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Realizing representations on generalized flag manifolds

TIM BRATTEN
FaMAF UNC (5000) Córdoba, Argentina. e-mail: bratten@mate.uncor.edu

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Abstract. Let G be a complex reductive linear algebraic group and $G_0 \subseteq G$ a real form. Suppose P is a parabolic subgroup of G and assume that P has a Levi factor L such that $G_0 \cap L = L_0$ is a real form of L. Using the minimal globalization $V_{\min}$ of a finite length admissible representation for L_0, one can define a homogeneous analytic vector bundle on the G_0 orbit S of P in the generalized flag manifold $Y = G/P$. Let $\mathcal{A}(P, V_{\min})$ denote the corresponding sheaf of polarized sections. In this article we analyze the G_0 representations obtained on the compactly supported sheaf cohomology groups $H^p_c(S, \mathcal{A}(P, V_{\min}))$.

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1. Introduction

By the mid-1970s there had emerged two important geometric constructions for producing irreducible representations of real reductive Lie groups. On the one hand some irreducible representations could be realized as the global sections of certain real analytic vector bundles defined over various compact homogeneous spaces (this is the real parabolic induction) [28–30]. A second method obtained some other irreducible representations as actions on the sheaf cohomology groups of holomorphic vector bundles defined over complex homogeneous spaces [4, 10, 21].

By now it is becoming better understood how these constructions fit into the larger scheme of equivariant sheaves defined on orbits in complex flag manifolds [13] [16] [23]. Still, there are few results of a general nature about the topological representations obtained from these sorts of constructions in the setting of a generalized flag manifold [31]. Indeed, the two historically significant models mentioned above have yet to be analyzed under one conceptual heading. In this paper we do just that: beginning with polarized homogeneous vector bundles defined over a large class of orbits in a generalized flag manifold, we then characterize the representations obtained on the sheaf cohomology groups. Moreover, our methods make it possible to analyze representations originating from vector bundles with infinite dimensional geometric fibers. According to Chang's amplification [8] of a result by Hecht, Miličić, Schmid and Wolf [15], duality relates the representations realized in this paper to those studied in Vogan's book [28, Definition 6.3.1]. In some cases this duality can be made geometrically explicit (essentially because we can apply Serre duality [24]). This allows us to treat a certain conjecture about the geometric realization of Zuckerman modules [28, 29, Conjecture 6.11]. In case of finite dimensional geometric fibers, we obtain a new proof of a result due to Wong [31].

In order to make a precise statement of the main result we now specify the basic context for all that follows. Throughout this paper G will denote a connected reductive complex linear algebraic group with Lie algebra $\mathfrak{g}$. A *real form* of G means a closed subgroup whose Lie algebra is a real form for $\mathfrak{g}$. Suppose $G_0 \subset G$ is a real form and assume as well that G_0 has finitely many connected components. The purpose of this paper is the geometric realization of some representations for G_0.

A *parabolic subgroup* of G is defined to be any algebraic subgroup $P \subset G$ such that the corresponding quotient variety G/P is complete. On the other hand, a *generalized complex flag manifold* on which G acts is simply a complete homogeneous space for G. Suppose that $P \subset G$ is a parabolic subgroup and consider the generalized flag manifold G/P. A *Levi factor* of P refers to any maximal reductive algebraic subgroup $L \subset P$. We say that the G_0 orbit S of P in G/P is a *Levi orbit* provided P has a Levi factor L such that $L_0 = G_0 \cap L$ is a real form of L. Suppose that S is a Levi orbit and fix a maximal compact subgroup $K_0 \subset G_0$. By moving to a new point of S, if need be, we may assume $K_0 \cap L_0$ is a maximal compact subgroup in L_0. Assume $K \subset G$ is the complexification of K_0 and let Q denote the K orbit of P in G/P.

When a finite length, admissible representation for L_0 has an infinitesimal character then it will determine two geometrically defined objects as follows. On the one hand the underlying Harish–Chandra module V of the representation carries an algebraic action of $K \cap L$. Allowing the unipotent radical of $K \cap P$ to act trivially, we can thus obtain a K equivariant algebraic vector bundle with fiber V defined over the K orbit Q. Since V has infinitesimal character we can next apply a certain direct image construction, analogous to the direct image for $\mathcal{D}$ modules, to the sections of the bundle. The resulting object $\mathcal{I}(P, V)$ is a K equivariant sheaf of $\mathfrak{g}$ modules defined on all of Y.

On the other hand the minimal globalization $V_{\min}$ of V gives a global topological representation for the group L_0 [16] [22]. Similar to the above, we view $G_0 \cap P$ as acting on $V_{\min}$ by allowing the unipotent radical to act trivially. Since this continuous representation consists entirely of real analytic vectors, it determines a G_0 equivariant real analytic vector bundle with fiber $V_{\min}$ defined over the G_0 orbit S. The Lie algebra $\mathfrak{p}$ of P determines an equivariant polarization for the homogeneous vector bundle. Let $\mathcal{A}(P, V_{\min})$ denote the subsheaf of sections of the bundle which are annihilated by this polarization. The main result established in this paper is the following

THEOREM. *Let q denote the codimension of Q in Y.*

(a) *The compactly supported sheaf cohomology groups $H_c^p(S, \mathcal{A}(P, V_{\min}))$ have naturally defined dual nuclear Fréchet topologies and continuous G_0 actions. The resulting representations are admissible and have finite length.*

(b) *If the Harish–Chandra module V has an infinitesimal character which is antidominant for Y [Section 3] then the compactly supported sheaf cohomology groups $H_c^p(S, \mathcal{A}(P, V_{\min}))$ vanish unless $p = q$, in which case the G_0 module $H_c^q(S, \mathcal{A}(P, V_{\min}))$ is naturally and topologically isomorphic to the minimal globalization of $\Gamma(Y, \mathcal{I}(P, V))$.*

(c) *In any case the sheaf cohomology group $H^{p-q}(Y, \mathcal{I}(P, V))$ is a Harish–Chandra module for $(\mathfrak{g}, K)$ whose minimal globalization is naturally isomorphic to the topological G_0 module $H_c^p(S, \mathcal{A}(P, V_{\min}))$ for all p.*

If the G_0 orbit S is open and if the inducing representation V is finite dimensional, then $\mathcal{A}(P, V)$ is the sheaf of sections for a homogeneous holomorphic bundle with fiber V. Hence, duality coupled with our main theorem allows us to give a new proof of a conjecture by Vogan about Zuckerman modules, as mentioned in the end of the second paragraph.

Our main difficulty in seeing that this duality should be geometric when $V_{\min}$ is infinite dimensional occurs because the standard methods cannot show that $\mathcal{A}(P, V_{\min})$ is something like the sections of a holomorphic vector bundle. Apparently this difficulty was not anticipated in [28, 29], therefore we conclude the paper with a brief consideration of the problem.

This paper is divided into ten sections and is structured according to the following outline.

The first section is the introduction. In the following three sections we introduce the basic geometric setting and the essential functorality used for establishing the main result. Some of the relevant facts pertaining to analytic localization are reviewed and the theory is expanded somewhat to the setting of a generalized flag manifold. In addition we briefly recall some points in the algebraic theory for sheaves of twisted differential operators as well as develop a few analogs for the generalized counterpart. The fifth section introduces the Levi orbits and their duals, which provides a geometric setting where the current technology readily facilitates an understanding of the analytic sheaf.

In the sixth section we begin to consider the induced sheaves. Subsequently, we consider how an analytic group action effects the structure of the localization. It turns out that the analytic localization of the minimal globalization of a Harish–Chandra module provides a certain (weak) equivariant complex of sheaves whose hypercohomology is known. The aim of the forthcoming argument is to show (at least in some cases) that the hypercohomology of this complex of sheaves in fact computes the sheaf cohomology groups for the induced sheaf. A fundamental tool used for establishing this fact is a comparison theorem for geometric fibers which we prove in the seventh section.

In the case of regular antidominant infinitesimal character, it is now a simple matter to establish the main result, which we do in Section 8. Using tensoring arguments, the complete result is obtained in Section 9. Finally, Section 10 examines the special case of an open orbit, as mentioned previously.

Before beginning the main body of the paper, we would like to establish the following conventions and notations. A smooth algebraic variety X will at times be viewed as a complex manifold. Typically we let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, we let $\mathcal{D}_X$ denote the sheaf of differential operators with holomorphic coefficients and so on. The sheaf of regular functions on X is denoted $\mathcal{O}_X^{\mathrm{alg}}$, the sheaf of differential operators with regular coefficients is labeled $\mathcal{D}_X^{\mathrm{alg}}$ and so on. Given a morphism $\varphi: X \to Y$ of topological spaces we let φ_* denote the direct image in the category of sheaves, we let $\varphi_!$ denote direct image with proper supports and we let φ^{-1} denote the inverse image in the category of sheaves. A morphism $\varphi: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces has inverse image φ^* in the category of $\mathcal{O}$ modules. We refer the reader to Section 4 of this paper for a discussion of the functor φ_+.

2. Localizing to a flag manifold

In this section we briefly review some relevant points about localizing to a flag manifold.

Geometrically defined, a complex generalized flag manifold is any complete algebraic variety which carries a transitive action by a connected, complex affine algebraic group. These are precisely the spaces which can be realized as the quotient of a connected algebraic reductive group modulo a parabolic subgroup. On the other hand a *Borel subgroup* of a connected, linear algebraic group is a maximal solvable connected subgroup. The theory of affine algebraic groups reveals that the Borel subgroups are all conjugate, that the parabolic subgroups equal their own normalizers and that a subgroup is parabolic if and only if it contains a Borel subgroup.

We use the notation $Y = G/P$ where $P \subset G$ is a parabolic subgroup to denote a member of the family of generalized flag manifolds on which G acts. In case the stabilizer of a point in the complete homogeneous space is a Borel subgroup $B \subset G$ we use the notation $X = G/B$ and call X the full flag manifold. For each point $y \in Y$ let $\mathfrak{p}_y$ be the Lie algebra of the stabilizer P_y of y. Since parabolic subgroups equal their own normalizers, it follows that Y can be naturally identified with the G conjugates of $\mathfrak{p}_y$. In particular, X can be identified as the variety of maximal solvable subalgebras of $\mathfrak{g}$.

Let $x \in X$ and let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ be the nilradical of $\mathfrak{b}_x$. The adjoint actions of B_x on $\mathfrak{b}_x$ and $\mathfrak{n}_x$ determine homogeneous holomorphic vector bundles on X, with corresponding sheaves of sections $\mathfrak{b}^{\cdot}$ and $\mathfrak{n}^{\cdot}$. Since B_x acts trivially on $\mathfrak{b}_x/\mathfrak{n}_x$, the sheaf $\mathfrak{h}^{\cdot} = \mathfrak{b}^{\cdot}/\mathfrak{n}^{\cdot}$ is a free $\mathcal{O}_X$ module and the global sections $\mathfrak{h} = \Gamma(X, \mathfrak{h}^{\cdot})$ form what we call the *universal Cartan subalgebra*. For any point $x \in X$, if $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$, then the linear isomorphisms: $\mathfrak{h} \to \mathfrak{b}_x/\mathfrak{n}_x \leftarrow \mathfrak{c}$ determine an identification of $\mathfrak{h}$ with $\mathfrak{c}$, called a *specialization* of $\mathfrak{h}$ to $\mathfrak{c}$ at x. The specialization maps allow one to define a *universal root system* $\Sigma \subset \mathfrak{h}^*$ and a set of *universal positive roots* $\Sigma^+ \subset \Sigma$; we adopt the following positivity convention: if $\mathfrak{c}$ is a Cartan subalgebra in some Borel subalgebra $\mathfrak{b}_x$, then the positive roots at $x \in X$ are identified with the roots of $\mathfrak{c}$ in $\mathfrak{b}_x$.

A choice of $\lambda \in \mathfrak{h}^*$ determines a twisted sheaf of differential operators $\mathcal{D}_\lambda$, on X [See Prop. 3.1]. This sheaf has a certain topological structure inherited from the fact that it is locally free as an $\mathcal{O}_X$ module. In particular, sections over compact sets carry dnF(= dual nuclear Fréchet) topologies which give the structures for topological algebras. The restriction morphisms for nested compact sets provide continuous maps for these topologies. We are led to consider the category $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$ of dnF $\mathcal{D}_\lambda$ modules. The objects here are some sheaves of $\mathcal{D}_\lambda$ modules which carry a reasonable dnF topology over the compact sets in X. A $\mathcal{D}_\lambda$ morphism $\varphi\colon \mathcal{M} \to \mathfrak{N}$ of two dnF $\mathcal{D}_\lambda$ modules is a morphism in $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$ precisely when the induced map $\varphi_x\colon \mathcal{M}_x \to \mathfrak{N}_x$ on stalks is continuous, at each $x \in X$. An argument then shows that the image and cokernel of a continuous morphism belong to $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$, if the induced maps on stalks have closed ranges. One interesting point is that the category $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$ has enough acyclic objects for the functor of global sections: each object has a Γ_X acyclic resolution constructed from within the category [13].

It is known that $\mathcal{D}_\lambda$ is acyclic for Γ_X [19]. Let $U_\lambda = \Gamma(X, \mathcal{D}_\lambda)$. Since X is compact, U_λ is a dnF algebra: in fact it is an inductive limit of its finite dimensional subspaces. Because of this fact, $U_\lambda \otimes M$ is complete in the projective topology, when M is dnF, and hence it is a dnF space. Analogous to the sheaf side, we consider the category $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$, of dnF U_λ modules. A U_λ module belongs to this category precisely when it carries a dnF topology which allows U_λ to act by continuous operators. We observe that whenever M is an object in $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$, then the free left U_λ module $U_\lambda \otimes M$ is also a dnF module and there is a continuous surjection $U_\lambda \otimes M \to M$. Hence for each object in $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$ we can construct a free resolution of U_λ modules within the dnF category.

Let $\widehat{\otimes}$ denote the completed projective tensor product. In order to relate the categories $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$ and $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$, we first consider the object:

$$\Delta_X(M) = \mathcal{D}_\lambda \widehat{\otimes}_{U_\lambda} M.$$

Since $\Delta_X(M)$ is the cokernel of a morphism in $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$, it need not be an object in this category. A more important consideration is this: for many interesting objects M in $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$, examples show that $\Delta_X(M) = 0$.

It turns out that the program of analytic localization makes sense in the context of derived categories. In particular, it is possible to define derived categories $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda))$ and $D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda))$, of dnF U_λ modules and dnF $\mathcal{D}_\lambda$ modules. Both categories are triangulated in the usual way and Δ_X induces an exact functor $L\Delta_X\colon D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda))$ of triangulated categories. We will make a subtle use of the following fundamental result [13].

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THEOREM 2.1 [Hecht and Taylor]. *If λ is regular, then $L\Delta_X\colon D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda))$ is an equivalence of categories, with inverse $R\Gamma_X\colon D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda))$.*

At this point we briefly recall the structure of U_λ. Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$. At a point $x \in X$, an element $z \in Z(\mathfrak{g})$ agrees, mod the right ideal in $U(\mathfrak{g})$ generated by $\mathfrak{n}_x$, with a unique element $\psi_x(z) \in U(\mathfrak{b}_x/\mathfrak{n}_x)$. The resulting element in $U(\mathfrak{h})$ determined by this scheme, remains unchanged for any choice of point x. In this way we obtain an unnormalized *Harish Chandra map* $\psi\colon Z(\mathfrak{g}) \to U(\mathfrak{h})$. Let W be the *Weyl group* of Σ, put ρ = *one half the sum of the positive roots* and let χ_λ denote the composite: $Z(\mathfrak{g}) \xrightarrow{\psi} U(\mathfrak{h}) \xrightarrow{\lambda+\rho} \mathbb{C}$. Then $\chi_{w\lambda} = \chi_\lambda$, for each $w \in W$. Put $J_\theta = \ker \chi_\lambda$. It is known that U_λ is isomorphic to $U(\mathfrak{g})/J_\theta U(\mathfrak{g})$, as an algebra [2] [19]. In spite of this, it is convenient to keep the dependence on λ specific.

Before treating the case of a generalized flag manifold, a few remarks relating algebraic localization and analytic localization are in order. For $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein introduced a twisted sheaf of differential operators $\mathcal{D}_\lambda^{\mathrm{alg}}$, with regular coefficients, on X. Let $\mathcal{M}(U_\lambda)$ denote the category of U_λ modules and let $\mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}})$ be the category of quasicoherent $\mathcal{D}_\lambda^{\mathrm{alg}}$ modules. If M is an object in $\mathcal{M}(U_\lambda)$, define

$$\Delta_X^{\mathrm{alg}}(M) = \mathcal{D}_\lambda^{\mathrm{alg}} \otimes_{U_\lambda} M.$$

Recall that $\lambda \in \mathfrak{h}^*$ is called *antidominant* if $\check{\alpha}(\lambda)$ is not a positive integer, for each $\alpha \in \Sigma^+$. In [2] the following celebrated theorem is established.

THEOREM 2.2 [Beilinson and Bernstein]. *If λ is regular and antidominant, then $\Delta_X^{\mathrm{alg}}\colon \mathcal{M}(U_\lambda) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}})$ is an equivalence of categories, with inverse $\Gamma_X\colon \mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}}) \to \mathcal{M}(U_\lambda)$.*

In the context of representation theory for real reductive groups, the functor Δ_X^{alg} yields very interesting results, for certain finitely generated U_λ modules. This information is retained by the functor of analytic localization via the following scheme. A finitely generated U_λ module M is a dnF module, when it comes equipped with the inductive limit topology of its finite dimensional subspaces. By the very nature of its construction, $\mathcal{D}_\lambda \widehat{\otimes} M \simeq \mathcal{D}_\lambda \otimes M$ for this case. The Hochschild resolution of M is the free resolution $F.(M)$ given by

$$F_p(M) = \otimes^{p+1} U_\lambda \otimes M.$$

Observe that $\Delta_X(F_0(M)) \simeq \mathcal{D}_\lambda \otimes M$ and that $\Delta_X(F_p(M)) \simeq \mathcal{D}_\lambda \otimes (\otimes^p U_\lambda) \otimes M$ if $p \geq 1$. For a quasicoherent $\mathcal{O}_X^{\mathrm{alg}}$ module $\mathcal{F}$, let $\mathcal{F} \mapsto \mathcal{F}^{an}$ denote the application of Serre's GAGA functor [9] [25]. Since $(\mathcal{D}_\lambda^{\mathrm{alg}})^{an} \simeq \mathcal{D}_\lambda$ as a left $\mathcal{D}_\lambda$ module

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and using the fact that GAGA is an exact functor, it follows that $L_p\Delta_X(M) \simeq (L_p\Delta_X^{\text{alg}}(M))^{an}$ as a $\mathcal{D}_\lambda$ module, for each p.

3. Analytic localization on a generalized flag manifold

In this section we briefly treat the analytic localization to a generalized flag manifold. Some functorality is developed as needed for the proof of the main result.

Let Y be a generalized flag manifold on which G acts. If $\mathfrak{b}_x$ is a Borel subalgebra of $\mathfrak{g}$, there is a unique point $y \in Y$ with $\mathfrak{b}_x \subset \mathfrak{p}_y$. Hence, there is a G equivariant projection $\pi: X \to Y$. Since π is a proper morphism, the sheaf $\pi_*\mathcal{D}_\lambda$ is a dnF sheaf of algebras. Note that $\Gamma(Y, \pi_*\mathcal{D}_\lambda) = U_\lambda$.

We first review some key structural details about the sheaf $\pi_*\mathcal{D}_\lambda$. Let $\mathfrak{u}_y$ be the nilradical of $\mathfrak{p}_y$. The adjoint actions of P_y on $\mathfrak{u}_y$ and $\mathfrak{p}_y$ determine homogeneous holomorphic vector bundles on Y, with corresponding sheaves of sections $\mathfrak{u}^{\cdot}$ and $\mathfrak{p}^{\cdot}$. In this case P_y need not act trivially on the quotient $\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$. Nevertheless, $\mathfrak{l}^{\cdot} = \mathfrak{p}^{\cdot}/\mathfrak{u}^{\cdot}$ determines a sheaf of enveloping algebras $U(\mathfrak{l}^{\cdot})$. Note that $U(\mathfrak{l}^{\cdot})$ is the locally free G equivariant sheaf determined by the P_y action on $U(\mathfrak{l}_y)$, with algebra structure given by pointwise multiplication. Hence, if $Z(\mathfrak{l}^{\cdot}) =$ the center of $U(\mathfrak{l}^{\cdot})$, then $Z(\mathfrak{l}^{\cdot})$ is a free sheaf on Y. Put $\mathcal{Z}(\mathfrak{l}) = \Gamma(Y, Z(\mathfrak{l}^{\cdot}))$, *the universal center for the enveloping algebra of the Levi factor*. For any Levi factor $\mathfrak{l} \subset p_y$ there are morphisms of algebras $U(\mathfrak{l}) \to U(\mathfrak{l}_y) \leftarrow \Gamma(Y, U(\mathfrak{l}^{\cdot}))$. Letting $Z(\mathfrak{l})$ denote the center of $U(\mathfrak{l})$, we thus obtain the specialization isomorphism $Z(\mathfrak{l}) \to Z(\mathfrak{l}_y) \to \mathcal{Z}(\mathfrak{l})$.

Similar to the case of a flag manifold, we have an unnormalized *Harish Chandra map* $\mathcal{Z}(\mathfrak{l}) \to U(\mathfrak{h})$, obtained as follows. The preimage of a point $X_y = \pi^{-1}(y)$ is the flag manifold for the reductive Lie algebra $\mathfrak{l}_y$. An element of the universal Cartan for l_y is identified with a section of $\mathfrak{h}$ along X_y. Hence, via the Harish Chandra map previously defined, an element of $Z(\mathfrak{l}_y)$ determines an element of $U(\mathfrak{h})$. The desired map is obtained by evaluating an element from $\mathcal{Z}(\mathfrak{l})$, at y. The result of this map is the same for any choice of the point y. This identification of $\mathfrak{h}$ with the universal Cartan of $\mathfrak{l}_y$ allows one to define *the universal roots of* $\mathfrak{h}$ *in the Levi factor* $\Sigma(\mathfrak{l}) \subset \Sigma$, a corresponding set of positive roots $\Sigma(\mathfrak{l})^+ \subset \Sigma^+$ and a Weyl group $W_\mathfrak{l} \subset W$. Let χ_λ denote the composite: $\mathcal{Z}(\mathfrak{l}) \to U(\mathfrak{h}) \xrightarrow{\lambda+\rho} \mathbb{C}$. Then $\chi_\lambda = \chi_{w\lambda}$ for all $w \in W_\mathfrak{l}$. In order to compensate for the ambiguity in this parametrization of characters for $\mathcal{Z}(\mathfrak{l})$, the following definition is convenient. We call $\lambda \in \mathfrak{h}^*$ *antidominant for Y* if there is an element in the orbit $W_\mathfrak{l} \cdot \lambda$ which is antidominant. An equivalent condition is that $\check{\alpha}(\lambda)$ not be a positive integer for each $\alpha \in \Sigma^+ - \Sigma(\mathfrak{l})^+$. On the other hand we say λ is *antidominant for* $\mathfrak{l}$ if $\check{\alpha}(\lambda)$ is not a positive integer for each $\alpha \in \Sigma(\mathfrak{l})^+$. A basic fact is that each element in $\mathfrak{h}^*$ is conjugate under $W_\mathfrak{l}$ to an element that is antidominant for $\mathfrak{l}$.

The left action of $\mathfrak{g}$ on $\mathcal{O}_Y$, denoted by $f \mapsto \xi_l f$, for $\xi \in \mathfrak{g}$ and f a local holomorphic function, is utilized in defining an algebra structure on the sheaf

$U^{\cdot}(\mathfrak{g}) = \mathcal{O}_Y \otimes U(\mathfrak{g})$ as follows. To begin with, we require that $\mathcal{O}_Y \otimes 1$, and $1 \otimes U(\mathfrak{g})$ be subalgebras of $U^{\cdot}(\mathfrak{g})$ under the usual operations. This first requirement, together with the commutator relation: $[1 \otimes \xi, f \otimes 1] = \xi_l f \otimes 1$, for $f \in \mathcal{O}_Y$ and $\xi \in \mathfrak{g}$, determines the desired algebra structure. Note that $\mathfrak{u}^{\cdot} \subset U^{\cdot}(\mathfrak{g})$ is a sheaf of Lie subalgebras, under the operation of pointwise bracket. It turns out that $\mathfrak{u}^{\cdot} U^{\cdot}(\mathfrak{g})$ is a sheaf of two sided ideals in $U^{\cdot}(\mathfrak{g})$. Define:

$$\mathcal{D}_{\mathfrak{l}} = \frac{U^{\cdot}(\mathfrak{g})}{\mathfrak{u}^{\cdot} U^{\cdot}(\mathfrak{g})}.$$

There are natural inclusions of sheaves of algebras: $\mathcal{Z}(\mathfrak{l}) \to U(\mathfrak{l}^{\cdot}) \to \mathcal{D}_{\mathfrak{l}}$. One checks that $\mathcal{Z}(\mathfrak{l})$ is the center of $\mathcal{D}_{\mathfrak{l}}$. The following is established in [7, 13].

PROPOSITION 3.1 (Chang, Hecht and Taylor). *For each* $\lambda \in \mathfrak{h}^*$

(a) $\mathcal{D}_\lambda$ *is acyclic for* π_*
(b) $\pi_* \mathcal{D}_\lambda \simeq \mathcal{D}_{\mathfrak{l}} \otimes_{\mathcal{Z}(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}$, *as a sheaf of algebras.*

There is a category $\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda)$ of dnF $\pi_* \mathcal{D}_\lambda$ modules and a corresponding derived category $D(\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda))$. Using the same sort of topological Czech resolutions as employed on the flag manifold [13] we define a derived functor of global sections $R\Gamma_Y \colon D(\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda))$.

For M an object in $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$ consider the sheaf

$$\Delta_Y(M) = \pi_* \mathcal{D}_\lambda \widehat{\otimes}_{U_\lambda} M.$$

This definition determines a right exact functor Δ_Y into the category of sheaves of $\pi_* \mathcal{D}_\lambda$ modules. The functor carries continuous morphisms of dnF U_λ modules to morphisms of $\pi_* \mathcal{D}_\lambda$ modules, which are continuous for the quotient topologies induced on stalks. Using the same sort of construction previously employed on the flag manifold we obtain a derived functor $L\Delta_Y \colon D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda))$.

PROPOSITION 3.2. (a) *The functor* $R\Gamma_Y \circ L\Delta_Y$ *is isomorphic to the identity.*
(b) $L\Delta_Y \simeq R\pi_* \circ L\Delta_X$ *as functors* $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda))$.

Proof. With the help of Proposition 3.1, the argument for (a) becomes identical to the case of a flag manifold [13, Prop. 5.2].

To establish (b) suppose that M is a dnF U_λ module. Using the fact that $\pi \colon X \to Y$ is proper we can see that the map: $\pi_* \mathcal{D}_\lambda \widehat{\otimes} M \simeq \pi_*(\mathcal{D}_\lambda \widehat{\otimes} M) \to \pi_*(\mathcal{D}_\lambda \widehat{\otimes}_{U_\lambda} M)$ determines a natural transformation of functors $\Delta_Y \to \pi_* \circ \Delta_X$, which is an isomorphism on free objects from $\mathcal{M}_{\mathrm{dnF}}(U_\lambda)$. Since free dnF $\mathcal{D}_\lambda$ modules are π_* acyclic and since π_* has finite cohomological dimension, a simple formal argument allows us to conclude the desired result $\square$

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The next task is to develop a simple but crucial functorial relationship which we call the *analytic base change*. The result we need can be easily and elegantly expressed in the language of derived categories.

Let $y \in Y$ put $X_y = \pi^{-1}(y)$ and consider the following diagram

$$\begin{tikzcd} X_y \arrow[r, "i"] \arrow[d] & X \arrow[d, "\pi"] \\ \{y\} \arrow[r] & Y \end{tikzcd}$$

Let $\mathfrak{l}$ be a Levi factor for $\mathfrak{p}_y$. When $\mathcal{F}$ is a $\mathcal{D}_\lambda$ module, then $i^*\mathcal{F}$ is a sheaf of $\mathfrak{l}$ modules via a tensor product action. This $\mathfrak{l}$ action induces an action of the sheaf of algebras $U^{\cdot}(\mathfrak{l}) = \mathcal{O}_{X_y} \otimes U(\mathfrak{l})$. In turn, this action factors through a sheaf of ideals, so that the quotient $\mathcal{D}^i_\lambda$ is a twisted sheaf of differential operators for the flag manifold X_y, acting on $i^*\mathcal{F}$. In particular, $\Gamma(X_y, i^*\mathcal{F})$ is a sheaf of modules for

$$U_\lambda(\mathfrak{l}) = U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho} = \Gamma(X_y, \mathcal{D}^i_\lambda),$$

where $Z(\mathfrak{l})$ acts through the Harish–Chandra map: $Z(\mathfrak{l}) \to Z(\mathfrak{l}_y) \to U(\mathfrak{h})$. On the other hand if $\mathcal{F}$ is a sheaf of $\pi_*\mathcal{D}_\lambda$ modules then the morphism of algebras $U(\mathfrak{l}) \to \Gamma(Y, \pi_*\mathcal{D}_\lambda)$ determines an action of $U_\lambda(\mathfrak{l})$ on the geometric fiber $T_y\mathcal{F}$. Let $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda(\mathfrak{l})))$ and $D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}^i_\lambda))$ denote the derived categories of appropriately defined dnF modules.

PROPOSITION 3.3. (a) *The functors i^* and T_y determine derived functors: Li^*: $D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}^i_\lambda))$ and LT_y: $D(\mathcal{M}_{\mathrm{dnF}}(\pi_*\ \mathcal{D}_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda(\mathfrak{l})))$.*

(b) *$LT_y \circ L\Delta_Y \simeq R\Gamma_{X_y} \circ Li^* \circ L\Delta_X$ as functors $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda)) \to D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda(\mathfrak{l})))$.*

Proof. Let $x \in X_y$. Since the stalk $\mathcal{O}_{X_y,x}$ is a finitely generated module for $\mathcal{O}_{X,x}$, the natural map: $\mathcal{O}_{X_y,x} \otimes_{\mathcal{O}_{X,x}} M \to \mathcal{O}_{X_y,x} \widehat{\otimes}_{\mathcal{O}_{X,x}} M$ is a bijection, whenever M is a dnF module for $\mathcal{O}_{X,x}$. In fact, resolving $\mathcal{O}_{X_y,x}$ by finitely generated free $\mathcal{O}_{X,x}$ modules and applying the natural transformation: $(\cdots) \otimes_{\mathcal{O}_{X,x}} M \to (\cdots) \widehat{\otimes}_{\mathcal{O}_{X,x}} M$ shows at once that free dnF $\mathcal{O}_{X,x}$ modules are acyclic for the functor: $\mathcal{O}_{X_y,x} \otimes_{\mathcal{O}_{X,x}} (\cdots)$. Hence, free dnF $\mathcal{D}_\lambda$ modules are acylic for i^* and there is an isomorphism of sheaves: $i^*(\mathcal{D}_\lambda \widehat{\otimes} M) \simeq (i^*\mathcal{D}_\lambda) \widehat{\otimes} M$, where $i^*\mathcal{D}_\lambda$ is a dnF sheaf on X_y in a natural fashion and M is any dnF space. In more generality, if $\mathcal{D}_\lambda \widehat{\otimes} \mathcal{F}$ is a quasifree object from $\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda)$ [13] then its stalk at x is isomorphic to $\mathcal{D}_{\lambda,x} \widehat{\otimes} \mathcal{F}_x$. It follows that quasifree objects are acyclic for i^* and there is an isomorphism of sheaves: $i^*(\mathcal{D}_\lambda \widehat{\otimes} \mathcal{F}) \simeq (i^*\mathcal{D}_\lambda) \widehat{\otimes} i^{-1}(\mathcal{F})$. Since the functor i^* has finite cohomological dimension we can now see there is a naturally induced derived functor $Li^*: D(\mathcal{M}_{\text{dnF}}(\mathcal{D}_\lambda)) \to D(\mathcal{M}_{\text{dnF}}(\mathcal{D}^i_\lambda))$

Using the fact that $\pi_*\mathcal{D}_\lambda$ is a locally free $\mathcal{O}_Y$ module, nearly identical considerations show that quasifree dnF $\pi_*\mathcal{D}_\lambda$ modules are acyclic for T_y and there is an isomorphism $T_y(\pi_*\mathcal{D}_\lambda\widehat{\otimes}\mathcal{F}) \simeq (T_y\pi_*\mathcal{D}_\lambda)\widehat{\otimes}\mathcal{F}_y$. In particular, we obtain a derived functor $LT_y: D(\mathcal{M}_{\text{dnF}}(\pi_*\mathcal{D}_\lambda)) \to D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})))$.

To establish (b) assume M is a dnF U_λ module. Then the morphisms $(T_y\pi_*\mathcal{D}_\lambda)\widehat{\otimes}M \simeq \Gamma(X_y, i^*\mathcal{D}_\lambda)\widehat{\otimes}M \simeq \Gamma_{X_y} \circ i^*(\mathcal{D}_\lambda\widehat{\otimes}M) \to \Gamma_{X_y} \circ i^* \circ \Delta_X(M)$ determine a natural transformation: $T_y \circ \Delta_Y \to \Gamma_{X_y} \circ i^* \circ \Delta_X$, which is an isomorphism on free dnF U_λ modules. Indeed, because $\Gamma(X_y, i^*\mathcal{D}_\lambda) \simeq T_y\pi_*\mathcal{D}_\lambda$ is a direct limit of its finite dimensional subpaces, it follows that $\Gamma(X_y, i^*\mathcal{D}_\lambda)\widehat{\otimes}M \simeq \Gamma(X_y, i^*\mathcal{D}_\lambda) \otimes M$. Recalling that $i^*\mathcal{D}_\lambda$ is acyclic for Γ_{X_y} [13], it now follows that $i^*(\mathcal{D}_\lambda\widehat{\otimes}M)$ is acyclic for Γ_{X_y} as well. □

4. The category $M_{qc}(\pi_*D^{\text{alg}}_\lambda)$

In this section we recall a few facts about the algebraic localization to a generalized flag manifold. We also consider the generalized direct image functor, which is a natural analog of the direct image for $\mathcal{D}$ modules in the setting of a generalized flag manifold.

If we think of X and Y as algebraic varieties then $\pi: X \to Y$ is a morphism. Let $\pi_*\mathcal{D}^{\text{alg}}_\lambda$ denote the sheaf of algebras $\pi_*(\mathcal{D}^{\text{alg}}_\lambda)$ and let $\mathcal{M}_{qc}(\pi_*\mathcal{D}^{\text{alg}}_\lambda)$ be the category of quasicoherent $\pi_*\mathcal{D}^{\text{alg}}_\lambda$ modules.

If M is a U_λ module then the algebraic localization to the generalized flag manifold is the quasicoherent sheaf of $\pi_*\mathcal{D}^{\text{alg}}_\lambda$ modules defined as follows

$$\Delta^{\text{alg}}_Y(M) = \pi_*\mathcal{D}^{\text{alg}}_\lambda \otimes_{U_\lambda} M.$$

LEMMA 4.1. *If λ is antidominant, then the following hold*

(a) $\pi_*: \mathcal{M}_{qc}(\mathcal{D}^{\text{alg}}_\lambda) \to \mathcal{M}_{qc}(\pi_*\mathcal{D}^{\text{alg}}_\lambda)$ *is exact.*
(b) $\Delta^{\text{alg}}_Y \simeq \pi_* \circ \Delta^{\text{alg}}_X$.
(c) $\Gamma_Y \circ \Delta^{\text{alg}}_Y \simeq id$.

Proof. The first claim is shown in [7, Theorem 4.16]. To establish (b), observe there is an exact sequence $U_\lambda \otimes K \to U_\lambda \otimes M \to M \to 0$, for M from $\mathcal{M}(U_\lambda)$. Now use the natural transformation $\Delta^{\text{alg}}_Y \to \pi_* \circ \Delta^{\text{alg}}_X$ and apply part (a). Part (c) is a formal consequence of Part (b) and the identity $\Gamma_Y \circ \pi_* \simeq \Gamma_X$. □

We will make use of the following analog of Theorem 2.2, which was observed by Chang in his thesis [7].

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THEOREM 4.2 [Chang]. *If λ is regular and antidominant for Y, then $\Delta_Y^{\mathrm{alg}}: \mathcal{M}(U_\lambda) \to \mathcal{M}_{qc}(\pi_* \mathcal{D}_\lambda^{\mathrm{alg}})$ is an equivalence with inverse Γ_Y.*

We make note here that the remarks following Theorem 2.2 hold in the context of the generalized flag manifold as well. In particular, if $\mathcal{F}$ is a quasicoherent $\mathcal{O}_Y^{\mathrm{alg}}$ module and if $\mathcal{F} \mapsto \mathcal{F}^{an}$ denotes the application of Serre's GAGA functor then the canonical morphism $H^p(Y, \mathcal{F}) \to H^p(Y, \mathcal{F}^{an})$ is an isomorphism [9]. Observe that the natural map $(\pi_* \mathcal{D}_\lambda^{\mathrm{alg}})^{an} \to \pi_* \mathcal{D}_\lambda$ induces an isomorphism on geometric fibers. Since both sheaves are locally free, the map is an isomorphism of left $\pi_* \mathcal{D}_\lambda$ modules. Let M be a finitely generated U_λ module and let $F.(M)$ denote its Hochschild resolution. As before, when M is regarded as limit of its finite dimensional subspaces, there is an isomorphism $\Delta_Y(F.(M)) \simeq (\Delta_Y^{\mathrm{alg}}(F.(M)))^{an}$ of complexes of $\pi_* \mathcal{D}_\lambda$ modules.

We now briefly consider a certain direct image functor for some sheaves on the generalized flag manifold which gives the analog for the direct image of algebraic $\mathcal{D}$ modules [3]. We alter the notations momentarily to streamline the exposition. In particular, let $\mathcal{A}_\lambda = \pi_* \mathcal{D}_\lambda^{\mathrm{alg}}$ and assume that *all objects are defined in the algebraic category for the remainder of this section*.

Suppose that $Q \subset Y$ is a smooth subvariety with inclusion morphism $Q \xrightarrow{j} Y$. Let $\mathcal{A}_\lambda^j$ be the sheaf of differential endomorphisms of the $\mathcal{O}_Q$ module $j^* \mathcal{A}_\lambda$ which commute with the right $j^{-1} \mathcal{A}_\lambda$ action. Let Ω_Q and Ω_Y be the canonical bundles for Q and Y. Put $\Omega_{Q|Y} = \Omega_Q \otimes_{\mathcal{O}_Q} j^* \Omega_Y^{-1}$ and $(\mathcal{A}_\lambda)_{Y \leftarrow Q} = \Omega_{Q|Y} \otimes_{\mathcal{O}_Q} j^* \mathcal{A}_{-\lambda}$, where $\mathcal{A}_{-\lambda}$ is the sheaf of algebras opposite to $\mathcal{A}_\lambda$. Then $(\mathcal{A}_\lambda)_{Y \leftarrow Q}$ is a left $j^{-1} \mathcal{A}_\lambda$, right $\mathcal{A}_\lambda^j$ bimodule. If $\mathcal{V}$ is a module for $\mathcal{A}_\lambda^j$ we define

$$j_+(\mathcal{V}) = j_*((\mathcal{A}_\lambda)_{Y \leftarrow Q} \otimes_{\mathcal{A}_\lambda^j} \mathcal{V})$$

In slightly more generality, assume we have nested inclusions of smooth subvarieties $Q_1 \xrightarrow{j_1} Q_2 \xrightarrow{j_2} Y$ and let $j = j_2 \circ j_1$. In the manner preceding, it is possible to define a sheaf of algebras $\mathcal{A}_\lambda^{j_1}$ which acts on $j_1^*(\mathcal{A}_\lambda^{j_2})$ and a direct image $j_{1+}: \mathcal{M}(\mathcal{A}_\lambda^{j_1}) \to \mathcal{M}(\mathcal{A}_\lambda^{j_2})$. The following extensions of the usual $\mathcal{D}$ module case appear in [7, 8].

THEOREM 4.3 [Chang]. (a) *The above definitions give left exact functors $j_{1+}: \mathcal{M}_{qc}(\mathcal{A}_\lambda^{j_1}) \to \mathcal{M}_{qc}(\mathcal{A}_\lambda^{j_2})$ and $j_{2+}: \mathcal{M}_{qc}(\mathcal{A}_\lambda^{j_2}) \to \mathcal{M}_{qc}(\mathcal{A}_\lambda)$.*

(b) *$\mathcal{A}_\lambda^{j_1} \simeq \mathcal{A}_\lambda^j$ and $j_+ \simeq j_{2+} \circ j_{1+}$ as functors $\mathcal{M}_{qc}(\mathcal{A}_\lambda^j) \to \mathcal{M}_{qc}(\mathcal{A}_\lambda)$.*

(c) *When Q_1 is open in Q_2 then j_1+ can be naturally identified with the direct image in the category of sheaves.*

(d) *If Q_1 is closed in Q_2 then Kashiwara's equivalence of categories holds* [3, *Thm.* 7.11]. *In particular, j_{1+} gives an equivalence of $\mathcal{M}_{qc}(\mathcal{A}_\lambda^{j_1})$ with the full subcategory of sheaves in $\mathcal{M}_{qc}(\mathcal{A}_\lambda^{j_2})$ supported in Q_1.*

A relatively simple and formal consequence of Kashiwara's theorem is a certain algebraic base change which will play a crucial role in the main argument. In particular, suppose y is a point in Q and let T_y^Q denote the geometric fiber at y relative to the smooth subvariety Q. T_y will denote the geometric fiber at y relative to the global space Y. We consider the following diagram:

$$\begin{tikzcd}
 & \{y\} \arrow[dl] \arrow[dr] & \\
Q \arrow[rr, "j"'] & & Y
\end{tikzcd}$$

If $\mathcal{V}$ is an $\mathcal{A}_\lambda^j$ module then $U_\lambda(\mathfrak{l}_y)$ acts on the geometric fiber $T_y^Q\mathcal{V}$. Similarly, if $\mathfrak{W}$ is a module for $\mathcal{A}_\lambda$ then $T_y\mathfrak{W}$ is also a module for $U_\lambda(\mathfrak{l}_y)$. The functors T_y^Q and T_y determine derived functors: $LT_y^Q\colon D(\mathcal{M}_{qc}(\mathcal{A}_\lambda^j)) \to D(\mathcal{M}(U_\lambda(\mathfrak{l}_y)))$ and $LT_y\colon D(\mathcal{M}_{qc}(\mathcal{A}_\lambda)) \to D(\mathcal{M}(U_\lambda(\mathfrak{l}_y)))$. By first applying j_+ and following this with the fully faithful embedding $\mathcal{M}_{qc}(\mathcal{A}_\lambda) \to D(\mathcal{M}_{qc}(\mathcal{A}_\lambda))$ we obtain a functor $LT_y \circ j_+\colon \mathcal{M}_{qc}(\mathcal{A}_\lambda^j) \to D(\mathcal{M}(U_\lambda(\mathfrak{l}_y)))$. On the other hand let q denote the codimension of Q in Y and let $[q]$ denote q applications of the translation functor on $D(\mathcal{M}_{qc}(\mathcal{A}_\lambda^j))$. By first utilizing the fully faithful embedding $\mathcal{M}_{qc}(\mathcal{A}_\lambda^j) \to D(\mathcal{M}_{qc}(\mathcal{A}_\lambda^j))$ we obtain the functor $LT_y^Q \circ [q]\colon \mathcal{M}_{qc}(\mathcal{A}_\lambda^j) \to D(\mathcal{M}(U_\lambda(\mathfrak{l}_y)))$.

PROPOSITION 4.4. (a) *For each* $y \in Q$, $LT_y \circ j_+ \simeq LT_y^Q \circ [q]$.

(b) *If* y *is not in* Q *and if* Q *is affinely embedded in* Y *then* $LT_y \circ j_+$ *is isomorphic to zero.*

Proof. The result can be deduced from Kashiwara's theorem in exactly the same manner as the base change for algebraic $\mathcal{D}$ modules [3, Theorem 8.4 and Corollary 8.5]. □

5. Special points, Levi orbits and dual Levi orbits

Let $G_0 \subset G$ be a real form and assume that G_0 has finitely many components. In this section we consider some simple geometry that relates the action (on Y) of the real form to the action of a certain linear algebraic group $K \subset G$.

The Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ of G_0 uniquely determines a complex conjugation $\tau\colon \mathfrak{g} \to \mathfrak{g}$ whose fixed point set is $\mathfrak{g}_0$. Fix a maximal compact subgroup $K_0 \subset G_0$ and let $K \subset G$ be the complexification of K_0. Let $\theta\colon \mathfrak{g}_0 \to \mathfrak{g}_0$ be a Cartan involution given by the maximal compact subgroup K_0 [11, Sect. 3] and extend θ to a complex linear involution on $\mathfrak{g}$.

A point y in the generalized flag manifold Y is called *special* if the parabolic subalgebra $\mathfrak{p}_y$ has a Levi factor $\mathfrak{l}$ which is stable under both τ and θ. Observe that the

Lie algebra $\tau\theta\mathfrak{p}_y \cap \mathfrak{p}_y$ is reductive in $\mathfrak{p}_y$ since the corresponding integral subgroup in the adjoint group $Ad(G)$ is the complexification of a compact subgroup. Because $\mathfrak{l} \subset \tau\theta\mathfrak{p}_y \cap \mathfrak{p}_y$ is a subalgebra which is maximal with respect to the property of being reductive in $\mathfrak{p}_y$ it follows there is exactly one θ, τ stable Levi factor associated to a special point. This θ, τ stable Levi factor will be refered to as the *stable Levi factor*.

In the ensuing discussion we fix a special point $y \in Y$. Let $P_y \subset G$ denote the corresponding subgroup and let $\mathfrak{l}$ be the stable Levi factor. Put L = the normalizer of $\mathfrak{l}$ in P_y. Then L is a connected reductive complex linear algebraic group: it is the connected subgroup in G with Lie algebra $\mathfrak{l}$. Furthermore P_y is a semidirect product $P_y = L \cdot U_y$, where $U_y = \exp(\mathfrak{u}_y)$ is the unipotent radical for P_y. Let $G_{0y} = P_y \cap G_0$ denote the stabilizer of y for the G_0 action on Y and let $L_0 = L \cap G_0$ be the normalizer of $\mathfrak{l}$ in G_{0y}. Then L_0 is a real form for L having finitely many components. In fact, L_0 has maximal compact subgroup $K_0 \cap L_0$ [28, Lemma 3.2.14].

The fiber $X_y = \pi^{-1}(y)$ is naturally identified with the flag manifold for $\mathfrak{l}$ (Borel subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}_y$ intersect with $\mathfrak{l}$ to give the Borel subalgebras of $\mathfrak{l}$). Observe that when a G_0 orbit S_X on X has nontrivial intersection with the fiber X_y then L_0 acts transitively on the intersection $X_y \cap S_X$. In particular, G_0 orbits having nonempty intersection with X_y correspond to L_0 orbits on the fiber.

On the other hand, consider the stablizer $K_y = P_y \cap K$ of y for the K action on Y. Then the normalizer $L \cap K$ of $\mathfrak{l}$ in K_y is the complexification of the compact group $K_0 \cap L_0$ (the argument is formally the same as the argument that L_0 has maximal compact subgroup $K_0 \cap L_0$). Similar to the above considerations, if Q_X is a K orbit on X that intersects the fiber X_y nontrivially then $L \cap K$ acts transitively on the intersection $X_y \cap Q_X$.

The following fundamental result is the special case of a more general result established by Matsuki and is often refered to as Matsuki duality. We state the following lemma [17, Sect. 1] for the purpose of establishing Proposition 5.3.

LEMMA 5.1 [Matsuki]. (a) *Each Borel subalgebra contains a θ stable* (*also a τ stable*) *Cartan subalgebra*.

(b) *Each θ stable* (*or each τ stable*) *Cartan subalgebra is conjugate under an element of the identity component of K* (*or G_0*) *to a θ, τ stable Cartan subalgebra*.

(c) *If two special points in the flag manifold are conjugate under K* (*or under G_0*) *then they are conjugate under K_0*.

The duality theorem is now an immediate corollary of the lemma. Let $\mathcal{U}$ denote the set of special points in the flag manifold X.

THEOREM 5.2 [Matsuki]. *The inclusion $\mathcal{U} \to X$ defines a one to one correspondence between the following*:

(i) *The K_0 orbits on $\mathcal{U}$ and the K orbits on X*
(ii) *The K_0 orbits on $\mathcal{U}$ and the G_0 orbits on X*.

In particular, for a special point $y \in Y$ the Matsuki duality applied to X_y determines a one to one correspondence between the L_0 and the $L \cap K$ orbits on X_y. This in turn defines a correspondence between some G_0 and some K orbits on Y (the general case has also been solved by Matsuki [18]). In order to make this correspondence explicit we have the following definitions. A G_0 orbit $S \subset Y$ is called a *Levi orbit* if it contains a parabolic subalgebra with a τ stable Levi factor. A K orbit $Q \subset Y$ is called a *dual Levi orbit* if it contains a parabolic subalgebra that has a θ stable Levi factor. The G_0 orbit S is said to be dual to the K orbit Q if $S \cap Q$ contains a special point.

PROPOSITION 5.3. *The following conditions hold on* Y

- (a) *Each Levi orbit (also each dual Levi orbit) contains a special point.*
- (b) *If S is a Levi orbit (or if Q is a dual Levi orbit) then the compact subgroup K_0 acts transitively on the special points in S (or in Q)*
- (c) *The relationship of duality establishes a one to one correspondence between the Levi orbits and the dual Levi orbits*

Proof. To establish (a) let Q be a dual Levi orbit. Then for some $y \in Q$ there is a parabolic subalgebra $\mathfrak{p}_y$ which has a θ stable Levi factor $\mathfrak{l}$. In turn, there is a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{l}$ which is θ stable [Lemma 5.1]. For some $k \in K$ the Cartan subalgebra $\mathrm{Ad}(k)\mathfrak{c}$ is stable under τ and θ [Lemma 5.1]. This implies that the Levi factor $\mathrm{Ad}(k)\mathfrak{l}$ is stable under the product $\theta\tau$ (because a root vector for $\mathrm{Ad}(k)\mathfrak{c}$ in $\mathrm{Ad}(k)\mathfrak{l}$ is sent by $\theta\tau$ to a root vector for the negative root). Since $\mathrm{Ad}(k)\mathfrak{l}$ is θ stable we have the desired result.

To establish (b) let y_1 and y_2 be two special points in a dual Levi orbit Q. By applying Lemma 5.1 to the fiber X_{y_1} we obtain a special point $x_1 \in X_{y_1}$. For some $k \in K$ the point $k \cdot x_1$ is in the fiber X_{y_2}. Now applying Lemma 5.1 to the fiber X_{y_2} we obtain some $k' \in K \cap P_{y_2}$ such that $k'k \cdot x_1$ is special. This implies that the points x_1 and $k'k \cdot x_1$ are in the same K_0 orbit $\square$

6. Induction and analytic localization of group representations

The underlying set of K_0 finite vectors in an admissible, finite length representation for G_0 yields a certain algebraic object whose formal properties define what is called a Harish–Chandra module. The study of these objects has greatly facilitated the understanding of topological representations for G_0. In turn, it is known that each Harish–Chandra module arises as the K_0 finite vectors in some global, topological representation for G_0 [5]. In this section we begin by briefly recalling a few relevant points about Harish–Chandra modules and their globalizations.

Suppose that M is a complex vector space that comes equipped with actions of K and $\mathfrak{g}$. Then we call M a *Harish–Chandra module for* $(\mathfrak{g}, K)$ provided the following conditions hold

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(a) M is a union of finite dimensional algebraic K modules.
(b) The differential of the K action agrees with the $\mathfrak{g}$ action.
(c) The action map $U(\mathfrak{g}) \otimes M \to M$ is K equivariant.
Here K is acting on $U(\mathfrak{g}) \otimes M$ via the tensor product of the adjoint action with the action on M.
(d) M has a finite composition series.

On the other hand, suppose we have a continuous representation: $G_0 \times M_\omega \to M_\omega$ given by $(g, m) \mapsto \omega(g)m$ where M_ω is some complete locally convex space. A vector $m \in M_\omega$ is called *analytic* if the function $G_0 \to M_\omega$ defined by $g \mapsto \omega(g)m$ is given locally by a convergent power series. When each vector in M_ω is analytic then the Lie algebra $\mathfrak{g}$ acts on M_ω by complexifying the derivative of the G_0 action. Because $G_0 \subset G$ is a real form it turns out that M_ω is an *analytic* dnF $(\mathfrak{g}, G_0)$ module [13] precisely when M_ω is a dnF space consisting of analytic vectors such that the operators $\omega(\xi)$ for $\xi \in \mathfrak{g}$ are continuous. If in addition, the center of the enveloping algebra acts by the infinitesimal character $\lambda \in \mathfrak{h}^*$ then we call M_ω an analytic (U_λ, G_0) module.

The K_0 finite vectors in any analytic $(\mathfrak{g}, G_0)$ module M_ω will form a $(\mathfrak{g}, K)$ module M satisfying (a) through (c) of the above definition. In case M also satisfies (d) then M_ω is said to *globalize* (or *complete*) the Harish–Chandra module M and we refer to M_ω as an *analytic globalization* (or *analytic completion*) of the Harish–Chandra module M.

It follows from work of Casselman and Wallach [6] that each Harish–Chandra module has a canonical and functorial analytic globalization (which carries the topology of a dnF space [13]). In fact, Schmid has shown that this canonical analytic completion coincides with the minimal globalization of the Harish–Chandra module [16, 22].

The induced sheaves

Suppose that $y \in Y$ is a special point and adopt the notations of the previous section. In particular, let S be the G_0 orbit of y and let $\psi: G_0 \to S$ denote the projection $\psi(g) = g \cdot y$. Assume V is a Harish–Chandra module for $(\mathfrak{l}, K \cap L)$ where $\mathfrak{l}$ is the stable Levi factor and let V_ω denote the minimal globalization of V. We now introduce the notion of the corresponding *standard analytic* (or *analytic induced*) sheaf $\mathcal{A}(y, V_\omega)$.

The action of L_0 on V_ω extends to the full stabilizer G_{0y} by allowing the unipotent radical to act trivially. Since V_ω is an analytic module, this representation in fact determines a homogeneous real analytic vector bundle defined over the G_0 orbit S. The sections of this vector bundle over an open set $U \subset S$ are precisely the real analytic functions $F: \psi^{-1}(U) \to V_\omega$ which satisfy:

$$F(gb) = \omega(b^{-1})F(g)$$

for $g \in \psi^{-1}(U)$ and $b \in G_{0y}$.

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The Lie algebra $\mathfrak{p}_y$ defines what is sometimes refered to as an *equivariant polarization* for the homogeneous bundle determined by V_ω. We describe this as follows. Extend the action of $\mathfrak{l}$ on V_ω to all of $\mathfrak{p}_y$ by allowing the unipotent radical to act trivially. Then the locally free equivariant sheaf $\mathfrak{p}^{\cdot}$ [Sect. 3] acts on the sections of the vector bundle via two actions. Differentiating the left translation of real analytic functions determines a $\mathfrak{g}$ action and hence one corresponding action for $\mathfrak{p}^{\cdot}$. On the other hand, the action of $\mathfrak{p}_y$ on V_ω determines a pointwise action for $\mathfrak{p}^{\cdot}$ on the sections. Then the induced analytic sheaf (on S) is nothing but the subsheaf of sections for the homogeneous bundle which are annihilated by the difference of these two $\mathfrak{p}^{\cdot}$ actions.

In other words, the induced analytic sheaf $\mathcal{A}_S(y, V_\omega)$ can be defined (on S) as follows. For an element $\xi \in \ g$ let ξ_r denote the operator obtained by differentiating the right translation of real analytic functions. If $U \subset S$ is open, then $\Gamma(U, \mathcal{A}_S(y, V_\omega))$ = the set of real analytic functions $F: \psi^{-1}(U) \to V_\omega$ which satisfy

(a) $F(gb) = \omega(b^{-1})F(g), \quad \text{for } g \in \psi^{-1}(U) \quad \text{and } b \in L_0,$

(b) $\xi_r F = 0, \quad \text{for } \xi \in \mathfrak{u}_y.$

The left translation of real analytic functions defines a G_0 action on $\mathcal{A}_S(y, V_\omega)$. Let $i: S \to Y$ denote the inclusion. Extension by zero provides a global sheaf $\mathcal{A}(y, V_\omega) = i_! \mathcal{A}_S(y, V_\omega)$ defined on all of Y. Differentiating the G_0 action and complexifying defines a $\mathfrak{g}$ action and in fact $\mathcal{A}(y, V_\omega)$ is a sheaf of $\mathcal{D}_\mathfrak{l}$ modules.

Let $\gamma: U \subset Y \to G$ be a holomorphic cross section for the fibration $g \mapsto g \cdot y$ with $\gamma(y) = e$ (the identity element of G). The following result is discussed in [13].

LEMMA 6.1 [Hecht and Taylor]. (a) $\mathcal{A}(y, V_\omega)$ *is a* dnF *sheaf of* $(\mathcal{D}_\mathfrak{l}, G_0)$ *modules. In particular, when* V_ω *is an analytic* $(U_\lambda(\mathfrak{l}), G_0)$ *module, then* $\mathcal{A}(y, V_\omega)$ *is a* dnF *sheaf of* $(\pi_* \mathcal{D}_\lambda, G_0)$ *modules.*

(b) *The section* γ *determines an isomorphism of* dnF $\mathcal{O}_{Y,y}$ *modules between the stalk of the induced sheaf at* y *and* $\mathcal{O}_{Y,y} \widehat{\otimes} V_\omega$.

Since the complexification K of the maximal compact subgroup $K_0 \subset G_0$ is a linear algebraic subgroup of G, the K orbits on Y are smooth algebraic subvarieties. In particular this is true for the K orbit Q of the special point y. Similar to the above construction, we extend the Harish–Chandra module V to a module for the stabilizer K_y by allowing the unipotent radical to act trivially. In this way V determines a K homogeneous algebraic vector bundle defined over the K orbit Q. Let $\mathcal{V}$ denote the corresponding sheaf of sections. More precisely: let $\phi: K \to Q$ be the projection $\phi(k) = k \cdot y$ and let $U \subset Q$ be an open set. Then $\Gamma(U, \mathcal{V})$ consists of all finite rank regular functions $F: \phi^{-1}(U) \to V$ which satisfy

$$F(kb) = \omega(b^{-1})F(k),$$

for $k \in \phi^{-1}(U)$ and $b \in K_y$.

We assume that V has infinitesimal character $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{k}$ denote the Lie algebra of K. The group K acts on $\mathcal{V}$ through the left translations. This in turn determines an action for the sheaf of algebras $U^{\cdot}(\mathfrak{k})^{\mathrm{alg}}$ [Sect. 3]. On the other hand, the adjoint action of K_y on $\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$ determines a locally free K equivariant sheaf $U_\lambda(\mathfrak{l}^{\cdot})_Q^{\mathrm{alg}}$ with fiber $U_\lambda(\mathfrak{l}_y)$ [Sect. 3]. Via the $\mathfrak{l}$ action on V the sheaf $\mathcal{V}$ is a module for $U_\lambda(\mathfrak{l}^{\cdot})_Q^{\mathrm{alg}}$.

Let $j\colon Q \to Y$ be the inclusion and recall the notations of Section 4. The following extensions from the setting of a flag manifold to the setting of a generalized flag manifold are treated in [7, 8].

LEMMA 6.2 [Miličić, Chang]. (a) *The orbit Q is affinely embedded in Y.*

(b) *The actions of $U^{\cdot}(\mathfrak{k})^{\mathrm{alg}}$ and $U_\lambda(\mathfrak{l}^{\cdot})_Q^{\mathrm{alg}}$ on $\mathcal{V}$ determine an action of $\mathcal{A}_\lambda^j$.*

(c) *$j_+\mathcal{V}$ is a coherent sheaf of modules for $(\pi_*\mathcal{D}_\lambda^{\mathrm{alg}}, K)$. In particular, $\Gamma(Y, j_+\mathcal{V})$ is a Harish–Chandra module for (U_λ, K).*

We use the notation $\mathcal{I}(y, V)$ to denote the sheaf $j_+\mathcal{V}$ and we refer to $\mathcal{I}(y, V)$ as the *standard Harish–Chandra* (or *algebraic induced*) sheaf corresponding to V.

Analytic localization of group representations

Suppose that M is a Harish–Chandra module for (U_λ, K) and that M_ω is an analytic completion on which G_0 acts. At first glance it may not seem so clear how this information could possibly effect the structure of the analytic localization. For regular λ the *Koszul complex* can be used to facilitate a description of the analytic localizations to a flag manifold [13]. For our purposes, it seems more advantageous to utilize a certain canonical free resolution. When these resolutions (for M and M_ω respectively) are localized then the respective groups K and G_0 will act on the resulting complexes of sheaves. These actions provide complexes of $(\mathcal{D}_\lambda, K)$ (respectively $(\mathcal{D}_\lambda, G_0)$) modules which carry a certain structure sometimes refered to as *weak* [1, Defn. 1.3.1]. The point is that the derivative of the group action and the algebra action do not agree at the level of the complex. Nevertheless these two actions are homotopic [12]. Because the module M_ω is analytic, it turns out we can understand the basic structure of the localizations (i.e. the homologies of the derived localization) on a G_0 orbit if only we know the geometric fibers (as topological representations for the stabilizer) at a point in that orbit.

In particular, we recall the Hochschild resolution $F.(M_\omega)$ given by

$$F_p(M_\omega) = \otimes^{p+1} U_\lambda \otimes M_\omega.$$

The analytic localization of M_ω is realized as the complex $\Delta_Y(F.(M_\omega))$. The group G_0 acts on this complex of sheaves $\Delta_Y(F.(M_\omega))$ via: the action on $\pi_*\mathcal{D}_\lambda \mathbin{\widehat{\otimes}}$ the adjoint action on $U_\lambda \mathbin{\widehat{\otimes}}$ the action on M_ω. Coupling this G_0 action with the left action of $\pi_*\mathcal{D}_\lambda$ one obtains the weak equivariant complex refered to above.

Choose $y \in Y$ and let $\Delta_Y(F.(M_\omega))_y$ denote the induced complex on stalks. Choose a holomorphic cross section $\gamma: U \subset Y \to G$ for the projection $g \mapsto g \cdot y$ with $\gamma(y) = e$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$ modules let $T_y(\mathcal{F})$ denote the geometric fiber of $\mathcal{F}$ at y. Put G_{0y} = the stabilizer of y in G_0. The existence of the previously mentioned homotopies establishes that the pth homology $h_p(T_y \circ \Delta_Y(F.(M_\omega)))$ will be an analytic $(\mathfrak{p}_y, G_{0y})$ module provided it is Hausdorff. Let S be the G_0 orbit of y. The following lemma can be established in a similar manner as the case for the flag manifold [13, Prop. 8.3 and Prop. 8.7].

LEMMA 6.3. *Assume M_ω is an analytic (U_λ, G_0) module.*

(a) *The section γ and the action of G_0 determines an isomorphism of complexes of* dnF *$\mathcal{O}_{Y,y}$ modules $\Delta_Y(F.(M_\omega))_y \simeq \mathcal{O}_{Y,y}\widehat{\otimes} T_y \circ \Delta_Y(F.(M_\omega))$.*

(b) *If $h_p(T_y \circ \Delta_Y(F.(M_\omega)))$ is Hausdorff for each p then $L_p\Delta_Y(M_\omega)$ restricted to S is isomorphic, as a $(\pi_*\mathcal{D}_\lambda\mid_S, G_0)$ module, to the sheaf analytically induced from $h_p(T_y \circ \Delta_Y(F.(M_\omega)))$.*

7. The comparison theorem

When M is a Harish Chandra module for $(\mathfrak{g}, K)$ with minimal globalization $M_{\min}$ there is a natural equivariant inclusion $M \to M_{\min}$ onto the K_0 finite vectors. Suppose M has infinitesimal character $\lambda \in \mathfrak{h}^*$ and consider the inclusion $M \to M_{\min}$ as a morphism between objects in $D(\mathcal{M}_{\text{dnF}}(U_\lambda))$ via the usual fully faithful embedding. On the sheaf side, the relation between a Harish Chandra module and its minimal globalization is captured as a morphism $L\Delta_Y(M) \to L\Delta_Y(M_{\min})$ in $D(\mathcal{M}_{\text{dnF}}(\pi_*\mathcal{D}_\lambda))$. Examples show that the induced morphisms on homologies $L_p\Delta_Y(M) \to L_p\Delta_Y(M_{\min})$ are often all zero. So the question is raised: what is the geometric content to this relation between a Harish–Chandra module and its minimal globalization on the sheaf side? It is the aim of the forthcoming comparison theorem to answer exactly this question.

In the setting of a full flag manifold X, Hecht and Taylor have established the following result [14], which turns out to be the key technical point in the development of our argument. In particular, suppose $x \in X$ is a special point and let $\mathfrak{c} \subset \mathfrak{b}_x$ be the stable Cartan subalgebra. Assume T_x denotes the functor for the geometric fiber at x. Then there is a derived functor $LT_x: D(\mathcal{M}_{\text{dnF}}(\mathcal{D}_\lambda)) \to D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{c})))$ [Sect. 3]. Hence we obtain a morphism $LT_x \circ L\Delta_X(M)) \to LT_x \circ L\Delta_X(M_{\min})$ in the derived category $D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{c})))$. The comparison theorem for a flag manifold says the following

THEOREM 7.1 [Hecht and Taylor]. *Assume that $\lambda \in \mathfrak{h}^*$ is regular and that M is a Harish–Chandra module for (U_λ, K). Let $M \to M_{\min}$ be an equivariant inclusion onto the K_0 finite vectors. Then the corresponding morphism $LT_x \circ L\Delta_X(M)) \to LT_x \circ L\Delta_X(M_{\min})$ in $D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{c})))$ is an isomorphism.*

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In particular, let $F.(M)$ and $F.(M_{\min})$ denote the respective Hochschild resolutions. Then the inclusion $M \to M_{\min}$ determines a morphism of the complexes $\Delta_X(F.(M)) \to \Delta_X(F.(M_{\min})$ which is equivariant for the K_0 actions [Sect. 6]. Thm. 7.1 implies that this morphism of complexes is in fact a quasi-isomorphism. Hence the induced morphisms on the homologies $h_p(T_x \circ \Delta_X(F.(M))) \to h_p(T_x \circ \Delta_X(F.(M_{\min})))$ are equivariant isomorphisms for the associated $(\mathfrak{c}, C_0)$ actions.

The rest of this section is concerned with proving the following extension of the comparison theorem to a generalized flag manifold.

THEOREM 7.2. *Comparison theorem. Assume λ is regular and antidominant for Y. Let M be a Harish Chandra module for (U_λ, K_0) with minimal globalization $M_{\min}$. Suppose $y \in Y$ is special. Then $h_p(T_y \circ \Delta_Y(F.(M)))$ is a Harish–Chandra module for $(\mathfrak{l}, K_0 \cap L_0)$ whose minimal globalization is naturally isomorphic to the toplogical L_0 module $h_p(T_y \circ \Delta_Y(F.(M_{\min})))$. In particular, let $M \to M_{\min}$ be an equivariant inclusion onto the K_0 finite vectors. Then corresponding morphism $h_p(T_y \circ \Delta_Y(F.(M))) \to h_p(T_y \circ \Delta_Y(F.(M_{\min})))$ is an an equivariant inclusion onto the $K_0 \cap L_0$ finite vectors.*

Proof. Without loss of generality assume λ is regular and antidominant. Let S be the G_0 orbit of y and let $Q = K \cdot y$ be the corresponding K orbit. Put $q = \dim X - \dim Q$. Assume $x \in X$ is a special point, $\mathfrak{c} \subset \mathfrak{b}_x$ is the stable Cartan subalgebra and $C_0 \subset G_0$ is the corresponding Cartan subgroup. A Harish–Chandra module M is called a *Beilinson Bernstein standard module* if $M = \Gamma(X, \mathcal{I}(x, V))$ where V is an irreducible module for $K_0 \cap C_0$, $\mathfrak{c}$ acts on V by $\lambda + \rho$ and $\mathcal{I}(x, V)$ is the corresponding standard Harish–Chandra sheaf originating from the K orbit $Q_X = K \cdot x$. Put $q_X = \dim X - \dim Q_X$ and let S_X be the G_0 orbit of x. The module V is $(\mathfrak{c}, C_0)$ module (that is: $V = V_{\min}$), so we obtain an induced analytic sheaf $\mathcal{A}(x, V)$ originating from S_X. We first check that the theorem holds when M is a Beilinson Bernstein standard module.

LEMMA 7.3. *Assume $M = \Gamma(X, \mathcal{I}(x, V))$ is a Beilinson Bernstein standard module with λ regular and antidominant. Then the following hold.*

(a) $L\Delta_X(M_{\min}) \simeq \mathcal{A}(x, V)[q_X]$ in $\mathcal{D}(\mathcal{M}_{\mathrm{dnF}}(\mathcal{D}_\lambda))$.
(b) If x does not belong to $\pi^{-1}(Q \cap S)$ then $LT_y \circ L\Delta_Y(M) \simeq LT_y \circ L\Delta_Y(M_{\min}) \simeq 0$ in $\mathcal{D}(\mathcal{M}_{\mathrm{dnF}}(\pi_* \mathcal{D}_\lambda))$.

Proof. Observe that (a) is a slight generalization of [13, Prop. 10.8]. We use the comparison theorem on X [Thm. 7.1] and the description of $L\Delta_X(M_{\min})$ [Lemma 6.3] to derive our result. This illustrates the key idea for establishing the main theorem in the next section. Let $F.(M)$ and $F.(M_{\min})$ be the Hochschild resolutions. The inclusion $M \to M_{\min}$ induces a morphism of complexes: $F.(M) \to F.(M_{\min})$. Since λ is regular and antidominant, it follows that $L_p\Delta_X^{\mathrm{alg}}(M) = 0$ unless $p = 0$ in which case $\Delta_X^{\mathrm{alg}}(M) \simeq \mathcal{I}(x, V)$. Let x' be a special point not in $Q_X \cap S_X$ and let $T_{x'}$ be the geometric fiber functor at x'. Observe x' is not in Q_X [Thm. 5.2].

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Since the complex $T_{x'} \circ \Delta_X(F.(M))$ computes the geometric fibers of $\mathcal{I}(x,V)$ at x' it follows that this complex has vanishing homology [Prop. 4.4] (this uses the fact that Q_X is affinely embedded in X [15]). Hence, the comparison theorem on X implies that the complex $T_{x'} \circ \Delta_X(F.(M_{\min}))$ has vanishing homology. Now one concludes that the restriction of $L\Delta_X(M_{\min})$ to the complement of S_X is zero [Lemma 6.3]. For the special point x we see that $h_p(T_x \circ \Delta_X(F.(M))) = 0$ unless $p = q_X$ where $h_{q_X}(T_y \circ \Delta_X(F.(M))) \simeq V$ [Prop. 4.4]. Hence, another application of the comparison theorem shows that $h_p(T_x \circ \Delta_X(F.(M_{\min}))) = 0$ unless $p = q_X$, in which case the map: $h_{q_X}(T_x \circ \Delta_X(F.(M))) \to h_{q_X}(T_y \circ \Delta_X(F.(M_{\min})))$ is an isomorphism. Since this map is evidently equivariant for the $(\mathfrak{c}, K_0 \cap C_0)$ actions, the desired result follows.

To establish (b) observe when x does not belong to $\pi^{-1}(Q \cap S)$ then the orbits S_X and Q_X cannot intersect the fiber $X_y = \pi^{-1}(y)$, since x and y are special [Sect. 5]. Let $i\colon X_y \to X$ be the inclusion. Then the homology of $LT_y \circ L\Delta_Y(M_{\min})$ is isomorphic to the hypercohomology (on X_y) of $i^*\mathcal{A}(x,V)[q_X]$ [Prop. 3.3]. Thus $LT_y \circ L\Delta_Y(M_{\min}) = 0$. The analogous statement holds for M once we see that $Li^*(\mathcal{I}(x,V))$ has vanishing homologies. To establish this last fact it suffices to apply the base change for twisted sheaves of differential operators [3, Thm. 8.4] to the diagram

$$\begin{array}{ccc} Q_X \cap X_y & \longrightarrow & Q_X \\ \downarrow & & \downarrow \\ X_y & \xrightarrow[i]{} & X \end{array}$$

Since $Q_X \cap X_y$ is empty and since Q_X is affinely embedded in X we have the desired result [3, Cor. 8.5]. $\square$

We now consider the case where $\pi(x) \in Q \cap S$. Since K_0 acts transitively on $Q_X \cap S_X$, we may assume $\pi(x) = y$. Adopting the earlier notations, recall the twisted sheaf of differential operators $\mathcal{D}^i_\lambda$ which acts on $i^*\mathcal{D}_\lambda$ [Sect. 3], the real stable Levi factor $L_0 \subset P_y$ and the complexification $K \cap L$ of $K_0 \cap L_0$. Since $C_0 \subset L_0$, the module V determines a standard Harish–Chandra sheaf $\mathcal{I}_{\text{fiber}}(x,V)$ on X_y originating from the $K \cap L$ orbit $Q_X \cap X_y$. Then $\Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))$ is a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K \cap L)$.

LEMMA 7.4. *Maintain the assumptions on λ and M as specified in Lemma 7.3. Assume $\pi(x) = y$. Then the following hold.*

(a) $LT_y \circ L\Delta_X(M) \simeq \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))[q]$ in $D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})))$.
(b) $LT_y \circ L\Delta_X(M_{\min}) \simeq \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))_{\min}[q]$ in $D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})))$.

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Proof. To establish (a) note that the Harish–Chandra module $\Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))$ determines a standard Harish–Chandra sheaf $\mathcal{I}(y, \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V)))$ originating from the K orbit Q on Y. Using the 'induction in stages' for the standard Harish–Chandra sheaves [7, Thm. 4.14 and 5.4] one knows that $\pi_*\mathcal{I}(x,V) \simeq \mathcal{I}(y, \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V)))$ as a sheaf of $(\pi_*\mathcal{D}_\lambda^{\text{alg}}, K)$ modules. Since λ is regular and antidominant, it follows that $\Delta_Y^{\text{alg}}(M) \simeq \mathcal{I}(y, \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V)))$ [Lemma 4.1 and Thm. 4.2]. In addition, $L_p\Delta_Y(M) = 0$ for p different from zero. Hence, the homology of the complex $T_y \circ \Delta_Y(F.(M))$ computes the geometric fibers of $\mathcal{I}(y, \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V)))$. It follows that $h_p(T_y \circ \Delta_Y(F.(M))) = 0$ unless $p = q$ in which case $h_q(T_y \circ \Delta_Y(F.(M))) \simeq \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))$ as a Harish–Chandra module for $(\mathfrak{l}, K \cap L)$ [Prop. 4.4]. This establishes part (a).

On the analytic side, the module V determines a standard analytic sheaf $\mathcal{A}_{\text{fiber}}(x,V) \simeq i^*\mathcal{A}(x,V)$ on X_y. Since $L\Delta_X(M) \simeq \mathcal{A}(x,V)[q_X]$ and since $\mathcal{A}(x,V)$ is acyclic for i^* it follows that $Li^* \circ L\Delta_X(M_{\text{min}}) \simeq \mathcal{A}_{\text{fiber}}(x,V)[q_X]$ in $D(\mathcal{M}_{\text{dnF}}(\mathcal{D}_\lambda^i))$. Since the parameter for the sheaf $\mathcal{D}_\lambda^i$ is antidominant and regular with respect to the Levi factor $\mathfrak{l}$ we see that $L\Delta_{X_y}(\Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))_{\text{min}}[q]) \simeq \mathcal{A}_{\text{fiber}}(x,V)[q_X]$ [Lemma 7.3]. Thus, $R\Gamma_{X_y} \circ Li^* \circ L\Delta_X(M_{\text{min}}) \simeq \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))_{\text{min}}[q]$ [Prop. 3.2]. The desired result follows by Proposition 3.3 □

We need to check that the morphism $h_q(T_y \circ \Delta_Y(F.(M))) \to h_q(T_y \circ \Delta_Y(F.(M_{\text{min}}))$ induced from the inclusion $M \to M_{\text{min}}$ is an isomorphism onto the $K_0 \cap L_0$ finite vectors. Put $W = \Gamma(X_y, \mathcal{I}_{\text{fiber}}(x,V))$ and observe that the morphism in question belongs to $\text{Hom}_{(U_\lambda(\mathfrak{l}), K_0 \cap L_0)}(W, W_{\text{min}})$. The following lemma shows that it's enough to check that this morphism is nonzero

LEMMA 7.5. $\text{Hom}_{(U_\lambda(\mathfrak{l}), K_0 \cap L_0)}(W, W_{\text{min}}) \simeq \text{Hom}_{(\mathfrak{c}, K_0 \cap C_0)}(V,V) \simeq \mathbb{C}$.

Proof. The $(\mathfrak{c}, K \cap C)$ module V determines an induced sheaf $\mathcal{V}$ on the $K \cap L$ orbit $Q_X \cap X_y$. Let $j\colon Q_X \cap X_y \to X_y$ be the inclusion. Then $\mathcal{V}$ is a sheaf of modules for twisted sheaf of algebraic differential operators $\mathcal{D}_\lambda^{alg,\, i\circ j}$ and our defintion of $\mathcal{I}_{\text{fiber}}(x,V)$ is given by $\mathcal{I}_{\text{fiber}}(x,V) = j_+\mathcal{V}$. Since an element of $\text{Hom}_{(U_\lambda(\mathfrak{l}), K_0 \cap L_0)}(W, W_{\text{min}})$ has range inside the $(K_0 \cap L_0)$ finite vectors, we see that

$$\begin{aligned}
&\text{Hom}_{(U_\lambda(\mathfrak{l}), K_0 \cap L_0)}(W, W_{\text{min}}) \simeq \text{Hom}_{(U_\lambda(\mathfrak{l}), K_0 \cap L_0)}(W, W)\\
&\quad\simeq \text{Hom}_{(U_\lambda(\mathfrak{l}), K \cap L)}(W, W) \simeq \text{Hom}_{(\mathcal{D}_\lambda^{alg,\, i}, K \cap L)}(j_+\mathcal{V}, j_+\mathcal{V})\\
&\quad\simeq \text{Hom}_{(\mathcal{D}_\lambda^{alg,\, i\circ j}, K \cap L)}(\mathcal{V}, \mathcal{V}) \simeq \text{Hom}_{(\mathfrak{s}, K \cap C)}(V,V),
\end{aligned}$$

from which the desired result follows □

Let $\phi\colon M \to M_{\text{min}}$ denote the inclusion. Via the fully faithful embedding $\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})) \to D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})))$ we obtain the isomorphism $\text{Hom}_{U_\lambda(\mathfrak{l})}(W, W_{\text{min}}) \simeq \text{Hom}_{D(\mathcal{M}_{\text{dnF}}(U_\lambda(\mathfrak{l})))}(W, W_{\text{min}})$. Hence if the morphism $h_q(T_y \circ \Delta_Y(F.(M))) \to$

$h_q(T_y \circ \Delta_Y(F.(M_{\min}))$ induced from ϕ is zero, then the morphism $LT_y \circ L\Delta_Y(\phi)$ is zero in $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda(\mathfrak{l})))$ [Lemma 7.4]. It follows from this that the morphism $R\Gamma_{X_y} \circ Li^* \circ L\Delta_X(\phi)$ is zero [Prop. 3.3]. This in turn implies that the morphism $Li^* \circ L\Delta_X(\phi)$ is zero, by the equivalence of derived categories on X_y [Thm. 2.1]. Let $T_x^{X_y}$ denote the geometric fiber functor at the special point x in relation to $\mathcal{O}_{X_y}$. Then we see that $LT_x^{X_y} \circ Li^* \circ L\Delta_X(\phi) \simeq LT_x \circ L\Delta_X(\phi)$ is zero as well. Since this contradicts the comparison theorem on X, we have the desired result for Beilinson–Bernstein standard modules.

To establish the general case, let M be a Harish–Chandra module for (U_λ, K_0) and adopt the notation $h_p(y, M)$ to denote the $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$ module $h_p(T_y \circ \Delta_Y(F.(M)))$. The standard sort of considerations show that $h_p(y, M)$ is a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$. Changing notations briefly let $\widehat{M}$ be the minimal globalization of M and put $h_p(y, \widehat{M}) = h_p(T_y \circ \Delta_Y(F.(\widehat{M})))$. Since $h_p(y, \widehat{M}) \simeq h_p(\Gamma_{X_y} \circ i^* \circ \Delta_X(F.(\widehat{M})))$ as a module for $(U_\lambda(\mathfrak{l}), L_0)$ it follows, using [13, Lemma 10.11] applied to the flag manifold X_y, that $h_p(y, \widehat{M})$ is a minimal globalization for L_0. The following lemma is only a slight modification of [14, Lemma 3.1].

LEMMA 7.6. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of Harish–Chandra modules for* (U_λ, K_0) *and assume that Theorem 7.2 holds for any two of the modules. Then it also holds for the third.*

Proof. Using the fact that minimal globalization is an exact functor [16, 22] we have a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & h_{p+1}(y, M'') & \longrightarrow & h_p(y, M') & \longrightarrow & h_p(y, M) & \longrightarrow & h_p(y, M'') & \longrightarrow & h_{p-1}(y, M') & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & h_{p+1}(y, \widehat{M''}) & \longrightarrow & h_p(y, \widehat{M'}) & \longrightarrow & h_p(y, \widehat{M}) & \longrightarrow & h_p(y, \widehat{M''}) & \longrightarrow & h_{p-1}(y, \widehat{M'}) & \longrightarrow
\end{array}$$

Since the functor of minimal globalization is exact, we can apply it to the top row and then use the five lemma together with the open mapping theorem for the desired result □

To complete the proof of the comparison theorem, one can now argue exactly as in [14, Prop. 3.2] □

8. Realizing representations from the Levi orbits: The case of regular antidominant infinitesimal character

Let $y \in Y$ be a special point and let V be a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$. Put $q = \dim\, Y - \dim\, Q$, where $Q = K \cdot y$ is the K orbit of y. We will utilize the following terminology. If $\mathcal{F}^{\cdot}$ is a complex of sheaves on Y and if $S \subset Y$ is

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a locally closed subset then we say $\mathcal{F}^{\cdot}$ *is supported in* S provided the stalks for each of the cohomology sheaves $h^p(\mathcal{F}^{\cdot})$ vanish at all points outside of S.

This section is devoted to establishing the following result.

THEOREM 8.1. *Assume λ is regular and antidominant for Y. Then $H^p(Y, \mathcal{A}(y, V_{\min}))$ vanishes for p different from q in which case $H^q(Y, \mathcal{A}(y, V_{\min}))$ is the minimal globalization of $\Gamma(Y, \mathcal{I}(y, V))$. More specifically, there is a naturally defined* dnF *topology as well as a continuous G_0 action on $H^q(Y, \mathcal{A}(y, V_{\min}))$ such that the resulting functor $V \to H^q(Y, \mathcal{A}(y, V_{\min}))$ from Harish–Chandra modules for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$ to topological G_0 modules is isomorphic to the functor $V \to \Gamma(Y, \mathcal{I}(y, V))_{\min}$.*

Proof. Choose an element $\lambda \in \mathfrak{h}^*$, representing the infinitesimal character of V, which is both regular and antidominant. Let $M = \Gamma(Y, \mathcal{I}(y, V))$. Think of M and $M_{\min}$ as objects in $D(\mathcal{M}_{\mathrm{dnF}}(U_\lambda))$. The point of the argument is to see there is an isomorphism $L\Delta_Y(M_{\min}) \simeq \mathcal{A}(y, V_{\min})[q]$ in $D(\mathcal{M}_{\mathrm{dnF}}(\pi_*\mathcal{D}_\lambda))$. The desired result then follows since $R\Gamma_Y \circ L\Delta_Y$ is isomorphic to the identity [Prop. 3.2]. We consider the naturality (which will be crucial later on) at the end of the proof.

The argument proceeds in a similar fashion as the proof of Lemma 7.3. Let $S = G_0 \cdot y$ be the G_0 orbit of y and let $F.(M)$ be the Hochschild resolution of M. Applying the equivalence of categories [Thm. 4.2] as well as the algebraic base change [Prop. 4.4] we see that the homologies of the complex $T_y \circ \Delta_Y(F.(M))$ vanish except in degree q where we obtain the $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$ module V. Now applying the Comparison Theorem [Thm. 7.2] in conjunction with Lemma 6.3 it follows that $L_p\Delta_Y(M_{\min}) \mid_S = 0$ unless $p = q$ in which case $L_q\Delta_Y(M_{\min}) \mid_S \simeq \mathcal{A}(y, V_{\min}) \mid_S$. Hence to complete the argument it suffices to show that $L\Delta_Y(M_{\min})$ is supported in S.

Because $\pi: X \to Y$ is a proper morphism it follows that $R\pi_*$ sends a complex of sheaves supported in $\pi^{-1}(S)$ to a complex of sheaves supported in S. Using the identity [Prop. 3.2]

$$L\Delta_Y \simeq R\pi_* L\Delta_X,$$

we can thus establish the desired result provided we show that $L\Delta_X(M_{\min})$ is supported in $\pi^{-1}(S)$.

Let $x \in X$ be a special point and assume $\pi(x)$ is not in S. Let T_x denote the functor of geometric fiber at x. In order to check that $L\Delta_X(M_{\min})$ is supported in $\pi^{-1}(S)$ it is enough to see that $T_x \circ L_p\Delta_X(M_{\min}) = 0$ for each p. Applying the Comparison Theorem on X we only need to check that $h_p(LT_x(\Delta_X(M))) = 0$ for each p. To establish this last point let $Q = K \cdot y$ be the K orbit of y and observe that the discussion in Section 5 implies that $y' = \pi(x)$ does not belong to Q. Suppose that $X_{y'} \xrightarrow{i} X$ is the inclusion of the fiber over y' into the flag manifold. To see that $h_p(LT_x(\Delta_X(M))) = 0$ for each p it is enough to check that $Li^* \circ L\Delta_X(M)$

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is isomorphic to zero in $D(\mathcal{M}_{\text{dnF}}(\mathcal{D}^i_\lambda))$. Since λ is regular and antidominant an application of Theorem 2.1 shows that it is enough to check that

$$R\Gamma_{X_{y'}} \circ Li^* \circ L\Delta_X(M) \simeq 0.$$

By the analytic base change [Prop. 3.3] we only need to see that $LT_{y'} \circ L\Delta_Y(M) \simeq 0$. Using the fact that Q is affinely embedded in Y, this result follows from algebraic base change formula [Prop. 4.4].

The proof that our construction is natural parallels the above considerations. In particular suppose that V and W are Harish–Chandra modules for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$. Let M and N be the corresponding Harish–Chandra modules for (U_λ, K_0) obtained by taking global sections of the induced algebraic sheaves $\mathcal{I}(y, V)$ and $\mathcal{I}(y, W)$. Assume we have a morphism $M \to N$ obtained functorially from a morphism of the Harish–Chandra modules $V \to W$. Then we have the following communative diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \big\downarrow & & \big\downarrow \\ M_{\text{min}} & \longrightarrow & N_{\text{min}} \end{array}$$

where the vertical arrows are equivariant inclusions onto the K_0 finite vectors and the bottom morphism is the completion of the top morphism. Applying the functors of analytic localization and geometric fiber to the respective Hochschild resolutions provides a commutative diagram of complexes. Taking the qth homolgy we recover the original morphism $V \to W$ of Harish–Chandra modules for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$ [Thm. 4.2 and Prop. 4.4]. Thus the comparison theorem [Thm. 7.2] together with the above description of the localizations $L\Delta_Y(M_{\text{min}})$ and $L\Delta_Y(N_{\text{min}})$ implies that the bottom row $M_{\text{min}} \to N_{\text{min}}$ in the diagram above localizes to the morphism $\mathcal{A}(y, V_{\text{min}}) \to \mathcal{A}(y, W_{\text{min}})$ obtained via the completion of the original morphism $V \to W$. Hence we recover the morphism $M_{\text{min}} \to N_{\text{min}}$ by applying $R\Gamma_Y$ to $\mathcal{A}(y, V_{\text{min}}) \to \mathcal{A}(y, W_{\text{min}})$ [Prop. 3.2] $\square$

9. The tensoring argument

We divide the argument into two parts. The first part establishes the main theorem when the infinitesimal character for the stable Levi factor is antidominant for Y. In turn, this result for the antidominant case then becomes the initial step in an induction argument based on the length of an infinitesimal character. We supply details for this induction argument in the second part of the section.

Tensoring to the walls

In this part we argue that, for our purposes, the case of antidominant infinitesimal character reduces to the case of regular antidominant infinitesimal character. Under certain conditions tensoring on the geometric fibers commutes with both the analytic and the algebraic inductions. This fact can be utilized to understand the antidominant case once we see that the stable Levi factor has enough modules with regular antidominant infinitesimal character. In order to establish this last point we begin with a geometric description of the translation functor on the fiber.

Let $y \in Y$, let $L \subset P_y$ be a Levi factor and suppose that $C \subset L$ is a Cartan subgroup. Via specialization to a point $x \in X_y$ the differential of a holomorphic character for C determines a weight $\mu \in \mathfrak{h}^*$. Induction at the point x determines the sheaf of sections $\mathcal{O}(\mu)$ for an L homogeneous algebraic line bundle on X_y. More generally, we refer to an element $\mu \in \mathfrak{h}^*$ as an *integral weight* provided $\check{\alpha}(\lambda)$ is an integer for each $\alpha \in \Sigma$. One basic fact is that each integral weight μ determines a unique irreducible finite dimensional $\mathfrak{g}$ module F^μ with extremal weight μ [28].

Let $i\colon X_y \to X$ be the inclusion. Suppose $\lambda \in \mathfrak{h}^*$ and let $\mathcal{D}_\lambda^{\mathrm{alg},\, i}$ be the corresponding sheaf of twisted differential operators on X_y. If $\mathcal{V}$ is a sheaf of modules for $\mathcal{D}_\lambda^{\mathrm{alg},\, i}$ and if μ is an integral weight corresponding to the differential of a holomorphic character for $C \subset G$ then $\mathcal{D}_{\lambda-\mu}^{\mathrm{alg},\, i}$ acts on $\mathcal{V}(-\mu) = \mathcal{V} \otimes_{\mathcal{O}_{X_y}^{\mathrm{alg}}} \mathcal{O}(-\mu)$. On the other hand, if F^μ is the finite dimensional G module with extremal weight μ then $\mathcal{V}(-\mu) \otimes F^\mu$ is a sheaf of modules for $U^{\cdot}(\mathfrak{p}_y)^{\mathrm{alg}} = \mathcal{O}_{X_y}^{\mathrm{alg}} \otimes U(\mathfrak{p}_y)$ [Sect. 3]. Given a sheaf $\mathcal{M}$ of modules for $U(\mathfrak{l})$ let $\mathcal{M}_{(\lambda)}$ denote the generalized $Z(\mathfrak{l})$ eigensheaf and let $\mathcal{M}_{[\lambda]}$ denote the $Z(\mathfrak{l})$ eigensheaf, both corresponding to the parameter $\lambda \in \mathfrak{h}^*$. An element in $\mathfrak{h}^*$ is called *dominant* if the negative of that element is antidominant. The following result from [19] is in fact a version of the ‘key lemma’ in [2].

LEMMA 9.1 [Miličić]. *Let $\lambda \in \mathfrak{h}^*$ and suppose that μ corresponds to the differential of a holomorphic character for a Cartan subgroup $C \subset L$. Suppose there is a $w \in W$ such that $w\lambda$ is antidominant and $w\mu$ is dominant. Then we have the following.*

(a) *For each weight ν of F^μ and for each $s \in W$ if $s\lambda = \lambda - \mu + \nu$ then $s\lambda = \lambda$ and $\mu = \nu$.*
(b) *If $\mathcal{V}$ is a $\mathcal{D}_\lambda^{\mathrm{alg},\, i}$ module then $(\mathcal{V}(-\mu) \otimes F^\mu)_{(\lambda)}$ is a $\mathcal{D}_\lambda^{alg,\, i}$ module naturally isomorphic to $\mathcal{V}$.*

We now apply this result to show that, for our purposes, there are enough $\mathfrak{l}$ modules with regular antidominant infinitesimal character.

Assume $\mathfrak{l}$ is the stable Levi factor associated to a special point $y \in Y$.

LEMMA 9.2. *Let V be a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K \cap L)$ and suppose that λ is antidominant. Let μ be an integral weight corresponding to the differential*

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of a holomorphic character for $C \subset L$. Assume that μ is so very dominant that $\lambda - \mu$ is antidominant and regular. Then there exists a Harish–Chandra module M for $(\mathfrak{l}, K \cap L)$ with infinitesimal character $\lambda - \mu$ such that V is naturally isomorphic to $(M \otimes F^{\mu})_{(\lambda)}$.

Proof. Let $\Delta_{\lambda}^{\mathrm{alg}}$ denote the localization functor from $U_{\lambda}(\mathfrak{l})$ to $\mathcal{D}_{\lambda}^{alg,\,i}$ and let $\mathcal{M}$ be the sheaf of $(\mathcal{D}_{\lambda-\mu}^{alg,\,i}, K \cap L)$ defined by $\mathcal{M} = \Delta_{\lambda}^{\mathrm{alg}}(V)(-\mu)$. Then $\Delta_{\lambda}^{\mathrm{alg}}(V) \simeq (\mathcal{M} \otimes F^{\mu})_{(\lambda)}$ [Lemma 9.1]. Put $M = \Gamma(X_y, \mathcal{M})$. Then $(M \otimes F^{\mu})_{(\lambda)} \simeq \Gamma(X_y, (\mathcal{M} \otimes F^{\mu})_{(\lambda)}) \simeq \Gamma(X_y, \Delta_{\lambda}^{\mathrm{alg}}(V) \simeq V$ [Lemma 4.1] □

For the moment assume M is an arbitrary Harish–Chandra module for $(\mathfrak{l}, L \cap K)$ and suppose that F is a finite dimensional G module. Using Schmid's results on minimal globalizations [16, 22] one can see that $M_{\min} \otimes F$ is the minimal globalization of $M \otimes F$ and that the eigenspace space $(M_{\min} \otimes F)_{[\lambda]}$ (which is a closed subspace of $M_{\min} \otimes F$) is in fact the minimal globalization of $(M \otimes F)_{[\lambda]}$. Since $M \otimes F$ splits into a finite direct sum of generalized $Z(\mathfrak{l})$ eigenspaces [28] one can deduce in a similar fashion that the minimal globalization of the generalized eigenspace $(M \otimes F)_{(\lambda)}$ is naturally isomorphic with $(M_{\min} \otimes F)_{(\lambda)}$

LEMMA 9.3. *Make the same assumptions on λ and μ and w as in Lemma 9.2 and let M be a Harish–Chandra module for $(U_{\lambda-\mu}(\mathfrak{l}), K \cap L)$.*

(a) *Let $(\mathcal{A}(y, M_{\min}) \otimes F^{\mu})_{(\lambda)}$ denote the generalized $Z(\mathfrak{g})$ eigensheaf on Y corresponding to the infinitesimal character $\lambda \in \mathfrak{h}^*$. Then $(\mathcal{A}(y, M_{\min}) \otimes F^{\mu})_{(\lambda)}$ is a* dnF *sheaf of $(\pi_* \mathcal{D}_{\lambda}, G_0)$ modules naturally isomorphic with $\mathcal{A}(y, (M_{\min} \otimes F^{\mu})_{[\lambda]})$.*

(b) *Let $(\mathcal{I}(y, M) \otimes F^{\mu})_{(\lambda)}$ be the generalized $Z(\mathfrak{g})$ eigensheaf corresponding to λ. Then $(\mathcal{I}(y, M) \otimes F^{\mu})_{(\lambda)}$ is a sheaf of $(\pi_* \mathcal{D}_{\lambda}^{\mathrm{alg}}, K)$ modules naturally isomorphic to $\mathcal{I}(y, (M \otimes F^{\mu})_{[\lambda]})$*

Proof. At this point we briely modify the general setup established in Sect. 6. In particular, we view $M_{\min} \otimes F^{\mu}$ as a module for $(\mathfrak{p}_y, G_{0y})$ where the respective nilradical and unipotent radical act via the tensor product action [28, Defn. 4.1.11(b)]. Then $M_{\min} \otimes F^{\mu}$ is an analytic $(\mathfrak{p}_y, G_{0y})$ module and $\mathcal{A}(y, M_{\min} \otimes F^{\mu})$ is a dnF sheaf of analytic modules for $(U^{\cdot}(\mathfrak{g}), G_0)$ [13]. Indeed, there is a natural isomorphism $\mathcal{A}(y, M_{\min}) \otimes F^{\mu} \simeq \mathcal{A}(y, M_{\min} \otimes F^{\mu})$ [28, Lemma 4.5.2].

To see that the generalized $Z(\mathfrak{l})$ eigenspace in $M_{\min} \otimes F^{\mu}$ corresponding to λ coincides with the eigenspace $(M_{\min} \otimes F^{\mu})_{[\lambda]}$ let t be an element of the Weyl group $W_{\mathfrak{l}}$ of $\mathfrak{l}$ such that $t(\lambda - \mu)$ is antidominant for $\mathfrak{l}$ [Sect. 3]. Consider the *algebraic* localization $\Delta_{t(\lambda-\mu)}^{\mathrm{alg}}(M_{\min})$ of $M_{\min}$ to X_y and let $\mathcal{V}$ denote the sheaf of $\mathcal{D}_{t\lambda}^{\mathrm{alg},\,i}$ modules defined by the equation $\mathcal{V} = \Delta_{t(\lambda-\mu)}^{\mathrm{alg}}(M_{\min}) \otimes_{\mathcal{O}_{X_y}} \mathcal{O}(t\mu)$. Now apply Lemma 9.1 using the parameters $t\lambda$ and $t\mu$.

To complete the argument, filter F^{μ} as in [28, Lemma 7.2.3(b)]. Using Lemma 9.1, we can argue as in [28, Prop. 7.4.1] to obtain the desired result.

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The proof for Part (b) is similar, but with a slight twist. The naturality depends on the naturality of Kashiwara's theorem. View $\mathcal{I}(y,M)\otimes F^{\mu}$ as a sheaf of modules for $(U^{\cdot}(\mathfrak{g})^{\mathrm{alg}},K)$ via the tensor product action and argue, using the obvious filtration coupled with Lemma 9.1, that $(\mathcal{I}(y,M)\otimes F^{\mu})_{(\lambda)}$ is in fact a sheaf of $(\pi_*\mathcal{D}_{\lambda}^{\mathrm{alg}},K)$ modules on which $Z(\mathfrak{g})$ acts via the Harish–Chandra morphism $Z(\mathfrak{g})\to\mathcal{Z}(\mathfrak{l})$ [Sect. 3]. To complete the argument, fix a resolution $\mathcal{R}^{\cdot}\to\mathcal{I}(y,M)$ of $\mathcal{I}(y,M)$ by flat, quasicoherent $\pi_*\mathcal{D}_{\lambda}^{\mathrm{alg}}$ modules. Then we can apply the above sort of filtration arguments to the complex of $\mathcal{Z}(\mathfrak{l})$ finite sheaves of modules $i^*(\mathcal{R}^{\cdot}\otimes F^{\mu})$. The final result follows via some applications of Kashiwara's theorem [7, Thm. 7.11]. $\square$

We are now prepared to conclude Part (b) of the main theorem described in the introduction. It turns out that for our proof of the general case (in the next section) we need (and can obtain) a slightly stronger naturality than alluded to in the introduction.

THEOREM 9.4. *Assume y is special and let $q=\dim Y-\dim Q$ be the codimension of the K orbit $Q=K\cdot y$. Suppose V is a Harish–Chandra module for $(\mathfrak{l},K_0\cap L_0)$ with an infinitesimal character that is antidominant for Y. Then the sheaf cohomology groups $H^p(Y,\mathcal{A}(y,V_{\min}))$ vanish unless $p=q$ in which case $H^q(Y,\mathcal{A}(y,V_{\min}))$ is a topological G_0 module naturally isomorphic to the minimal globalization of $\Gamma(Y,\mathcal{I}(y,V))$. More specifically: let $\mathcal{I}(y,V))\mapsto\mathcal{I}(y,V))^{an}$ denote the application of Serre's GAGA functor* [*Sect.* 2 *and Sect.* 4]. *Then there is a natural morphism $\mathcal{I}(y,V))^{an}\to\mathcal{A}(y,V_{\min})[q]$ in $D(\mathcal{M}(\pi_*\mathcal{D}_{\lambda}))$ such that the resulting morphism $h^0(R\Gamma_Y(\mathcal{I}(y,V)^{an}))\to h^0(R\Gamma_Y(\mathcal{A}(y,V_{\min})[q]))$ gives the desired isomorphism of functors.*

Proof. Let λ be an antidominant element of $\mathfrak{h}^*$ representing the infinitesimal character of V. There exists a μ corresponding to the differential of a holomorphic character for $C\subset L$ and which is so dominant that $\lambda-\mu$ is antidominant and regular. According to Lemma 9.2 there is a finite dimensional G module F^{μ} of highest weight μ and there is a Harish–Chandra module M with infinitesimal character $\lambda-\mu$ such that V is naturally isomorphic to $(M\otimes F^{\mu})_{(\lambda)}$. Hence we see that $H^p(Y,\mathcal{A}(y,V_{\min}))\simeq H^p(Y,(\mathcal{A}(y,M_{\min})\otimes F^{\mu})_{(\lambda)})$ [Lemma 9.3]. A standard argument for sheaf cohomology shows that $H^p(Y,(\mathcal{A}(y,M_{\min})\otimes F^{\mu})_{(\lambda)})\simeq(H^p(Y,\mathcal{A}(y,M_{\min}))\otimes F^{\mu})_{(\lambda)}$. Thus an application of Theorem 8.1 makes vanishing clear and shows that $H^q(Y,\mathcal{A}(y,V_{\min}))$ is the minimal globalization of $(\Gamma(Y,\mathcal{I}(y,M))\otimes F^{\mu})_{(\lambda)}$. The desired result is obtained by another application of Lemma 9.3.

To obtain the morphism $\mathcal{I}(y,V))^{an}\to\mathcal{A}(y,V_{\min})[q]$ in the case of regular antidominant λ simply localize the equivariant inclusion $\Gamma(Y,\mathcal{I}(y,V)\to H^q(Y,\mathcal{A}(y,V_{\min}))$. Use the tensoring [Lemma 9.3] for the general case. The naturality of the construction is apparent from Theorem 8.1 and Lemma 9.3. $\square$

Tensoring down the length

Our first task is to define a notion of length for an infinitesimal character of a Levi factor. Suppose $y \in Y$ and let $\mathfrak{l} \subset \mathfrak{p}_y$ be a Levi factor. Let $\Sigma(\mathfrak{l})^+ \subset \Sigma^+$ be the set of positive roots for $\mathfrak{h}$ in $\mathfrak{l}$ and let $\Sigma(\mathfrak{u}) = \Sigma^+ - \Sigma(\mathfrak{l})^+$ denote the roots of $\mathfrak{h}$ in the nilradical. Assume χ is a character for $Z(\mathfrak{l})$ and suppose $\lambda \in \mathfrak{h}^*$ is a parameter representing χ. We consider the nonnegative integer $n(\chi) =$ the number of roots $\alpha \in \Sigma(\mathfrak{u})$ such that $\check{\alpha}(\lambda)$ is a positive integer. Since the Weyl group $W_{\mathfrak{l}}$ for $\mathfrak{l}$ is contained in the set of $w \in W$ which map $\Sigma(\mathfrak{u})$ to itself, we see that the number $n(\chi)$ is independent of the choice of parameter λ representing χ. We refer to this nonnegative integer $n(\chi)$ as the *length* of χ.

In order to apply a certain result of Miličić [19] we also introduce a notion of length for elements of $\mathfrak{h}^*$ that is closely related to the previous definition. Specifically, for $\lambda \in \mathfrak{h}^*$ define $n(\lambda) =$ the number of roots $\alpha \in \Sigma^+$ such that $\check{\alpha}(\lambda)$ is a positive integer. Observe that if χ_λ is the character corresponding to λ then we have the inequality

$$n(\chi_\lambda) \leq n(\lambda).$$

For any root α let s_α denote the corresponding reflection on $\mathfrak{h}^*$.

LEMMA 9.5 [Miličić]. *Let $\lambda \in \mathfrak{h}^*$ and suppose that $n(\lambda)$ is positive. Then there exists an integral weight μ and an $\alpha \in \Sigma^+$ such that*

(i) $\check{\alpha}(\lambda) = \check{\alpha}(\mu)$ *is positive*
(ii) $n(\lambda - \mu)$ *and* $n(s_\alpha\lambda)$ *are both less than* $n(\lambda)$
(iii) *If ν is any weight of F^μ and if $s \in W$ then the equation $\lambda - \mu + \nu = s\lambda$ is satisfied if and only if either $s\lambda = \lambda$ and $\nu = \mu$ or else if $s\lambda = s_\alpha\lambda$ and $\nu = s_\alpha\mu$.*

The next lemma simply points out that the result by Miličić is sufficient for an induction on the length of infinitesimal characters for a Levi factor.

LEMMA 9.6. *Suppose that $\lambda \in \mathfrak{h}^*$ is antidominant for $\mathfrak{l}$ and that $n(\lambda)$ is positive. Suppose we have an integral weight μ and an $\alpha \in \Sigma^+$ satisfying the properties spelled out in the previous lemma. Then $n(\chi_{\lambda-\mu})$ and $n(\chi_{s_\alpha\lambda})$ are both less than $n(\chi_\lambda)$. In particular $\alpha \in \Sigma(\mathfrak{u})$.*

Proof. Since λ is antidominant for $\mathfrak{l}$ it follows that $n(\chi_\lambda) = (\lambda)$. Hence the claims follow from the above inequality. □

There is a slight complication that occurs in our argument because it may be the case that the μ we need from Lemma 9.5 does not correspond to the differential of a holomorphic character for a Cartan subgroup of G. For that reason we introduce a certain covering group defined in [28, Lemma 7.3.5].

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LEMMA 9.7 [Vogan]. *There is a finite covering group $\widetilde{G} \to G$ with the property that for each weight $\mu \in \mathfrak{h}^*$ there exits $\widetilde{\mu} \in \mathfrak{h}^*$ which corresponds to the differential of a holomorphic character for a Cartan subgroup $\widetilde{C} \subset \widetilde{G}$ and which satisfies: $\check{\alpha}(\tilde{\mu}) = \check{\alpha}(\mu)$ for all $\alpha \in \Sigma$. In particular, we may replace μ by $\widetilde{\mu}$ in Lemma 9.5 and retain the stated properties.*

The following result is the analog to Lemma 9.2 and shows that, for our purposes there are enough Harish–Chandra modules with shorter infinitesimal characters.

Assume that $\mathfrak{l}$ is the stable Levi factor associated to a special point $y \in Y$ and let $K \widetilde{\cap} L$ be the preimage of $K \cap L$ in $\widetilde{G}$.

LEMMA 9.8. *Suppose that V is a Harish–Chandra module for $(\mathfrak{l}, K \cap L)$ with a given infinitesimal character of positive length. Let λ be a parameter for this infinitesimal character that is antidominant for $\mathfrak{l}$. Fix an integral weight μ corresponding to the differential of a holomorphic character for $\widetilde{C} \subset \widetilde{G}$ and a root α having the properties spelled out in Lemma* 9.5*. Then there exists a Harish–Chandra module M for $(\mathfrak{l}, K \widetilde{\cap} L)$ with infinitesimal character $\lambda - \mu$ and a natural isomorphism $V \simeq (M \otimes F^{\mu})_{(\lambda)}$ of $(\mathfrak{l}, K \widetilde{\cap} L)$ modules where $(M \otimes F^{\mu})_{(\lambda)}$ denotes the generalized $Z(\mathfrak{l})$ eigenspace. In addition the $K \widetilde{\cap} L$ action on $(M \otimes F^{\mu})_{(s_\alpha \lambda)}$ factors to an action of $K \cap L$.*

Proof. Let $\mathcal{M}$ be the sheaf of $(\mathcal{D}^{alg,\,i}_{\lambda-\mu}, K \widetilde{\cap} L)$ modules defined by $\mathcal{M} = \Delta^{\mathrm{alg}}_{\lambda}(V)(-\mu)$ and let $M = \Gamma(X_y, \mathcal{M})$. An argument as in the proof of Lemma 9.2 via an application of Lemma 9.5 shows that $\Delta^{\mathrm{alg}}_{\lambda}(V) \simeq (\mathcal{M} \otimes F^{\mu})_{(\lambda)}$. Hence we recover the first claim by taking global sections [Lemma 4.1].

Another application of Lemma 9.5 to the (by now) standard filtration argument establishes an isomorphism of $(\mathcal{D}^{alg,\,i}_{\lambda-\mu}, K \widetilde{\cap} L)$ modules: $(\mathcal{M} \otimes F^{\mu})_{(s_\alpha \lambda)} \simeq \Delta^{\mathrm{alg}}_{\lambda}(V)(-\check{\alpha}(\mu)\alpha)$. Since the $K \widetilde{\cap} L$ action on $\mathcal{O}(-\check{\alpha}(\mu)\alpha)$ factors through to $K \cap L$ we recover the second claim by taking global sections □

LEMMA 9.9. *Let V be a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K \cap L)$. Suppose that λ has positive length and is antidominant for $\mathfrak{l}$. Fix μ, α and M as in the previous lemma.*

(a) *Let $(\mathcal{I}(y, M) \otimes F^{\mu})_{(\lambda)}$ denote the generalized $Z(\mathfrak{g})$ eigensheaf on Y corresponding to the infinitesimal character $\lambda \in \mathfrak{h}^*$. Then there is the following short exact sequence of $(U^{\cdot}(\mathfrak{g})^{\mathrm{alg}}, K)$ modules:*

$$0 \to \mathcal{I}(y, V) \to (\mathcal{I}(y, M) \otimes F^{\mu})_{(\lambda)} \to \mathcal{I}(y, (M \otimes F^{\mu})_{[s_\alpha \lambda]}) \to 0$$

(b) *Let $(\mathcal{A}(y, M_{\mathrm{min}}) \otimes F^{\mu})_{(\lambda)}$ be the generalized $Z(\mathfrak{g})$ eigensheaf on Y corresponding to the infinitesimal character λ. Then there is the following short exact sequence of* dnF $(U^{\cdot}(\mathfrak{g}), G_0)$ *modules.*

$$\begin{aligned} 0 &\to \mathcal{A}(y, V_{\mathrm{min}}) \to (\mathcal{A}(y, M_{\mathrm{min}}) \otimes F^{\mu})_{(\lambda)} \\ &\to \mathcal{A}(y, (M_{\mathrm{min}} \otimes F^{\mu})_{[s_\alpha \lambda]}) \to 0. \end{aligned}$$

Proof. To establish part (b) use Lemma 9.5 and argue as in [28, Prop. 7.4.3]. Lemma 9.8 shows that the generalized $Z(\mathfrak{l})$ eigenspace $(M_{\min} \otimes F^{\mu})_{(s_\alpha \lambda)}$ agrees with the the eigenspace $(M_{\min} \otimes F^{\mu})_{[s_\alpha \lambda]}$. These considerations show we obtain an exact sequence of sheaves of dnF $(U^{\cdot}(\mathfrak{g}), \widetilde{G_0})$ modules where $\widetilde{G_0} \subset \widetilde{G}$ is the preimage of G_0 in $\widetilde{G}$. Another application of Lemma 9.8 shows that $(\mathcal{A}(y, M_{\min}) \otimes F^{\mu})_{(\lambda)}$ is an extension of two G_0 modules. Hence the action of $\widetilde{G_0}$ on $(\mathcal{A}(y, M_{\min}) \otimes F^{\mu})_{(\lambda)}$ factors through to G_0.

The above considerations coupled with the techniques utilized in the proof of Lemma 9.3 make the proof of part (a) straightforward. □

The main result in the paper is the following theorem. As mentioned earlier it turns out we need (and can obtain) a slightly stronger version of naturality than alluded to in the introduction.

THEOREM 9.10. *Let $y \in Y$ be a special point and let V be a Harish–Chandra module for $(\mathfrak{l}, K \cap L)$ with infinitesimal character. Let q denote the codimension for the K orbit of y. Then there is a naturally defined* dnF *topology and a continuous G_0 action defined on the sheaf cohomolgy group $H^p(Y, \mathcal{A}(y, V_{\min}))$ such that for each p the resulting G_0 module is naturally and topologically isomorphic to the minimal globalization of $H^{p-q}(Y, \mathcal{I}(y, V))$. More specifically there is a natural morphism $\mathcal{I}(y, V)^{an} \to \mathcal{A}(y, V_{\min})[q]$ in $D(\mathcal{M}(\pi_* \mathcal{D}_\lambda))$ such that the resulting morphisms:*

$$h^{p-q}(R\Gamma_Y(\mathcal{I}(y, V)^{an})) \to h^{p-q}(R\Gamma_Y(\mathcal{A}(y, V_{\min})[q]))$$

provide the desired isomorphisms of functors.

Proof. We induct on the length of the infinitesimal character χ of V. When χ has length zero then the result reduces to Theorem 9.4. So assume χ has positive length and suppose the theorem holds for all Harish–Chandra modules with shorter infinitesimal characters. Suppose λ is an $\mathfrak{l}$ antidominant element of $\mathfrak{h}^*$ representing the infinitesimal character χ. Fix μ, α and M as in the previous lemma. Using the inductive hypothesis as well as Lemma 9.9 we obtain the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}(y, V) & \longrightarrow & (\mathcal{I}(y, M) \otimes F^{\mu})_{(\lambda)} & \longrightarrow & \mathcal{I}(y, (M \otimes F^{\mu})_{[s_\alpha \lambda]}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}(y, V_{\min})[q] & \longrightarrow & (\mathcal{A}(y, M_{\min}) \otimes F^{\mu})_{(\lambda)}[q] & \longrightarrow & \mathcal{A}(y, (M_{\min} \otimes F^{\mu})_{[s_\alpha \lambda]})[q] & \longrightarrow & 0
\end{array}$$

Because of the naturality the vertical morphisms coming from the inductive assumption are such that the square commutes in $D(\mathcal{M}(\pi_* \mathcal{D}_\lambda))$. Thus we obtain a morphism $\mathcal{I}(y, V) \to \mathcal{A}(y, V_{\min})[q]$ in $D(\mathcal{M}(\pi_* \mathcal{D}_\lambda))$ such that the above diagram completes to a morphism of distinguished triangles. To see that the morphism $\mathcal{I}(y, V) \to \mathcal{A}(y, V_{\min})[q]$ satisfies the necessary naturality simply argue using the functorality of the short exact sequences in Lemma 9.9.

We change notations briefly. Let $H^{p-q}(V)$ denote the sheaf cohomology group $H^{p-q}(Y, \mathcal{I}(y, V))$, let $H^{p-q}(W)$ stand for $(H^{p-q}(Y, \mathcal{I}(y, M)) \otimes F^{\mu})_{(\lambda)}$ and put $H^{p-q}(Q) = H^{p-q}(Y, \mathcal{I}(y, (M \otimes F^{\mu})_{[s_\alpha \lambda]}))$. Similarly for the analytic sheaves let $H^p(V_{\min})$ denote $H^p(Y, \mathcal{A}(y, V_{\min}))$ and likewise for the rest. Applying the derived functor of global sections to the above morphism of distinguished triangles gives the resulting morphism of long exact sequences in sheaf cohomology

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^{p-1-q}(Q) & \longrightarrow & H^{p-q}(V) & \longrightarrow & H^{p-q}(W) & \longrightarrow & H^{p-q}(Q) & \longrightarrow & H^{p+1-q}(V) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^{p-1}(Q_{\min}) & \longrightarrow & H^{p}(V_{\min}) & \longrightarrow & H^{p}(W_{\min}) & \longrightarrow & H^{p}(Q_{\min}) & \longrightarrow & H^{p+1}(V_{\min}) & \longrightarrow \end{array}$$

The bottom row is a long exact of sequence of topological $U(\mathfrak{g})$ modules and continuous morphisms. Using the inductive assumption and a standard argument [13, Lemma 9.1 and Corollary A.11] we see that this bottom row is in fact a long exact sequence of dnF $U(\mathfrak{g})$ modules. Because the relevant categories we consider are not closed under extensions by $U(\mathfrak{g})$ modules, we refer to the formalism of group actions on sheaf cohomology in order to complete the argument [8, 13].

LEMMA 9.11 [Chang, Hecht and Taylor]. (a) *The* U_λ *modules* $H^{p-q}(Y, \mathcal{I}(y, V))$ *have naturally defined compatible algebraic* K *actions*.

(b) *The* dnF U_λ *modules* $H^p(Y, \mathcal{A}(y, V_{\min}))$ *have naturally defined compatible analytic* G_0 *actions*.

Observe that the K and G_0 actions are each uniquely determined for the identity components by the $\mathfrak{g}$ action.

Using Lemma 9.9 together with the inductive hypothesis it follows that the top row is a long exact sequence of Harish–Chandra modules while the bottom row is long exact sequence of minimal globalizations [13, Lemma 10.11] such that the vertical morphisms are $(\mathfrak{g}, K_0)$ equivariant. In particular the morphisms $H^{p-q}(V) \to H^p(V_{\min})$ lift to continuous G_0 equivariant morphisms $(H^{p-q}(V))_{\min} \to H^p(V_{\min})$. Hence we can apply the functor of minimal globalization to the top row and use the five lemma to obtain the desired result. $\square$

10. An open orbit and duality

If $x \in X$ is a special point in the flag manifold then the G_0 orbit of x is open if and only if the corresponding Borel subalgebra $\mathfrak{b}_x$ is θ stable. On the generalized flag manifold, examples show that an open G_0 orbit need not contain a θ stable parabolic subalgebra. Nevertheless if $y \in Y$ is special then the G_0 orbit $S = G_0 \cdot y$ is open if and only if $\mathfrak{p}_y$ is θ stable. In turn this happens if and only if $G_0 \cap P_y = L_0$ is the real stable Levi factor associated to y. For the remainder of the section we fix a special point y and we assume $\mathfrak{p}_y$ is θ stable. Let Q be the K orbit of y and put $q = \dim Y - \dim Q$. Since the preimage $\pi^{-1}(Q) \subset X$ contains a closed K orbit [17] it follows that Q is closed in Y.

For the moment assume λ is antidominant for Y. Then we can apply the following argument exactly as in the case of a flag manifold. In particular, $\Gamma_Y \circ \Delta_Y^{\mathrm{alg}}$ is naturally isomorphic to the identity [Lemma 4.1]. Using these facts we can in turn conclude that whenever $\mathcal{V}$ is an irreducible, coherent sheaf of $(\pi_*\mathcal{D}_\lambda^{\mathrm{alg}}, K)$ modules then $\Gamma(Y, \mathcal{V})$ is an irreducible Harish–Chandra module provided it is nonzero. On the other hand, suppose that V is an irreducible Harish–Chandra module for $(\mathfrak{l}, K_0 \cap L_0)$. Since Q is closed, with the help of Kashiwara's theorem we conclude that $\mathcal{I}(y, V)$ is an irreducible, coherent sheaf of $(\pi_*\mathcal{D}_\lambda^{\mathrm{alg}}, K)$ modules. Thus we have the following

COROLLARY 10.1. *Suppose that y is a θ stable special point and that V_{min} is a topologically irreducible minimal globalization for L_0 whose infinitesimal character is antidominant for Y. Then $H^q(Y, \mathcal{A}(y, V_{\mathrm{min}}))$ is a topologically irreducible representation for G_0 whenever it is nonzero. If the infinitesimal character for V_{min} is regular as well as antidominant for Y then $H^q(Y, \mathcal{A}(y, V_{\mathrm{min}}))$ is not zero.*

Our final task in this paper is to briefly consider an application of the main results to a certain conjecture about the geometric realization of Zuckerman modules [28, 29, Conj. 6.11]. In particular, using a derived functor construction (which depends on the parabolic subalgebra $\mathfrak{p}_y$), each Harish–Chandra module V for $(\mathfrak{l}, L_0 \cap K_0)$ determines a family of Harish–Chandra modules: $R^p(y, V), p = 0, 1, 2, \ldots,$ for $(\mathfrak{g}, K_0)$ called *Zuckerman modules* [15, 28]. The conjecture we refer to proceeds as follows. Using a smooth globalization of V and the polarization $\mathfrak{p}_y$ define a G_0 equivariant holomorphic vector bundle over the complex manifold $S = G_0 \cdot y$. Let $\mathcal{V}$ denote the corresponding sheaf of sections. Then the sheaf cohomolgy groups $H^p(S, \mathcal{V})$ are conjectured to be globalizations of the Zuckerman modules $R^p(y, V)$.

To approach this problem we use a duality theorem relating the Harish–Chandra modules we have considered here to the Zuckerman modules, as follows. When M is a Harish–Chandra module for $(\mathfrak{g}, K_0)$ then the K_0 finite dual $M^\vee$ is again a Harish–Chandra module. Indeed, the continuous dual M'_{min} of the minimal globalization of M is a maximal globalization of $M^\vee$ [22]. Put $s = \dim Q$. Let $T_y(\Omega)$ be the geometric fiber of the canonical bundle at y. As an L_0 module $T_y(\Omega) \simeq \bigwedge^n \mathfrak{u}_y$ where n is the dimension of Y and $\mathfrak{u}_y$ is the nilradical of $\mathfrak{p}_y$. Using precisely the methods developed by Hecht, Milicić, Schmid and Wolf in [15], Chang has established the following result [8].

THEOREM 10.2 [Hecht, Milicić, Schmid and Wolf; Chang]. *If V is a Harish–Chandra module for $(U_\lambda(\mathfrak{l}), K_0 \cap L_0)$ then for each integer p there is a natural isomorphism of Harish–Chandra modules $H^p(Y, \mathcal{I}(y, V))^\vee \simeq R^{s-p}(y, V^\vee \otimes T_y(\Omega))$.*

COROLLARY 10.3. *If the Harish–Chandra module V has an infinitesimal character then $H^p(Y, \mathcal{A}(y, V_{\mathrm{min}}))'$ is the maximal globalization of $R^{n-p}(y, V^\vee \otimes T_y(\Omega))$ for each p.*

If V is a finite dimensional representation for L_0 and if $\mathcal{A}(y,V)$ denotes the corresponding induced analytic sheaf then $\mathcal{A}(y,V)|_S$ is the sheaf of sections for a homogeneous holomorphic vector bundle defined over S. In all cases the sheaf cohomology groups $H^p(Y,\mathcal{A}(y,V))$ are naturally identified with the compactly supported sheaf cohomology groups $H^p_c(S,\mathcal{A}(y,V)|_S)$. Nevertheless, even here the application of Serre duality [24] is not completely trivial, since we have not used the Dolbeault resolution to define topologies for the sheaf cohomology groups. Our approach for dealing with the topological duality utilizes some ideas developed by Hecht, Miličić and Taylor.

Assume now that V is a Harish–Chandra module for $(\mathfrak{l}, K_0 \cap L_0)$. To simplify notation let $\mathcal{A}$ denote the sheaf $\mathcal{A}(y,V_{\min})$ and let $\mathcal{C}^{\cdot}(\mathcal{A})$ denote the Czech resolution of $\mathcal{A}$. Consider the complex $\mathbb{D}^p(\mathcal{A})(Y) = \Gamma(Y,\mathcal{C}^{n-p}(\mathcal{A}))'$ obtained by shifting the global sections of the Czech complex and applying the functor of continuous dual. In particular $\mathbb{D}^{\cdot}(\mathcal{A})(Y)$ is a complex of nF (= nuclear Fréchet) modules for $U(\mathfrak{g})$. Since the cohomologies for the global sections of the Czech complex are Hausdorff, it follows that the cohomologies $h^p(\mathbb{D}^{\cdot}(\mathcal{A})(Y))$ provide the maximal globalizations refered to in Corollary 10.3. The question we are interested in is this: does the complex $\mathbb{D}^{\cdot}(\mathcal{A})(Y)$ compute the sheaf cohomology (on S) of a reasonably defined induced sheaf?

One thing is certain: the complex $\mathbb{D}^{\cdot}(\mathcal{A})(Y)$ does compute the hypercohomology of a certain complex of sheaves on Y. In particular, we can see that when $\mathcal{F}$ is a dnF sheaf on Y then the compactly supported sections of $\mathcal{F}$ in an open set $U \subset Y$ are identified with a closed subspace of the global sections, since we have the short exact sequence:

$$0 \to \Gamma_c(U,\mathcal{F}) \to \Gamma(Y,\mathcal{F}) \to \Gamma(Y-U,\mathcal{F}).$$

Hence, $\Gamma_c(U,\mathcal{F})$ is a dnF space. If $\mathcal{F}$ is a dnF sheaf of $\pi_*\mathcal{D}_\lambda$ modules then $\Gamma(U,\pi_*\mathcal{D}_{-\lambda})$ acts on the continuous dual $\Gamma_c(U,\mathcal{F})'$. Suppose $\mathcal{F}$ is a dnF sheaf and let $\mathcal{F}^{\cdot}(\mathcal{F})$ denote the Czech resolution of $\mathcal{F}$. For each p consider the presheaf defined by: $\mathbb{D}^p(\mathcal{F})(U) = \Gamma_c(U,\mathcal{F}^{n-p}(\mathcal{F}))'$. Using the fact that the sheaves $\mathcal{F}^{n-p}(\mathcal{F})$ are fine, one checks that the presheaf $\mathbb{D}^p(\mathcal{F})$ is in fact a sheaf, which is flabby since $\Gamma_c(U,\mathcal{F}^{n-p}(\mathcal{F}))$ injects onto a closed subspace of $\Gamma(X,\mathcal{F}^{n-p}(\mathcal{F}))$. The resulting complex of sheaves $\mathbb{D}^{\cdot}(\mathcal{F})$ will be refered to as *the dual complex*. We summarize the above remarks in the following proposition.

PROPOSITION 10.4. *To each* dnF *sheaf $\mathcal{F}$ on Y we can functorially assign a complex of flabby nF sheaves denoted $\mathbb{D}^{\cdot}(\mathcal{F})$ and called the dual complex for $\mathcal{F}$. When $\mathcal{F}$ is an object in $\mathcal{M}_{\mathrm{dnF}}(\pi_*\mathcal{D}_\lambda)$ then $\mathbb{D}^{\cdot}(\mathcal{F})$ is a complex of $\pi_*\mathcal{D}_{-\lambda}$ modules. If $H^p(Y,\mathcal{F})$ is Hausdorff for each p, then the dual complex has hypercohomologies: $\mathbb{H}^p(Y,\mathbb{D}^{\cdot}(\mathcal{F})) = h^p(\mathbb{D}^{\cdot}(\mathcal{F})(Y)) = H^{n-p}(Y,\mathcal{F})'$.*

Let V be a finite dimensional module for L_0. As above let $\mathcal{A} = \mathcal{A}(y,V)$ denote the corresponding induced analytic sheaf. Since $\mathcal{A}|_S$ is the sheaf of sections of a

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holomorphic vector bundle on S we can also consider the sheaf of sections $\mathcal{A}^*$ for the dual bundle (on S) extended by zero to all of Y. In particular, $\mathcal{A}^* \simeq \mathcal{A}(y, V^*)$ where L_0 acts on V^* in the standard fashion. Since $\mathcal{A}^* \otimes_{\mathcal{O}} \Omega$ is a $U^{\cdot}(\mathfrak{g})$ module the sheaf cohomologies $H^p(S, \mathcal{A}^* \otimes_{\mathcal{O}} \Omega)$ are $U(\mathfrak{g})$ modules. The following proposition assures us that Serre duality holds in this case.

PROPOSITION 10.5. (a) *$h^p(\mathbb{D}^{\cdot}(\mathcal{A}))|_S = 0$ unless $p = 0$, in which case $h^0(\mathbb{D}^{\cdot}(\mathcal{A}))|_S \simeq (\mathcal{A}^* \otimes_{\mathcal{O}} \Omega)|_S$ as $U^{\cdot}(\mathfrak{g})|_S$ modules.*

(b) *For each p, $h^p(\mathbb{D}^{\cdot}(\mathcal{A})(Y)) \simeq H^p(S, \mathcal{A}^* \otimes_{\mathcal{O}} \Omega)$ as a $U(\mathfrak{g})$ module.*

Proof. To establish (a), let $U \subset S$ be an open set. For each p, let $H_c^p(U, \mathcal{A})_{\text{Czech}}$ denote the pth compactly supported sheaf cohomology on U computed as a topological vector space using the Czech resolution and let $H_c^p(U, \mathcal{A})_{\text{Dolbeault}}$ denote the same sort of object constructed using the Dolbeault resolution with distribution coefficients. Then Taylor has shown [26] there is a continuous isomorphism $H_c^p(U, \mathcal{A})_{\text{Czech}} \to H_c^p(U, \mathcal{A})_{\text{Dolbeault}}$ which is topological whenever $H_c^p(U, \mathcal{A})_{\text{Dolbeault}}$ is Hausdorff. In particular, if $U \subset S$ is a Stein open set, then $H_c^p(U, \mathcal{A})$ vanishes unless $p = n$ in which case $H_c^n(U, \mathcal{A})_{\text{Dolbeault}}$ is Hausdorff with continuous dual (isomorphic to) $\Gamma(U, \mathcal{A}^* \otimes_{\mathcal{O}} \Omega)$ [24]. Hence $h^p(\mathbb{D}^{\cdot}(\mathcal{A}))|_S = 0$ unless $p = 0$. Indeed, the two isomorphisms mentioned above determine an isomorphism of presheaves $(\mathcal{A}^* \otimes_{\mathcal{O}} \Omega)|_S \to h^0(\mathbb{D}^{\cdot}(\mathcal{A}))|_S$ defined on a basis of Stein open sets.

To establish (b), note that part (a) together with the fact that the dual complex consists of flabby $U^{\cdot}(\mathfrak{g})$ modules implies $h^p(\mathbb{D}^{\cdot}(\mathcal{A})(S)) = H^p(S, \mathcal{A}^* \otimes_{\mathcal{O}} \Omega))$ as a $U(\mathfrak{g})$ module. Now consider the inclusion of complexes of DNF $U(\mathfrak{g})$ modules: $\Gamma_c(S, \mathcal{F}^{\cdot}(\mathcal{A})) \to \Gamma(Y, \mathcal{F}^{\cdot}(\mathcal{A}))$. By the very nature of the constructions involved this inclusion is a quasi-isomorphism. Since the cohomologies $h_p(\Gamma(Y, \mathcal{F}^{\cdot}(\mathcal{A})))$ are Hausdorff, one sees that the morphism of complexes $\mathbb{D}^{\cdot}(\mathcal{A})(Y) \to \mathbb{D}^{\cdot}(\mathcal{A})(S)$ is also a quasi-isomorphism $\square$

In particular, if $\mathcal{V}$ is the sheaf of sections of a holomorphic vector bundle defined on an open set $S \subset Y$ then the previous argument shows that Serre duality holds whenever the Czech resolution yields Hausdorff topologies for the compactly supported sheaf cohomology groups.

The following result was established in [13] using different methods.

COROLLARY 10.6. *Let $y \in Y$ be a θ stable special point and let V be a finite dimensional L_0 module. Assume $\mathcal{V}$ is the sheaf of sections of the corresponding homogeneous holomorphic vector bundle on $S = G_0 \cdot y$. Then, for each p, the sheaf cohomologies $H^p(S, \mathcal{V})$ are maximal globalizations of the Zuckerman modules $R^p(y, V)$.*

We conclude with a brief consideration of the difficulties involved in generalizing this result to the case where V is an infinite dimensional Harish–Chandra module for $(\mathfrak{l}, L_0 \cap K_0)$. Assume V has an infinitesimal character which is antidominant for Y. Let $(g, v) \to \omega(g)v$ denote the action of L_0 on $V_{\min}$ and as before, let $\mathcal{A}$ denote the sheaf $\mathcal{A}(y, V_{\min})$. Observe that Proposition 10.4 and Corollary 10.3 imply that the dual complex $\mathbb{D}^{\cdot}(\mathcal{A})$ has hypercohomologies $\mathbb{H}^p(Y, \mathbb{D}^{\cdot}(\mathcal{A})) = 0$ unless $p = s$ in which case $\mathbb{H}^s(Y, \mathbb{D}^{\cdot}(\mathcal{A}))$ is a maximal globalization of the Zuckerman module $R^s(y, V^\vee \otimes T_y(\Omega))$. The difficulty is in seeing that something like Proposition 10.5(a) should hold. In particular, we would like to know that for some open sets $U \subset Y$ forming a basis of the topology on Y that the compactly supported sheaf cohomologies $H^p_c(U, \mathcal{A})$ vanish unless $p = n$ in which case $H^n_c(U, \mathcal{A})$ is Hausdorff. This would allow us to conclude that the hypercohomology of the dual complex computes the cohomology of a sheaf on S. A sufficient reason this holds when V is finite dimensional is the following: the action of L_0 on $V_{\min}$ extends to a local holomorphic action of the complex group L. This in turn is equivalent to condition that each of the real analytic functions $g \mapsto \omega(g)v$ for $v \in V_{\min}$ extends to a holomorphic function on some (small) fixed open set in the complex group L (where this open set is not dependent on the choice of v). Hence, one is able to deduce that the sheaf $\mathcal{A}$ is locally free as an $\mathcal{O}$ module on S. Our final result shows that this line of argument works only if $V_{\min}$ is finite dimensional.

We establish the following terminology. A local action of the complex group L on $V_{\min}$ consists of an open set $U \subset L$ and a continuous map $\phi\colon U \times V_{\min} \to V_{\min}$ that satisfies

(a) $v \mapsto \phi(g)v$ is a linear map $V_{\min} \to V_{\min}$ for each $g \in U$.

(b) $\phi(g)\phi(h)v = \phi(gh)v$ if g, h and gh all belong to U.

This local action of L on $V_{\min}$ is said to be holomorphic if the functions $U \to V_{\min}$ by $g \mapsto \phi(g)v$ are holomorphic, for each $v \in V_{\min}$.

PROPOSITION 10.7. *Suppose the action of L_0 on $V_{\min}$ extends to a local holomorphic action of the complex group L. Then $V_{\min}$ is finite dimensional.*

Proof. The ingredients of the proof are classical. To begin with, simple length considerations reduce the argument to the case where L_0 is a connected linear semi-simple group and $V_{\min}$ is topologically irreducible. A slight modification of the argument given in [20, Section 51] shows that the local holomorphic action of L determines a global holomorphic action for the simply connected covering group $\widetilde{L}$ of L. Let $\widetilde{U_0} \subset \widetilde{L}$ be a compact real form ($\widetilde{L}$ is semisimple). Now a simple modification of the argument given in [27, Thm. 4.11.14] shows that the action of $\widetilde{U_0}$ on $V_{\min}$ is irreducible. Finally, one knows that the irreducible representations of a compact group on a complete, locally convex space are finite dimensional [30, Sect. 4.4.3]. □

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