Reducing this to a single fraction with denominator

$$(y^2+z^2)(z^2+x^2)(x^2+y^2)(x^2+y^2+z^2),$$

the numerator is

$$\begin{aligned} &2\Sigma x^2\Sigma(y^2+z^2)(z^2+x^2)-9(y^2+z^2)(z^2+x^2)(x^2+y^2)\\ &=2\Sigma x^2(\Sigma x^4+3\Sigma x^2y^2)-9(\Sigma x^4y^2+2x^2y^2z^2)\\ &=2(\Sigma x^6+\Sigma x^4y^2+3\Sigma x^4y^2+9x^2y^2z^2)-9(\Sigma x^4y^2+2x^2y^2z^2)\\ &=2\Sigma x^6-\Sigma x^4y^2=\Sigma(x^2-y^2)(x^4-y^4)\\ &\qquad\qquad=\Sigma(x^2+y^2)(x^2-y^2)^2 \end{aligned}$$

which is a sum of squares.

R. F. Muirhead

Cauchy's Condensation Test.—

Theorem.—If the terms of the series $\Sigma f(n)$ never increase as n increases, and if $\underset{n\to\infty}{\mathrm{L}} f(n)=0$ then $\Sigma f(n)$ converges or diverges with $\Sigma a^n f(a^n)$, $a>1$.

I. Suppose a an integer $\geqq 2$.

Then

$$\begin{aligned} \Sigma f(n)=&[f(1)+f(2)+\quad+f(a-1)]\\ &+[f(a)+f(a+1)+\quad+f(a^2-1)]\\ &+[f(a^2)+\quad+f(a^3-1)]\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &+[f(a^m)+\ldots+f(a^{m+1}-1)]\\ &+\text{ etc.} \end{aligned}$$

Hence $\Sigma f(n)$ converges or diverges with

$$\Sigma[f(a^m)+\ldots+f(a^{m+1}-1)].$$

Now $f(a^m)>f(a^m+1)>\ldots>f(a^{m+1}-1)>f(a^{m+1})$.

$\therefore\ (a^{m+1}-a^m)f(a^m)>f(a^m)+\ldots+f(a^{m+1}-1)>(a^{m+1}-a^m)f(a^{m+1})$.

i.e. $(a-1)a^m f(a^m)>f(a^m)+\ldots+f(a^{m+1}-1)>\dfrac{a-1}{a}a^{m+1}f(a^{m+1})$.

$\therefore\ \Sigma f(n)$ converges with $\Sigma a^m f(a^m)$ or $\Sigma a^n f(a^n)$

or diverges with $\Sigma a^{m+1}f(a^{m+1})$ or $\Sigma a^n f(a^n)$.

This proves the theorem for a integral.

II. Assume $a>1$, but not necessarily an integer

(1°) Suppose $\underset{x\to\infty}{\mathrm{L}} xf(x)\pm 0$ then ultimately $xf(x)>\mathrm{A}$.

i.e. $f(x)>\dfrac{\mathrm{A}}{x}$.

i.e. $f(n)>\dfrac{\mathrm{A}}{n}$.

Hence in this case $\Sigma f(n)$ is divergent.

(2°) Suppose $\underset{x\to\infty}{\mathrm{L}}\, xf(x=0$, and that the approach to the limit is steady.

We can choose r an integer so that $b_1 < a^r < b_2$ where b_1 and b_2 are integers $\geqq 2$.

$\Sigma a^n f(a^n)$.

$$
\begin{array}{rll}
 & =\{af(a)+\ldots+a^r f(a^r)\} & > rb_2 f(b_2) \\
rb_1 f(b_1) > & +\{a^{r+1}f(a^{r+1})+\ +a^{2r}f(a^{2r})\} & > rb_2^2 f(b_2^2) \\
rb_1^2 f(b_1^2) > & +\{a^{2r+1}f(a^{2r+1})+\ldots+a^{3r}f(a^{3r}\} & > rb_2^3 f(b_2^3) \\
 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
rb_1^{m-1}f(b_1^{m-1}) > & +\{a^{\overline{m-1}\,r+1}f(a^{\overline{m-1}\,r+1})+\ldots+a^{mr}f(a^{mr})\} & > rb_2^m f(b_2^m) \\
rb_1^m f(b_1^m) > & +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}
$$

Hence we have

$$\{af(a)+\ldots+a^r f(a^r)\}+r\Sigma b_1^m f(b_1^m) > \Sigma a^n f(a^n) > r\Sigma b_2^m f(b_2^m).$$

Hence if $\Sigma a^n f(a^n)$ converges so does $\Sigma b_2^m f(b_2^m)$ and $\therefore\ \Sigma f(m)$
and if $\Sigma a^n f(a^n)$ diverges so does $\Sigma b_1^m f(b_1^m)$ and $\therefore\ \Sigma f(m)$.

Hence $\Sigma f(n)$ converges or diverges with $\Sigma a^2 f(a^n)$ where $a > 1$.

Q.E.D.

ROBERT VICKERS

To show that the equation

$$\begin{vmatrix} x-a & f & e \\ f & x-b & d \\ e & d & x-c \end{vmatrix} = 0$$

has three real roots.—Let x_1 be any one root. Then ξ, η, ζ can be chosen uniquely to satisfy simultaneously the three equations

$$(x_1-a)\xi+f\eta+e\zeta=0 \ldots\ldots\ldots\ldots \text{(i)}$$
$$f\xi+(x_1-b)\eta+d\zeta=0 \ldots\ldots\ldots\ldots \text{(ii)}$$
$$e\xi+d\eta+(x_1-c)\zeta=0 \ldots\ldots\ldots\ldots \text{(iii)}$$

If x_1 is complex, so also will be ξ, η, ζ. Let their conjugate complexes be $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ so that

$$\xi=\xi_1+\iota\xi_2,\ \bar{\xi}=\xi_1-\iota\xi_2\ ;\ \text{etc.}$$

Multiplying (i), (ii), (iii) respectively by $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ and adding we have

$$\Sigma_{a,b,c}(x_1-a)\xi\bar{\xi}+\Sigma_{d,e,f}f(\eta\bar{\xi}+\xi\bar{\eta})=0 \ldots\ldots\ldots\ldots \text{(iv)}$$

https://doi.org/10.1017/S1757748900001031 Published online by Cambridge University Press