) : a = (0) : a^2$, we get $\operatorname{Ker} \varphi = X \cdot IA[X]$ whence

$$A[X]/(pA[X] + X \cdot IA[X]) \cong G$$

as A-algebras (recall that $R/pR = G$). Consequently we have isomorphisms

$$(\sharp) \quad A/(p + tI + J) \cong A[X]/((X - t) + pA[X] + X \cdot IA[X] + JA[X]) \cong G/(X - t, b_2, b_3, \cdots, b_d)G$$

of A-algebras. Thus, since $\dim A/(p + tI + J) = 0$ as is proved above,

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we find that $G/(X-t, b_2, b_3, \cdots, b_d)G$ is an Artinian local ring whence $X-t, b_2, b_3, \cdots, b_d$ is a system of parameters for the Cohen-Macaulay local ring G_M. In particular

$$r(G_M) = r(G/(X-t,\ b_2, b_3, \cdots, b_d)G)$$

by (2.9). Hence the equality

$$r(G) = r(A/(p + tI + J))$$

follows from the isomorphisms (♯), as $r(G) = r(G_M)$ (cf. [2]).

Claim 3. There exist exact sequences

$$(1) \qquad 0 \longrightarrow I/(b_1, b_2, \cdots, b_d)I \longrightarrow A/(p + tI + J) \xrightarrow{f} A/(p + I + J) \longrightarrow 0$$

$$(2) \qquad 0 \longrightarrow I/(b_1, b_2, \cdots, b_d)I \longrightarrow A/(b_1, b_2, \cdots, b_d) \xrightarrow{g} A/(p + I + J) \longrightarrow 0$$

of A-modules, where f and g are canonical epimorphisms.

Proof. (1) It suffices to show that

$$(p + I + J)/(p + tI + J) \cong I/(b_1, b_2, \cdots, b_d)I$$

as A-modules. Notice that $(p+I+J)/(p+tI+J) \cong I/(I \cap (p+J) + tI)$.

First of all, we shall prove that $I \cap (p+J) = JI$. Let $x \in I \cap (p+J)$ and write $x = ax_1 + \sum_{i=1}^{d} b_i x_i$ with $x_i \in A$ $(1 \leqq i \leqq d)$. Then as $at = ax = 0$, we get

$$ax_1 \cdot (a + t) + \sum_{i=2}^{d} b_i \cdot (ax_i) = 0\,.$$

Consequently $ax_1 \in J$ because $b_1 = a + t, b_2, \cdots, b_d$ is an A-regular sequence, whence we find that $x = ax_1 + \sum_{i=2}^{d} b_i x_i \in J \cap I$. As $J \cap I = JI$ (recall that $b_2, b_3, \cdots, b_d$ is an A/I-regular sequence), this guarantees that

$$I \cap (p + J) = JI$$

as claimed.

Now recall that $tI = (a+t)I$ and we have

$$(I \cap (p + J)) + tI = JI + (a + t)I = (b_1, b_2, \cdots, b_d)I\,.$$

Thus $(p + I + J)/(p + tI + J) \cong I/(b_1, b_2, \cdots, b_d)I$ as A-modules.

(2) Because $p + I + J = I + (a + t, b_2, \cdots, b_d)$ and $b_1 = a + t, b_2, \cdots, b_d$ is an A/I-regular sequence (cf. (3.2)), the required exact sequence (2) immediately follows from the short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 .$$

Now let us finish the proof of Theorem (3.4). We have only to prove that

$$r(A/(p + tI + J)) = r(A/(b_1, b_2, \cdots, b_d)) ,$$

because $r(G) = r(A/(p + tI + J))$ by Claim 2 and $r(A) = r(A/(b_1, b_2, \cdots, b_d))$ by (2.9). To do this, it suffices to show that

$$f([(0) : m]_{A/(p+tI+J)}) = (0)$$

and

$$g([(0) : m]_{A/(b_1, b_2, \cdots, b_d)}) = (0)$$

in the exact sequences obtained by Claim 3 (recall that

$$r(A/(p + tI + J)) = \dim_{A/m} [(0) : m]_{A/(p+tI+J)}$$

and

$$r(A/(b_1, b_2, \cdots, b_d)) = \dim_{A/m} [(0) : m]_{A/(b_1, b_2, \cdots, b_d)}$$

by definition).

Let $x \in A$ and assume that $mx \subset p + tI + J$. Then as $tx \in p + tI + J$, we may write $t(x - y) \in p + J$ with $y \in I$. Notice that $(a + t)(x - y) \in p + J$, as $a \in p$. Then we get $x - y \in p + J$, because $b_1 = a + t$ is regular on $A/(p + J)$. Thus $x \in p + I + J$, i.e.,

$$f(x \bmod p + tI + J) = 0 \qquad \text{in } A/(p + I + J) .$$

Let $x \in A$ and assume that $mx \subset (b_1, b_2, \cdots, b_d)$. Then $tx = (a + t)y + z$ for some $y \in A$ and $z \in J$. Because $t(x - y) \in p + J$, we get $x \in p + I + J$ as is mentioned above. Thus

$$g(x \bmod (b_1, b_2, \cdots, b_d)) = 0 \qquad \text{in } A/(p + I + J) ,$$

which completes the proof of Theorem (3.4).

COROLLARY (3.5). *Assume that A is a Gorenstein ring. Then the following three conditions are equivalent.*

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(1) *G is a Gorenstein ring.*

(2) *G is a Cohen-Macaulay ring.*

(3) *A/p is a Cohen-Macaulay ring.*

Proof. (3) $\Rightarrow$ (1) See (3.4).

(1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) By the same reason as in Proof of Theorem (3.4), we may assume that $r = 0$. Then as G is Cohen-Macaulay, we get $s \geqq d - 1$ by (2.8) whence A/I is, by (3.1), a Cohen-Macaulay ring of dimension d. Notice that $(0): I = p$, since I is contained in A/p as an ideal (cf. (3.3)). Then we find by [10, Proposition 3.1] that the ring A/p is Cohen-Macaulay, because A is Gorenstein by our standard assumption.

Proof of Theorem (1.1).

(2) $\Rightarrow$ (1) This follows from (3.4).

(1) $\Rightarrow$ (2) By (3.5) it is enough to show that A is Gorenstein. We put

$$R' = R[T^{-1}] .$$

Then as is well-known,

$$G \cong R'/T^{-1}R'$$

as graded A-algebras. Let $N' = mR' + T^{-1}R' + [R']_+$. Then N' is a unique graded maximal ideal in R' and the local ring $R'_{N'}/T^{-1}R'_{N'} = G_M$ is, by our assumption (1), Gorenstein. Therefore so is $R'_{N'}$, as T^{-1} is R'-regular. Hence we get by [2, Theorem 3.1] that R' is globally Gorenstein. Consequently the ring $A[T, T^{-1}] = R'[T]$ is still Gorenstein and so we conclude that A itself is a Gorenstein ring (notice that $A[T, T^{-1}]$ is a faithfully flat extension of A).

EXAMPLE (3.6). Let $A = k[[X, Y, Z]]$ be a formal power series ring over a field k and put

$$p = (X^3 - YZ,\ Y^2 - XZ,\ Z^3 - X^2Y)$$

in A. Then as A/p is a Cohen-Macaulay ring (cf. (2.2)), we get by (3.5) that the ring $G(p)$ is Gorenstein.

Similarly let $A = k[[X, Y, Z, W]]$ be a formal power series ring and put

$$p = (Z^2 - YW,\ XW - Y^2Z,\ Y^3 - XZ)$$

in A. Then p is a prime ideal of A with height 2 and is an almost complete intersection in A (cf. Section 2). Moreover the ring $G(p)$ is Gorenstein, as A/p is Cohen-Macaulay also in this case.

§ 4. The Gorensteinness of Rees rings $R(p)$ and Proof of Theorem (1.2)

We shall maintain the assumption and notation in the preface of Section 3.

First of all, we note the following

LEMMA (4.1). *The ring R is not Cohen-Macaulay, if $r = 0$.*

Proof. The local ring R_N is not Cohen-Macaulay, as depth $R_N \leqq d$ by (2.8) (2) (recall that $\dim R_N = d + 1$, cf., e.g., [17]).

PROPOSITION (4.2). *Suppose that $r \geqq 3$. Then the ring R is not Gorenstein.*

Proof. The ring $R_p = R(pA_p)$ cannot be Gorenstein, since A_p is a regular local ring of $\dim A_p = r \geqq 3$ (cf., e.g., [5, Theorem (1.2)]).

Let K be an A-module. Then K is called the canonical module of A, if

$$\hat{A} \otimes_A K \cong \mathrm{Hom}_A\,(H^d_m(A), E_A(A/m))$$

as $\hat{A}$-modules, where $\hat{A}$ (resp. $E_A(A/m)$) denotes the completion of A (resp. the injective hull of A/m). The canonical module of A is, if it exists, uniquely determined up to isomorphisms, which we denote by K_A. Various properties of canonical modules are discussed by [7]. Here let us summarize some of them, which we shall need later.

PROPOSITION (4.3). (1) *The ring A possesses the canonical module if and only if A is a homomorphic image of a Gorenstein ring.*

(2) *Suppose that A possesses the canonical module K_A. Then*

$$v_A(K_A) = r(A)\,.$$

(3) *Let K be an A-module. Then the following two conditions are equivalent*:

(a) $K = K_A$.

(b) (i) *K is a Cohen-Macaulay A-module of dimension d,*

(ii) $[(0): K]_A = (0)$, *and* (iii) $\dim_{A/m}[(0): m]_{H^d_m(K)} = 1$.

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(4) *Suppose that A contains the canonical module K_A as a proper ideal. Then A/K_A is a Gorenstein ring of dimension $d-1$.*

Proof. (1) The *if* part is due to [7, 5.19]. See [13] for the *only if* part.

(2) See [7, Satz 6.10].

(3) (a) $\Rightarrow$ (b) See [7, Satz 6.1 and Lemma 6.6].

(b) $\Rightarrow$ (a) We may assume that A is complete. First of all, recall that

$$(\sharp) \qquad K \cong \operatorname{Hom}_A(\operatorname{Hom}_A(K, K_A), K_A)$$

as A-modules since the A-module K is Cohen-Macaulay and of dimension d (cf. [7, Satz 6.1]). Notice that

$$\operatorname{Hom}_A(K, K_A) \cong \operatorname{Hom}_A(H_m^d(K), E_A(A/m))$$

(cf. [7, Satz 5.2]) and we find, by the condition (iii), that the A-module $\operatorname{Hom}_A(K, K_A)$ is cyclic. Accordingly it follows from the condition (ii) that

$$\operatorname{Hom}_A(K, K_A) \cong A$$

because $(0)\colon \operatorname{Hom}_A(K, K_A) = (0)\colon K$ (cf. ($\sharp$)), whence we get by the isomorphism ($\sharp$) that $K = K_A$ as claimed.

(4) See [7, Korollar 6.13].

THEOREM (4.4). *Assume that $r = 1$. Then the following two conditions are equivalent.*

(1) *R is a Gorenstein ring.*

(2) $K_A = (a_1)\colon a_2$.

(*Here a_1, a_2 are elements of A obtained by* (2.5).)

In order to prove Theorem (4.4) we need one more lemma. Let $A[X]$ be a polynomial ring over A and let $P = mA[X] + XA[X]$, the unique graded maximal ideal of $A[X]$. We denote by $H_P^i(\cdot)$, for each i, the local cohomology functor relative to P. Given a graded $A[X]$-module L, we consider $H_P^i(L)$ to be a graded $A[X]$-module whose homogeneous component of degree n shall be denoted by $[H_P^i(L)]_n$ ($n \in \mathbf{Z}$).

LEMMA (4.5). *Let E be a finitely generated A-module and i an integer. Then*

(1) $[H_P^{i+1}(A[X] \otimes_A E)]_n = (0)$ *for all* $n \geqq 0$.

(2) $[H_P^{i+1}(A[X] \otimes_A E)]_n \cong H_m^i(E)$ *as A-modules for all* $n < 0$.

Proof. For a given graded $A[X]$-module L, we denote by $L(-1)$ the graded $A[X]$-module whose underlying module coincides with that of L and whose graduation is given by $[L(-1)]_n = L_{n-1}$ $(n \in \boldsymbol{Z})$. With this notation, first of all, consider the exact sequence

$$0 \longrightarrow (A[X] \otimes_A E)(-1) \xrightarrow{X} A[X] \otimes_A E \longrightarrow E \longrightarrow 0$$

of graded $A[X]$-modules. Apply functors $H_P^i(\cdot)$ to this sequence and we get a long exact sequence

$$(\sharp) \qquad \cdots \longrightarrow H_m^i(E) \xrightarrow{\partial} H_P^{i+1}(A[X] \otimes_A E)(-1) \xrightarrow{X} H_P^{i+1}(A[X] \otimes_A E) \longrightarrow \cdots$$

of local cohomology modules.

Now assume that $[H_P^{i+1}(A[X] \otimes_A E)]_n \neq (0)$ for some integer n. We choose such n as large as possible. Then as

$$X \cdot [H_P^{i+1}(A[X] \otimes_A E)(-1)]_{n+1} = (0) ,$$

we get by the sequence $(\sharp)$ that

$$(0) \neq [H_P^{i+1}(A[X] \otimes_A E)(-1)]_{n+1} \subset \partial([H_m^i(E)]_{n+1}) ,$$

whence $[H_m^i(E)]_{n+1} \neq (0)$. Thereby we get $n + 1 = 0$, because $[H_m^i(E)]_q = (0)$ for all $q \neq 0$. Thus

$$[H_P^{i+1}(A[X] \otimes_A E)]_n = (0)$$

for all integers $n \geqq 0$ and i.

Let us prove the assertion (2). Let $n < 0$ and i be integers. Consider the homogeneous component of degree n (resp. degree 0) in the sequence $(\sharp)$ and, because $H_m^j(E) = [H_m^j(E)]_0$ (resp. $[H_P^j(A[X] \otimes_A E)]_0 = (0)$ by the assertion (1)) for all j, we get an isomorphism

$$[H_P^{i+1}(A[X] \otimes_A E)]_{n-1} \cong [H_P^{i+1}(A[X] \otimes_A E)]_n$$
$$(\text{resp. } H_m^i(E) \cong [H_P^{i+1}(A[X] \otimes_A E)]_{-1})$$

of A-modules. Hence

$$[H_P^{i+1}(A[X] \otimes_A E)]_n \cong H_m^i(E)$$

for all integers $n < 0$ and i.

Proof of Theorem (4.4).

Let $A[X, Y]$ be a polynomial ring over A and let $\varphi: A[X, Y] \to R$ be the epimorphism of graded A-algebras defined by $\varphi(X) = a_2T$ and $\varphi(Y) = a_1T$. Then by virtue of (2.7) we find that

$$\operatorname{Ker}\varphi = (b_2X + b_1Y \mid b_1, b_2 \in A \text{ such that } a_1b_1 + a_2b_2 = 0)$$

in $A[X, Y]$. Hence

(a) $$A[X]/X\cdot IA[X] \cong R/(a_1T)$$

as graded A-algebras, where $I = (a_1): a_2$.

(1) $\Rightarrow$ (2) First of all, notice that $s \geqq d-2$ by (2.8) (2), as R is Cohen-Macaulay and as $r = 1$. Then we get by (3.1) that A/I is a Cohen-Macaulay ring of dimension $d-1$. Hence the A-module I is Cohen-Macaulay and of dimension d.

Now apply functors $H_P^i(\cdot)$ (here $P = mA[X] + XA[X]$) to the exact sequence

$$0 \longrightarrow (IA[X])(-1) \longrightarrow A[X] \longrightarrow A[X]/X\cdot IA[X] \longrightarrow 0$$

of graded $A[X]$-modules. Then we obtain an exact sequence

(b) $$0 \longrightarrow H_P^d(A[X]/X\cdot IA[X]) \longrightarrow H_P^{d+1}(IA[X])(-1) \longrightarrow H_P^{d+1}(A[X])$$

of local cohomology modules. Recall that

$$[H_P^{d+1}(A[X])]_0 = (0) \quad \text{and} \quad [H_P^{d+1}(IA[X])]_n = (0) \quad (n \geqq 0)$$

by (4.5) (1). Then we get, by the above sequence (b) and (4.5) (2), that

(c) $$[H_P^d(A[X]/X\cdot IA[X])]_n = (0) \quad (n > 0) \quad \text{and}$$
$$[H_P^d(A[X]/X\cdot IA[X])]_0 = H_m^d(I)\,.$$

On the other hand, recalling that $R/(a_1T)$ is a Gorenstein ring as R is Gorenstein by our assumption (1) and as a_1T is a regular element of R, we find by the isomorphism (a) that the ring $A[X]/X\cdot IA[X]$ is also Gorenstein. Hence

$$\dim_{A[X]/P}[(0): P]_{H_P^d(A[X]/X\cdot IA[X])} = 1$$

and consequently, by the assertion (c), we get that

$$\dim_{A/m}[(0): m]_{H_m^d(I)} = 1\,.$$

Since I is a Cohen-Macaulay A-module of dimension d and since

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$[(0): I]_A = (0)$ (recall that I contains a non-zerodivisor a_1 of A), we conclude by (4.3) (3) that $I = K_A$ as required.

(2) $\Rightarrow$ (1) We have by the assumption (2) that

$$K_{A[X]_P} = X \cdot IA[X]_P ,$$

since $IA[X] \cong X \cdot IA[X]$ as $A[X]$-modules and $K_{A[X]_P} = A[X]_P \otimes_A K_A$ (cf. [7, Korollar 5.21]). Hence, by (4.3) (4), the local ring $A[X]_P/X \cdot IA[X]_P$ is Gorenstein and thereby we get, by the isomorphism (a), that so is the ring $R_N/(a_1T)R_N$. Because a_1T is R-regular, R_N is Gorenstein and thus we have by [2, Theorem 3.1] that R is globally a Gorenstein ring. This completes the proof of Theorem (4.4).

COROLLARY (4.6). *Assume that $r = 1$ and that A is Gorenstein. Then R is not a Gorenstein ring.*

Proof. Suppose that R is Gorenstein. Then we have, by (4.3) (2) and (4.4), that

$$v_A((a_1): a_2) = 1 .$$

Let $I = (a_1): a_2 = (b)$ with $b \in A$.

Claim. $b \notin p$.

Proof. If $b \in p$, then we get $I = (a_1)$ as $p \cap I = (a_1)$ by (3.3). Hence a_1, a_2 is an A-regular sequence and so we have $r = \dim A_p = 2$ — this is impossible.

We write $a_1 = bx$ and $a_2 b = a_1 y$ with $x, y \in A$. Then $x \in p = (a_1, a_2)$, since $b \notin p$ by the above claim. Express $x = a_1 z + a_2 w$ with $z, w \in A$. Then as a_1 is A-regular and as

$$\begin{aligned} a_1 &= bx \\ &= a_1 \cdot bz + a_2 b \cdot w \\ &= a_1 \cdot bz + a_1 \cdot yw , \end{aligned}$$

we get $1 = bz + yw$ whence y is a unit of A. Hence $a_1 \in (a_2)$, as $a_2 b = a_1 y$ by our choice and therefore $p = (a_2)$ — this contradicts our standard assumption that p is minimally generated by 2 elements. Thus R is not a Gorenstein ring.

EXAMPLE (4.7). Let $k[[s, t, u]]$ be a formal power series ring over a field k. We put

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$$A = k[[s^2, t, s^3, st, s^2u, tu]]$$

in $k[[s, t, u]]$ and

$$p = (s^2u, tu)A .$$

Then

(1) A is a Cohen-Macaulay local domain of $\dim A = 3$;

(2) p is a prime ideal of height 1 and is an almost complete intersection in A;

(3) the ring $R(p)$ is Gorenstein;

(4) the ring $G(p)$ is Cohen-Macaulay but not Gorenstein.

Proof. (1) As $B = k[[s^2, t, s^3, st]]$ is a Buchsbaum local domain of dimension 2 and as s^2, t is a system of parameters for B, we know by [15] that A is a Cohen-Macaulay ring of dimension 3.

(2) As $B = A/p$, the ideal p is prime and $\dim A_p = 1$. Clearly p is an almost complete intersection in A (recall that $pA_p = (s^2u)A_p$).

(3) Notice that the elements $a_1 = s^2u$ and $a_2 = tu$ fulfill the conditions (1), (2), and (3) in Lemma (2.5). Let $I = (s^2u): tu$ in A. Then as depth $A/p = 1$, we get by (3.1) that A/I is a Cohen-Macaulay ring of dimension 2. Hence the A-module I is Cohen-Macaulay. Therefore in order to prove that the ring $R(p)$ is Gorenstein, it suffices, by (4.3) (3) and (4.4), to show that

$$\dim_{A/m} [(0): m]_{H^3_m(I)} = 1 .$$

Let $r_A(I) = \dim_{A/m} \operatorname{Ext}^3_A (A/m, I)$. Then we get, by [7, Satz 5.2 and Satz 6.10], that

$$\begin{aligned}\dim_{A/m} [(0): m]_{H^3_m(I)} &= v_A (\operatorname{Hom}_A (H^3_m(I), E_A(A/m))) \\ &= v_A (\operatorname{Hom}_A (I, K_A)) \\ &= r_A(I) .\end{aligned}$$

Hence we have only to show that $r_A(I) = 1$. Notice that $s^2, t - s^2u, tu$ is a system of parameters for A and that

$$I = (s^2u, s^2, s^3)A .$$

Then as

$$r_A(I) = \dim_{A/m} [(0): m]_{I/(s^2, t-s^2u, tu)I}$$

by (2.9), it is now routine to check that $r_A(I) = 1$, which we leave to the reader.

(4) The assertion that $G(p)$ is not Gorenstein follows from (1.1), because A/p $(=B)$ is not a Cohen-Macaulay ring in this example (see also (4.6)). Of course $G(p)$ is Cohen-Macaulay by (2.8) (1).

The rest of this section is devoted to a proof of Theorem (1.2). Notice that the assertion (1) in Theorem (1.2) is already given by (4.2). To prove the assertion (2) we need a few preliminaries.

In what follows, we assume that $r = 2$ and put

$$\bar{A} = A/a_1A\,, \quad \bar{p} = p/a_1A\,, \quad \text{and} \quad \bar{R} = R(\bar{p})\,.$$

Let $\varphi\colon R \to \bar{R}$ be the epimorphism of Rees rings induced by the canonical map $A \to \bar{A}$. Let $A[X, Y, Z]$ be a polynomial ring over A and let $\psi\colon A[X, Y, Z] \to R$ denote the homomorphism of A-algebras defined by

$$\psi(X) = a_1T\,, \quad \psi(Y) = a_2T\,, \quad \text{and} \quad \psi(Z) = a_3T\,.$$

Then we have the following

LEMMA (4.8). (1) $\operatorname{Ker}\varphi = (a_1, a_1T)$.

(2) $\operatorname{Ker}\psi = (b_1X + b_2Y + b_3Z \mid b_1, b_2, b_3 \in A$ *such that* $a_1b_1 + a_2b_2 + a_3b_3 = 0)$.

Proof. (1) This follows from the equality $a_1A \cap p^n = a_1p^{n-1}$ $(n > 0)$, cf. (2.6).

(2) See (2.7).

COROLLARY (4.9). *There exists an exact sequence*

$$0 \longrightarrow G(-1) \longrightarrow R/a_1R \longrightarrow \bar{R} \longrightarrow 0$$

of graded R-modules.

Proof. Notice that

$$\begin{aligned}(a_1, a_1T)/a_1R &\cong (a_1T)/(a_1T) \cap a_1R \\ &= (a_1T)/a_1T\cdot pR \\ &\cong (R/pR)(-1)\end{aligned}$$

as graded R-modules. Then since $G = R/pR$, we get by (4.8) (1) the required exact sequence at once.

We put

$$Q = (a_1T,\ a_2T - a_1,\ a_3T - a_2)R$$

and

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$$q = (a_1b_2 + a_2b_3 \mid b_2, b_3 \in A \text{ such that } a_1b_1 + a_2b_2 + a_3b_3 = 0 \text{ for some } b_1 \in A)\,.$$

LEMMA (4.10). (1) $q \supset (a_1^2, a_2^2 - a_1a_3, a_1a_2)$.

(2) $A/q \cong R/Q$ *as* A-*algebras.*

(3) $\dim A/q = d - 2$.

Proof. (1) Notice that $a_1(-a_2) + a_2a_1 = 0$, $a_2a_3 + a_3(-a_2) = 0$, and $a_1(-a_3) + a_3a_1 = 0$.

(2) We have an isomorphism

$$A[X, Y, Z]/((X, Y - a_1, Z - a_2) + \operatorname{Ker}\psi) \cong R/Q$$

of A-algebras. Notice that, by (4.8) (2), A/q is isomorphic to the ring on the left hand side.

(3) We get $\dim A/q \leqq d - 2$ by the assertion (1), since $a_1^2, a_2^2 - a_1a_3$ is an A-regular sequence. Recall that $\dim R_N = d + 1$ and that the ideal Q is generated by 3 elements in N. Then we have

$$\dim R/Q \geqq \dim R_N/QR_N \geqq d - 2\,,$$

whence $\dim A/q = \dim R/Q = d - 2$ (recall that $A/q \cong R/Q$ by the assertion (2)).

Now we are ready to prove Theorem (1.2).

Proof of Theorem (1.2) (2).

Assume that R is Gorenstein. Then we have $s \geqq d - 3$ by (2.8) (2). Hence by (2.8) (2) (resp. (2.8) (1)), the ring $\bar{R}$ (resp. G) is a Cohen-Macaulay ring of dimension d. Therefore applying functors $H_N^i(\cdot)$ to the exact sequence obtained by (4.9), we get a short exact sequence

$$(\sharp) \qquad 0 \longrightarrow [H_M^d(G)](-1) \longrightarrow H_N^d(R/a_1R) \longrightarrow H_N^d(\bar{R}) \longrightarrow 0$$

of local cohomology modules. Recall that $\dim_{R/N} [(0):N]_{H_N^d(R/a_1R)} = 1$, as the ring R/a_1R is Gorenstein. Then we have by the sequence ($\sharp$) that

$$\dim_{G/M} [(0):M]_{H_M^d(G)} = 1\,,$$

whence G is a Gorenstein ring.

Conversely suppose that the ring G is Gorenstein. Then by (1.1) A is Gorenstein and A/p is Cohen-Macaulay. Hence R is, by (2.8), Cohen-Macaulay. Since $a_1T, a_2T - a_1, a_3T - a_2$ is a subsystem of parameters for the local ring R_N and since $A/q \cong R/(a_1T, a_2T - a_1, a_3T - a_2)$ (cf. (4.10),

(2) and (3)), in order to prove that the ring R is Gorenstein it suffices to show that so is the ring A/q.

Let $J = (a_1^2, a_2^2 - a_1a_3)A$. Then as A is Gorenstein and as $a_1^2, a_2^2 - a_1a_3$ is an A-regular sequence, the ring A/J is again a Gorenstein ring of dimension $d-2$. Recall that $q \supset J$ and that A/q $(=R/Q)$ is a Cohen-Macaulay ring of dimension $d-2$. Then we find by [10, Proposition (3.1) (c)] that

$$r(A/q) = v_A((J : q)/J) .$$

LEMMA (4.11). $J : q = J + (a_2)$.

Proof. Let b_1, b_2, b_3 be elements of A such that $a_1b_1 + a_2b_2 + a_3b_3 = 0$. Then as $a_2^2 \equiv a_1a_3 \bmod J$, we find that

$$\begin{aligned} a_2 \cdot (a_1b_2 + a_2b_3) &\equiv a_1 \cdot (a_2b_2 + a_3b_3) \\ &\equiv a_1 \cdot (-a_1b_1) \\ &\equiv 0 \end{aligned}$$

mod J. Hence $a_2q \subset J$.

Conversely let $x \in J : q$. First of all, we shall show that $x \in (a_1, a_2)$. In fact, as $a_1a_2 \in q$ (cf. (4.10) (1)), we may write

$$a_1a_2x = a_1^2y + (a_2^2 - a_1a_3)z$$

with $y, z \in A$. Hence $z \in (a_1)$, as $a_1, a_2^2 - a_1a_3$ is an A-regular sequence. Let $z = a_1v$ with $v \in A$. Then we get, as $a_2x = a_1y + (a_2^2 - a_1a_3)v$, that $a_2(x - a_2v) \in (a_1)$. Hence $x \in (a_1, a_2)$ as required.

Now let us write $x = a_1f + a_2g$ with $f, g \in A$. Then $a_1f \in J : q$, because $a_2 \in J : q$ as is proved earlier. Let t be an element of A such that $p = (a_1, a_2) : t$. We express $a_1b_1 + a_2b_2 + a_3t = 0$ with $b_1, b_2 \in A$. Then $a_1b_2 + a_2t \in q$ by definition, whence $a_1f \cdot (a_1b_2 + a_2t) \in J$. We write

$$a_1f \cdot (a_1b_2 + a_2t) = a_1^2h + (a_2^2 - a_1a_3)k$$

with $h, k \in A$. Notice that $k \in (a_1)$, because $a_1, a_2^2 - a_1a_3$ is an A-regular sequence. Let $k = a_1k'$ with $k' \in A$. Then

$$f \cdot (a_1b_2 + a_2t) = a_1h + (a_2^2 - a_1a_3)k'$$

clearly and so we find that

$$a_2(ft - a_2k') \in (a_1) .$$

Therefore $ft \in (a_1, a_2)$ as a_1, a_2 is an A-regular sequence, whence $f \in p = (a_1, a_2) : t$. Because $p = (a_1, a_2, a_3)$, we get that

$$x = a_1f + a_2g \in (a_1^2, a_2, a_1a_3) = J + (a_2) .$$

Thus $J: q = J + (a_2)$ as claimed.

By this lemma (4.11) we find that $r(A/q) = 1$ and hence the ring R is Gorenstein. This completes the proof of Theorem (1.2) (2).

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