Canad. Math. Bull. Vol. **31** (2), 1988

FURTHER THEOREMS OF THE ROGERS-RAMANUJAN TYPE THEOREMS*

BY
M. V. SUBBARAO** AND A. K. AGARWAL

ABSTRACT. We give three new partition theorems of the classical Rogers-Ramanujan type which are very much in the style of MacMahon. These are a continuation of four theorems of the same kind given recently by the second author. One of these new theorems, very similar to one of the original Rogers-Ramanujan-MacMahon type theorems is as follows: Let $C(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 4, ± 5, ± 6, ± 7 (mod 20). Let $D(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \ldots + b_t$, where $b_t \geqq 2$, $b_i \geqq b_{i+1}$, and if $1 \leqq i \leqq [(t-2)/2]$, $b_i - b_{i+1} \geqq 2$. Then $C(n) = D(n)$.

1. **Introduction, notations and the main results.** In the theory of partitions we find a number of identities which state that for each positive integer n the partitions of n with parts restricted to certain residue classes are equinumerous with the partitions of n on which certain difference conditions are imposed. Among the most striking results of this type are the Rogers-Ramanujan identities. These were stated combinatorially by P. A. MacMahon as follows (1, Theorems 364, 365, p. 291):

1.1. *The number of partitions of n into parts with minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* ± 1 (mod 5).

1.2. *The number of partitions of n with minimal part* 2 *and minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* ± 2 (mod 5).

Recently, Hirschhorn [2] using some of the Slater's identities [4] proved four theorems of the Rogers-Ramanujan type. Later, using the same identities of Slater, Subbarao [3] established entirely different combinatorial results. Subbarao's results bear striking resemblance with the Rogers-Ramanujan

Received by the editors November 3, 1986.
*This work was initiated during the second author's visit to the University of Alberta in July 1985.
**Supported partly by NSERC Grant.
Key words: Partitions, congruences, generating functions.
AMS Subject Classification (1980): 10A45.

identities. For instance, his Theorem 2.1.

1.3. *Let $A(n)$ denote the number of partitions of n into parts congruent to* ± 1, ± 2, ± 5, ± 6, ± 8, ± 9 (mod 20). *Let $B(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \ldots + b_t$,* where $b_i \geqq b_{i+1}$ *and, if*

$$1 \leqq i \leqq \left[\frac{t-1}{2}\right], b_i - b_{i+1} \geqq 2.$$

Then $A(n) = B(n)$ for all n.

This is very much analogous in structure to (1.1).

The object of this paper is to prove the following theorem:

1.4. THEOREM. *Let $C(n)$ denote the number of partitions of n into parts congruent to* ± 2, ± 3, ± 4, ± 5, ± 6, ± 7 (mod 20). *Let $D(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \ldots + b_t$, where $b_t \geqq 2$, $b_i \geqq b_{i+1}$, and, if*

$$1 \leqq i \leqq \left[\frac{t-2}{2}\right], b_i - b_{i+1} \geqq 2.$$

Then $C(n) = D(n)$ for all n.

It is worthwhile to remark here that (1.4) is an analogue to (1.2), in the same manner as (1.3) is to (1.1).

We shall also prove two more identities stated below:

1.5. THEOREM. *Let $P_1(n)$ denote the number of partitions of n into parts congruent to* ± 1, ± 4, ± 6, ± 7 (mod 16). *Let $P_2(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \ldots + b_{2s+1}$, where $b_i \geqq b_{i+1}$, $b_{s+1} \geqq s$, $b_s \neq b_{s+1}$, and, if $1 \leqq i \leqq s - 1$, $b_i - b_{i+1} \geqq 2$. Then $P_1(n) = P_2(n)$ for all n.*

1.6. THEOREM. *The number of partitions of n into odd parts equals the number of partitions of n into an odd number, say $2s + 1$, of parts, satisfying the conditions that the middle part is at least s and the first s parts have minimal difference* 1.

The rising q-factorial is denoted by

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}.$$

If n is a positive integer, then obviously

$$(a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1}),$$

and

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \ldots .$$

https://doi.org/10.4153/CMB-1988-032-3 Published online by Cambridge University Press

2. Proofs of the theorems.

2.1. PROOF OF THEOREM 1.4. Let $\pi_t(n)$ be a partition enumerated by $D(n)$. Then for some $s \geqq 1$, $t = 2s - 1$ or $t = 2s$. First suppose that $t = 2s$. Then

$$\pi_{2s}(n) = b_1 + b_2 + \ldots + b_{2s}$$

with

$$b_s \geqq 2, b_{s-1} \geqq 4, \ldots, b_1 \geqq 2s, \text{ and}$$

$$b_{s+1} \geqq b_{s+2} \geqq \ldots \geqq b_{2s} \geqq 2.$$

We subtract $2, 4, 6, \ldots, 2s$ from $b_s, b_{s-1}, \ldots, b_2, b_1$ respectively, and 2 from each of $b_{s+1}, \ldots, b_{2s}$. This produces a partition of

$$n - [(2 + 4 + \ldots + 2s) + 2s] = n - (s^2 + 3s)$$

into at most $2s$ parts. Thus the partitions of the type $\pi_{2s}(n)$ are generated by

$$\frac{q^{s^2+3s}}{(q; q)_{2s}} \cdot (s = 1, 2, \ldots).$$

Similarly, if $t = 2s - 1$, then

$$\pi_{2s-1}(n) = b_1 + b_2 + \ldots + b_{2s-1}$$

with

$$b_{s-1} \geqq 2, b_{s-2} \geqq 4, \ldots, b_1 \geqq 2s - 2,$$

$$b_s \geqq b_{s+1} \geqq b_{s+2} \geqq \ldots \geqq b_{2s-1} \geqq 2.$$

Subtracting $2, 4, \ldots, 2(s - 1)$ from $b_{s-1}, b_{s-2}, \ldots, b_1$, respectively, and 2 from each of $b_s, b_{s+1}, \ldots, b_{2s-1}$ we are left with a partition of

$$n - 2(1 + 2 + \ldots (s - 1)) - 2s = n - (s^2 + s)$$

into at most $2s - 1$ parts. This shows that the partitions of the type $\pi_{2s-1}(n)$ are generated by

$$\frac{q^{s^2+s}}{(q; q)_{2s-1}} \cdot (s = 1, 2, \ldots).$$

Thus

$$\sum_{n=0}^{\infty} D(n)q^n = 1 + \sum_{s=1}^{\infty} \frac{q^{s^2+s}}{(q; q)_{2s-1}} + \sum_{s=1}^{\infty} \frac{q^{s^2+3s}}{(q; q)_{2s}} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_{2n}}.$$

Now an appeal to Slater's identity [4, (99), p. 162].

$$\text{(2.1.1)} \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n-1})(1 - q^{10n-9})(1 - q^{20n-8})$$

$$\times (1 - q^{20n-12})(1 - q^{10n}),$$

leads to Theorem (1.4).

2.2. PROOF OF THEOREM 1.5. Let $\pi_{2s+1}(n)$ denote a partition enumerated by $P_2(n)$. Then

$$\pi_{2s+1}(n) = b_1 + b_2 + \ldots + b_s + b_{s+1} + \ldots + b_{2s+1}$$

with

$$b_{s+1} \geqq s, b_s \geqq s + 1, b_{s-1} \geqq s + 3, \ldots, b_1 \geqq s + (2s - 1),$$

and

$$b_{s+2} \geqq b_{s+3} \geqq \ldots \geqq b_{2s+1} \geqq 1.$$

Subtract $s, s + 1, s + 3, \ldots, s + (2s - 1)$ from $b_{s+1}, b_s, \ldots, b_1$ respectively and 1 from each of $b_{s+2}, b_{s+3}, \ldots, b_{2s+1}$. This produces a partition of $n - 2s(s + 1)$ into at most $2s + 1$ parts. This shows that the partitions of the type $\pi_{2s+1}(n)$ are generated by

$$\frac{q^{2s(s+1)}}{(q; q)_{2s+1}} (s = 1, 2, \ldots).$$

The theorem follows immediately once we recall the following identity of Slater [4, (86), p. 161]:

$$(2.2.1) \qquad \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_\infty} \prod_{n=1}^{\infty} (1 - q^{8n-3})(1 - q^{8n-5}) \times (1 - q^{16n-14})(1 - q^{16n-2})(1 - q^{8n}).$$

2.3. PROOF OF THEOREM 1.6. Let $\mu(n)$ denote the number of partitions of n of the type described in the second part of the theorem. By the usual argument it can be shown that

$$(2.3.1) \qquad \sum_{n=0}^{\infty} \mu(n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q; q)_{2n+1}}.$$

The theorem follows immediately once we note that the right-hand side of (2.3.1) equals $(-q; q)_\infty$ in view of the following identity due to Slater [4, (9), p. 153]:

$$(2.3.2) \quad \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q; q)_{2n+1}} = \prod_{n=1}^{\infty} (1 + q^{4n-1})(1 + q^{4n-3})(1 - q^{4n})/(1 - q^{2n}).$$

References

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press (5th Edition), 1983.

2. M. D. Hirschhorn, *Some partition theorems of the Rogers-Ramanujan type*, J. Combinatorial Theory, Series A, Vol. 27, No. 1 (1979), pp. 33-37.

3. M. V. Subbarao, *Some Rogers-Ramanujan type partition theorems*, Pacific J. Math., Vol. 120 (1985), pp. 431-435.

4. L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc., Vol. 54 (1951-52), pp. 147-167.

Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1
Canada

Department of Mathematics
The Pennsylvania State University
(Mont Alto Campus)
Mont Alto, PA 17237
U.S.A.

https://doi.org/10.4153/CMB-1988-032-3 Published online by Cambridge University Press