PARTIAL COMPLEMENTS IN FINITE GROUPS

INGRID CHEN

(Received 23 November 2008; accepted 6 January 2010)

Communicated by E. A. O’Brien

Abstract

Let $G$ be a finite group with normal subgroup $N$. A subgroup $K$ of $G$ is a partial complement of $N$ in $G$ if $N$ and $K$ intersect trivially. We study the partial complements of $N$ in the following case: $G$ is soluble, $N$ is a product of minimal normal subgroups of $G$, $N$ has a complement in $G$, and all such complements are $G$-conjugate.


Keywords and phrases: soluble groups, complements, partial complements.

1. Introduction

Let $G$ be a finite soluble group with a normal subgroup $N$. A subgroup $H$ of $G$ is a complement of $N$ in $G$ if $H$ intersects $N$ trivially and $G = NH$. We define $K$ to be a partial complement of $N$ in $G$ if $K$ is a subgroup of $G$ and $K$ and $N$ intersect trivially. Consider the following question: if $G$ is a finite soluble group, is each partial complement of $N$ in $G$ contained in a complement of $N$ in $G$?

Hall [4] proved that if $G = NH$, where $N$ is a $p$-group ($p$ a prime) and $p$ does not divide the order of $H$, then each partial complement of $N$ in $G$ is contained in a complement of $N$ in $G$.

Rose considered related problems in [6]. Assume that $p$ is a prime, $N$ is an abelian normal $p$-subgroup of $G$ and $P$ is a Sylow $p$-subgroup of $G$. He proves in [6, Theorem 4] that $P$ splits over $N$ and all the complements of $N$ in $P$ form a single conjugacy class in $P$ if and only if the following conditions hold: $G$ splits over $N$; all the complements of $N$ in $G$ form a single conjugacy class in $G$; and each partial complement of $N$ in $G$ is contained in some complement of $N$ in $G$. If $G$ is a finite primitive soluble group, then it splits over $N$ and all the complements of $N$ form a single conjugacy class in $G$. Rose gives examples of such groups $G$ where:

(i) each partial complement of $N$ is contained in a complement of $N$ [6, Theorem 11];
(ii) a partial complement of $N$ is not contained in a complement of $N$ [6, Example 16].

The problem was again raised by Doerk and Hawkes in [2] (see the discussion between Propositions 15.9 and 15.10). Like Rose, they provide an example [2, Example VIII, 2, 19] of a finite soluble primitive group $G$ where there is a partial complement of $N$ in $G$ that is not contained in any complement of $N$ in $G$.

Assume that $G$ is a finite soluble group, $N$ is a product of minimal normal subgroups of $G$, $N$ has a complement in $G$ and all its complements are conjugate. In this paper we give a necessary and sufficient condition for each partial complement of $N$ in $G$ to be contained in a complement of $N$ in $G$. Let GF($p$) denote the field with $p$ elements.

**Theorem 1.** Let $G$ be a finite soluble group. Let $N$ be a product of minimal normal subgroups of $G$ where $N$ is complemented in $G$ and all its complements are conjugate in $G$. Let $p$ be a prime and let $N_p$ be the Sylow $p$-subgroup of $N$. Then each partial complement of $N$ in $G$ is contained in some complement of $N$ in $G$ if and only if, for each prime $p$ dividing the order of $N$, $N_p$ is projective as a GF($p$)(G/N)-module.

2. Preliminaries

In this section we collect a number of results required in the proof of Theorem 1. Notation is mostly standard; in general, we use the same notation as [2]. We assume throughout the paper that $p$ is a prime number.

**Lemma 2** [6, Lemma 9(i)]. Suppose $G$ is a finite group. Let $N$ be a normal subgroup of $G$ and let $H$ be a complement of $N$ in $G$. Then the number of conjugates of $H$ in $G$ is $|N|/|C_N(H)|$.

Let $F$ be a field of characteristic $p$ and let $K$ be a group of order $p$. We define an $FK$-module $U$ to be uniserial if the successive quotients of the radical series of $U$ are simple.

The following lemma is a consequence of [5, Theorem VII, 5.3] and its proof.

**Lemma 3.** Let $F$ be a field of characteristic $p$ and let $K$ be a group of order $p$. Then the following hold.

(i) The regular $FK$-module is uniserial.
(ii) An $FK$-module generated by a single element is indecomposable.
(iii) $\text{Rad}^{p-1}(FK) \neq 0$ and $\text{Rad}^p(FK) = 0$.
(iv) $\text{Rad}^i(FK)/\text{Rad}^{i+1}(FK)$ is simple, so $\text{Rad}^i(FK)/\text{Rad}^{i+1}(FK)$ has order $|F|$.

**Lemma 4.** Let $F$ be a field of characteristic $p$ where $p$ is a prime. Let $K$ be a group of order $p$. Let the $FK$-module $V$ be the direct sum of $W_1$ and $W_2$ where $W_1$ contains no free submodule and $W_2$ is a free submodule or $W_2 = 0$. Then $\text{Rad}^{p-1}(V) = 0$ if and only if $W_2 = 0$. 


If $W_2 = 0$ then $V = W_1$. Let $U$ be an indecomposable submodule of $W_1$. By [5, Theorem VII, 5.3], there exists $0 < i < p$ such that $U \cong FK/\text{Rad}^i(FK)$. Therefore $\text{Rad}^{p-1}(U) = 0$. Since the radical of the direct sum of submodules is the sum of the radicals of each submodule, $\text{Rad}^{p-1}(V) = 0$.

If $W_2 \neq 0$ then $\text{Rad}^{p-1}(W_2) \neq 0$ since $W_2$ is free and $\text{Rad}^{p-1}(FK) \neq 0$. Therefore $\text{Rad}^{p-1}(V) = 0$. Note that in both cases, $\text{Rad}^p(V) = 0$. \hfill \Box

In the following, we will deal with the semidirect product $G = VK$ where $K$ is a group of order $p$ and $V$ is an elementary abelian $p$-subgroup of $G$. Of course $V$ is a $\text{GF}(p)K$-module, and we will denote by $\text{Rad}(V)$ the radical of $V$ as a $\text{GF}(p)K$-module so that we can apply the previous results; observe that $V$ is now viewed as a multiplicative group.

The next result is similar to [6, Lemma 9(ii)].

**Lemma 5.** Let $G$ be the semidirect product of a normal elementary abelian subgroup $V$ and a cyclic subgroup $K$ of order $p$. Assume that $V = W_1 \times W_2$ is the direct product of two $\text{GF}(p)K$-submodules of $V$ such that $W_1$ contains no free submodule and $W_2$ is a free submodule. Then the elements of order $p$ in $G$ but not in $V$ are those of the form $kw$, where $w \in W_1 \times \text{Rad}(W_2)$ and $1 \neq k \in K$.

**Proof.** Let $k \in K$ and $w \in V$. Define $r_1 = [w, k]$ and $r_{i+1} = [r_i, k]$. By induction,

$$(kw)^p = k^p w^p r_1^{s_1} \cdots r_{p-2}^{s_{p-2}} r_{p-1}^{s_{p-1}}$$

where $s_i = \binom{p}{i+1}$ for every $1 \leq j \leq p - 2$. Note that if $w \in W_1 \times \text{Rad}^j(W_2)$ for any $0 \leq j \leq p$ then $r_i \in \text{Rad}^i(W_1) \times \text{Rad}^{j+i}(W_2)$. Moreover, $k^p = v^p = 1$. On the other hand, since $r_i \in V$ and $p$ divides all the exponents $s_i$, then $r_1^{s_1} \cdots r_{p-2}^{s_{p-2}} = 1$. Observe that $r_{p-1}^{s_{p-1}} = 1$ if and only if $w \in W_1 \times \text{Rad}(W_2)$ (by Lemma 4). The result follows. \hfill \Box

**Corollary 6.** Let $K$ be a group of order $p$ where $p$ is a prime. Let the $\text{GF}(p)K$-module $V$ be the direct product of $W_1$ and $W_2$. Assume that $W_1$ has order $p^s$ and does not contain any free module and that $W_2$ is a free module of rank $l$. Then the number of subgroups of order $p$ in $VK$ that intersect $V$ trivially is $p^{s+l(p-1)}$.

**Proof.** First we apply Lemma 5 to find the number of elements of order $p$ that are in $VK$ but not in $V$. Since any indecomposable module of $V$ is uniserial,

$$|W_1 \times \text{Rad}(W_2)| = p^s p^l(p-1) = p^{l(p-1)+s}.$$ 

So there are $(p^{l(p-1)+s})(p-1)$ elements of order $p$ in $VK$ but not in $V$. Then the number of subgroups of order $p$ in $VK$ that intersect $V$ trivially is

$$\frac{(p^{l(p-1)+s})(p-1)}{p-1} = p^{l(p-1)+s}.$$ 

This concludes the proof. \hfill \Box
**Corollary 7.** Let $G$ be the semidirect product $NK$, where $K$ is a subgroup of order $p$ and $N$ is an elementary abelian $p$-subgroup of $G$. Then the number of subgroups of order $p$ in $G$ that intersect $N$ trivially is equal to the number of conjugates of $K$ in $G$ if and only if $N$ is free as a $\mathbb{GF}(p)K$-module.

**Proof.** Let the $\mathbb{GF}(p)K$-module $N$ be the direct product of $W_1$ and $W_2$, where $W_1$ has order $p^s$ and does not contain any free module, and $W_2$ is a free module of rank $l$. By Corollary 6, the number of subgroups of order $p$ in $NK$ that intersect $N$ trivially is $p^{s+l(p-1)}$. Let $W_1 = U_1 \times \cdots \times U_r$, where $U_i$ are indecomposable submodules. Observe that $S_i$, the minimal submodule of $U_i$, is in the centralizer $C_{U_i}(K)$ and if $w \in U_i$ is not contained in $S_i$, then $w$ is not centralized by $K$. Therefore $|C_{U_i}(K)| = |S_i| = p$. Hence $|C_{W_1}(K)| = p^r$. Similarly, $|C_{W_2}(K)| = p^l$. Hence $|C_N(K)| = p^l p^l = p^{l+r}$.

So, by Lemma 2, we obtain that the number of conjugates of $K$ in $NK$ is

$$\frac{|N|}{|C_N(K)|} = \frac{p^{s+p^l}}{p^{r+l}} = p^{s+l(p-1)-r}.$$

Comparing this result with that of Corollary 6, we see that the number of subgroups of order $p$ in $NK$ that intersect $N$ trivially is the same as the number of conjugates of $K$ in $NK$ if and only if $r=0$; that is, if and only if $N$ is free as a $\mathbb{GF}(p)K$-module.

**Lemma 8.** Let $G$ be a finite soluble group. Let $N$ be a product of minimal normal $p$-subgroups of $G$ where $p$ is a prime. Assume that $H$ is a complement of $N$ in $G$ and that all the complements of $N$ in $G$ are conjugates of $H$. Let $H_0$ be a subgroup of $H$ of order $p$. Then each subgroup of order $p$ in $NH_0$ (the semidirect product) but not in $N$ is contained in a complement of $N$ in $G$ if and only if the number of conjugates of $H_0$ in $NH_0$ is equal to the number of subgroups of order $p$ that are in $NH_0$ but not in $N$.

**Proof.** Let $L$ be a subgroup of order $p$ in $NH_0$ but not in $N$. Then $L$ is contained in a complement of $N$ in $G$ if and only if there exists an element $n$ in $N$ such that $L \subseteq NH_0 \cap H^n$. Observe that $NH_0 = (NH_0)^n$ for each $n \in N$. Hence $L \subseteq NH_0 \cap H^n$ if and only if

$$L \subseteq (NH_0)^n \cap H^n = (NH_0 \cap H)^n.$$

However,

$$(NH_0 \cap H)^n \cong N^n(NH_0 \cap H^n)/N^n \cong (NH_0 \cap NH)^n/N^n = (NH_0)^n/N^n \cong H_0^n.$$

Hence $|(NH_0 \cap H)^n| = |H_0^n| = p$. Therefore $L \subseteq (NH_0 \cap H)^n$ if and only if $L = H_0^n$. Hence each subgroup of order $p$ in $NH_0$ but not in $N$ is contained in some complement of $N$ in $G$ if and only if the number of conjugates of $H_0$ in $NH_0$ is equal to the number of subgroups of order $p$ that are in $NH_0$ but not in $N$. □
Before we prove Lemma 9, we first note a result needed in the proof. Let $H$ be a group, let $F$ be a finite field of prime characteristic $p$ and let $E$ be its algebraic closure. Let $V_E$ denote the $E H$-module $E \otimes_F V$ [5, Definition VII. 1.1]. By [5, Exercise VII. 7.19], the $F H$-module $V$ is projective if and only if the $E H$-module $V_E$ is projective.

**Lemma 9.** Let $F$ be a field of prime characteristic $p$. Let $H$ be a soluble group and let $V$ be a semisimple $F H$-module. Then $V$ restricted to each subgroup $C$ of $H$ of order $p$ is projective as a $F C$-module if and only if $V$ is projective.

**Proof.** If $V$ is projective as a $F (p) H$-module, then its restriction to a subgroup of $H$ is also projective [5, Theorem VII. 7.11(a)].

To prove the other direction, we use the main theorem from [1] which shows that, for a soluble group $H$ and an algebraically closed field $E$, an $E H$-module is primitive if and only if it is quasi-primitive.

Assume that $V$ is not projective. Since $V$ is not projective, $V_E$ is not projective. On the other hand, since $V$ is semisimple and the semisimplicity of $V$ is retained when changing fields (by [5, Theorem VII. 1.8]), $V_E$ is semisimple. Since $V_E$ is not projective, there exists a simple direct summand $U$ of $V_E$ which is not projective. Furthermore, since $H$ is soluble and $U$ is a simple $E H$-module, applying [1, main theorem], we deduce that there is a primitive $E A$-module $W$ where $A \leq H$ and $U \cong W^H$ ($A$ is called a stabilizer limit for $U$). We have that $W$ is simple as an $E A$-module by definition, because it is primitive.

First we show that if $X$ is a subgroup of $A$ having order $p$, then $W_X$ is projective. Let $X$ be a subgroup of $A$ of order $p$. Since $U \cong W^H$, we have $U_X \cong (W^H)_X$. On the other hand, let $\{1, g_2, \ldots, g_m\}$ be a full set of $(A, X)$-double coset representatives of $H$. Applying Mackey’s theorem [2, Theorem B. 6.20], we see that $(W^H)_X$ is isomorphic to the tensor product

$$((W \otimes 1)_{A \cap X})^X \oplus \bigoplus_{i=2}^m ((W \otimes g_i)_{A g_i \cap X})^X.$$

But observe that

$$((W \otimes 1)_{A \cap X})^X = (W_X)^X = W_X.$$

As $U_X \cong (W^H)_X$ is projective by assumption, $W_X$ is also projective by [2, Proposition B. 2.4]. Since $W_X$ is projective, it has dimension divisible by $p$. By [7, note after Theorem 12], the dimension of $W$ is coprime to $p$. Therefore $A$ does not contain any subgroup of order $p$ and so $W$ is projective as an $E A$-module.

By [2, Proposition B. 6.12], $W^H$ is projective. Hence, $U$ is projective since $U \cong W^H$. Therefore $V_E$ is projective and $V$ is also projective, which gives the final contradiction. \hfill $\square$

**3. Proof of main theorem**

Suppose that $G = NH$ where $N$ is a $p$-group. Suppose first that $N$ is not projective as a $F (p) H$-module. By Lemma 9, there exists a subgroup $H_0$ of $H$ of order $p$ such...
that $N$ is not projective as a $\text{GF}(p)H_0$-module. By [2, Theorem B. 4.12] and since $N_{H_0}$ is not projective, we know that $N_{H_0}$ is not free. Now by Corollary 7, the number of subgroups of order $p$ in $NH_0$ that intersect $N$ trivially is different from the number of conjugates of $H_0$ in $NH_0$. Therefore by Lemma 8, there exists a partial complement of $N$ in $G$ of order $p$ which is not contained in a complement of $N$ in $G$.

We now suppose that $N$ is projective as a $\text{GF}(p)H$-module. Let $K$ be a partial complement of $N$ in $G$. We find a subgroup $H_0 \leq H$ such that $K$ and $H_0$ are both complements of $N$ in $NK$. By Dedekind’s lemma [2, Lemma A. 1.3],

$$N(NK \cap H) \cong (NK \cap NH) = NK.$$ 

Observe that, by the isomorphism theorems,

$$(NK \cap H) \cong N(NK \cap H)/N \cong NK/N \cong K,$$

since $K$ and $N$ have trivial intersection. Hence $(NK \cap H) = H_0$ for some $H_0 \leq H$ where $K \cong H_0$.

By [5, Theorem VII. 7.11(a)], $N$ as a $\text{GF}(p)H_0$-module is projective. Furthermore, by [3, Section 2.2], all cohomologies (in particular, the first) vanish. On the other hand, by [2, Theorem A. 15.10], the number of conjugacy classes of complements of $NH_0$ is the order of the first cohomology group and so all complements are conjugate. Now $H_0$ and $K$ are both partial complements of $N$ in $NH_0$ so $K = H_0^n$ for some $n$ in $N$. As a consequence, $K = H_0^n \leq H^n$. That is, $K$ is in a complement of $N$ in $G$. Thus we have proved the theorem when $N$ is a $p$-group.

Now let $N = N_{p'}N_p$ and $G = NH$. Observe that $H$ is a complement of $N$ in $G$ if and only if $N_{p'}H$ is a complement of $N_p$ in $G$. Firstly, all subgroups $N_{p'}H^n$ are complements of $N_p$ in $G$. Secondly, assume that $C$ is any complement of $N_p$ in $G$. Let $q$ be a prime which divides the order of $N_{p'}$. Since the index of $C$ is prime to $q$, there is a Sylow $q$-subgroup of $G$ in $C$, and as a consequence $N_q \leq C$. Since this holds for any prime $q$ that divides the order of $N_{p'}$, $N_{p'} \leq C$. Now we have to show that all the complements of $N_p$ are conjugate. First observe that

$$C = C \cap N_{p'}N_pH = N_{p'}(C \cap N_pH) \quad \text{and} \quad N_{p'} \cap (C \cap N_pH) = 1.$$

Therefore $C \cap N_{p'}H$ is a complement of $N_{p'}$ in $C$ and so $C \cap N_pH$ is a complement of $N_pN_{p'}$ in $G$. Therefore $C \cap N_pH$ is a conjugate of $H$, and as a consequence all the complements of $N_p$ in $G$ are conjugates of $N_{p'}H$.

By [2, Theorem B. 4.11] if $N_p$ is projective as a $\text{GF}(p)H$-module then $N_p$ is projective as a $\text{GF}(p)N_{p'}H$-module. By [5, Theorem VII. 7.11(a)] if $N_p$ is projective as a $\text{GF}(p)N_{p'}H$-module then $N_p$ is projective as a $\text{GF}(p)H$-module. Therefore $N_p$ is projective as a $\text{GF}(p)H$-module if and only if $N_p$ is projective as a $\text{GF}(p)N_{p'}H$-module. Hence $G$ and $N_p$ satisfy the hypothesis of the theorem.

Now suppose that, for each prime $p$ that divides the order of $N$, the Sylow $p$-subgroup $N_p$ is projective as a $\text{GF}(p)N_{p'}H$-module (that is, $N_p$ is projective as a $\text{GF}(p)H$-module). By [6, Corollary 5] and the $p$-group case, every partial complement of $N$ in $G$ is contained in a complement of $N$ in $G$. 


Now suppose that there exists a prime $p$ that divides the order of $N$ such that the Sylow $p$-subgroup $N_p$ is not projective as a $\text{GF}(p)N'_pH$-module and so $N_p$ is not projective as a $\text{GF}(p)H$-module. By [6, Corollary 5] and the $p$-group case, there exists a partial complement of $N$ in $G$ that is not contained in a complement of $N$ in $G$. We have thus proved Theorem 1.

References


INGRID CHEN, Mathematical Sciences Institute, The Australian National University, Canberra, ACT 0200, Australia
e-mail: ingrid.chen@anu.edu.au