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DIMENSION THEORY VIA BISECTOR CHAINS

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ABSTRACT. For two subsets Z and Y of a metric space (X, d) the set Z is said to be a bisector in Y iff $Z \subset Y$ and there exist two distinct points $y_1, y_2 \in Y$ such that $Z = \{z : d(z, y_1) = d(z, y_2)$ and $z \in Y\}$. Considering chains of consecutive bisectors $X \supset X_1 \supset \cdots \supset X_k$ we denote by $b(X, d)$ the maximum of their length. The topological invariant $b(X)$ is defined as the minimum of $b(X, d)$ taken over the set of all metrizations of X. It is proved that if X is compact then $\dim(X) \leq b(X) \leq 2\dim(X)+1$, $b(X)=0$ iff $\dim(X)=0$ and $b(X)=n$ implies $\dim(X)=n$ for $n=1$ and ∞. The sharp result $b(E^n)=n$ for $n=1,2,\ldots$ is obtained for Euclidean space E^n.

1. **Introduction.** The idea of characterizing dimension of a metrizable topological space X by exhibiting a metric on X with certain particular properties goes back to J. de Groot and J. Nagata [4]. The known fact that a space X is zero-dimensional if and only if one can introduce on X a non-archimedian metric has been generalized by J. de Groot [1] to an arbitrary dimension. In our paper [2] we have shown that for separable metrizable spaces still another, and completely different metric characterization of zero-dimensionality is available. Calling a metric space (X, d) rigid iff no two distinct pairs of points in X have the same distance we have shown that a separable metrizable space X is zero-dimensional if and only if there exists a rigid metric on X. It was again J. de Groot who conjectured that the idea of rigidity is capable of generalization to characterize arbitrary dimension, but he never achieved to prove it as he died shortly after. The purpose of this paper is to show that the elementary and heavily metric dependent notion of bisector provides one such possible generalization.

2. **Bisectors in metric spaces.**

DEFINITION 2.1. For an ordered pair (x_1, x_2) of distinct points in a metric space (X, d) we define the set $H(x_1, x_2)$, "the half space determined by (x_1, x_2)", as $\{x : d(x, x_1) < d(x, x_2)\}$ and the bisector $B(x_1, x_2)$ as $\{x : d(x, x_1) = d(x, x_2)\}$.

The following facts about $H(x_1, x_2)$ and $B(x_1, x_2)$ follow readily from the definition:

(1) $H(x_1, x_2)$ is open, $x_1 \in H(x_1, x_2)$ but $x_2 \notin H(x_1, x_2) \cup B(x_1, x_2)$.

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(2) $B(x_1, x_2)$ is closed, $B(x_1, x_2) = B(x_2, x_1)$ and the sets $H(x_1, x_2)$, $B(x_1, x_2)$ and $H(x_2, x_1)$ form a disjoint partition of X.

(3) The boundary $Bd[H(x_1, x_2)]$ of $H(x_1, x_2)$ is contained in $B(x_1, x_2)$, the complement Compl $H(x_1, x_2)$ of $H(x_1, x_2)$ equals $H(x_2, x_1) \cup B(x_1, x_2)$ and the closure $\mathrm{Cl}[H(x_1, x_2)]$ of $H(x_1, x_2)$ is contained in $H(x_1, x_2) \cup B(x_1, x_2)$.

LEMMA 2.1. *In a compact metric space (X, d) the family $\{H(x_1, x_2): x_1, x_2 \in X$ and $x_1 \neq x_2\}$ of all half-spaces in (X, d) forms a sub-basis for the topology of X.*

Proof. It is enough to show that given an open ball $B(x_0, r) = \{x : d(x, x_0) < r\}$ for $x_0 \in X$ and $r > 0$ we can produce a finite family of half-spaces $\{H(x_0, x_i) : i = 1, 2, \ldots, n\}$ whose intersection $\bigcap_{i=1}^{n} H(x_0, x_i)$ is contained in $B(x_0, r)$. By the compactness of X this follows from the obvious fact that $\{x_0\} = \bigcap \{\mathrm{Cl}[H(x_0, x)] : x \in X,\ x \neq x_0\}$.

REMARK 2.1. In the sequel we shall deal with boundaries of elements of the basis considered above. We see easily that the boundary of an intersection $\bigcap_{i=1}^{n} H(x_i, y_i)$ is a closed set contained in $\bigcup_{i=1}^{n} B(x_i, y_i)$.

DEFINITION 2.2. For two subsets Y and Z of a metric space (X, d) we write $Y \rhd Z$ and say that Z is a bisector in Y iff $Z \subset Y$ and there are distinct elements $y_1, y_2 \in Y$ such that $Z = B(y_i, y_2) \cap Y$. For a singleton $\{x\}$ with $x \in X$ and the empty subset θ of X we postulate the relation $\{x\} \rhd \theta$ to be true.

We shall consider chains of consecutive bisectors in (X, d); for $X \supset X_1 \supset X_2 \supset \cdots \supset X_n$ we shall write $X_1 \rhd X_2 \rhd \cdots \rhd X_n$ iff $X_i \rhd X_{i+1}$ for $i = 1, 2, \ldots, n-1$.

DEFINITION 2.3. For a nonempty metric space (X, d) we define the number $b(X, d)$, "the maximal length of the bisector chains", in the following way: If there is an infinite chain $X = X_0 \rhd X_1 \rhd X_i \rhd \cdots$ or if the set of numbers k in the chains $X = X_0 \rhd X_1 \rhd \cdots \rhd X_k \rhd \theta$ terminating by the empty set θ is unbounded we set $b(X, d) = \infty$. In the opposite case we set $b(X, d) = \max\{k$: there is a chain $X = X_0 \rhd X_1 \rhd \cdots X_k \rhd \theta\}$.

REMARK 2.2. For the Euclidean space (E^n, e) equipped with the usual metric e, the number $b(E^n, e)$ equals n for $n = 1, 2, \ldots$, as can easily be shown by elementary geometrical arguments. It is also seen why the convention $\{x\} \rhd \theta$ was adopted.

The following two properties of $b(X, d)$ are easy to establish:

(1) If $Y \subset X$ is a nonempty subset of (X, d) then $b(Y, d) \leq b(X, d)$ where by (Y, d) is of course understood the metric space induced on Y by the metric d.

(2) If $Y \subset X$ is a nonempty bisector in (X, d) then $b(Y, d) \leq b(X, d) - 1$, i.e. $b(Y, d) < b(X, d)$ if $b(X, d)$ is finite.

DEFINITION 2.4. If X is a metrizable topological space we denote by $M(X)$ the set of all metrics on X compatible with the topology of X. We define the topological invariant $b(X)$ as the minimum of $b(X, d)$ for $d \in M(X)$.

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EXAMPLE. From this definition and Remark 2.2. follows that $b(E^n) \le n$ for $n = 1, 2, \ldots$. Later we shall prove that in fact $b(E)^n = n$ for $n = 1, 2, \ldots$.

Denoting by $\dim(X)$ the dimension of the space X we are now in a position to state our main theorem.

THEOREM 2.1. *If X is a nonempty compact metrizable space then* $\dim(X) \le b(X) \le 2\dim(X) + 1$ *and for $n = 0, 1$, and ∞ we have the following implications:*

$$b(X) = 0 \leftrightarrow \dim(X) = 0$$

$$b(X) = 1 \leftrightarrow \dim(X) = 1$$

$$b(X) = \infty \leftrightarrow \dim(X) = \infty.$$

3. Proof of the theorem.

DEFINITION 3.1. A metric space (X, d) is said to be metrically rigid iff for every $a > 0$ the equation $d(x_1, x_2) = a$ has at most one solution $\{x_1, x_2\} \subset X$.

DEFINITION 3.2. We say that a metric space (X, d) is bisector-empty iff $b(X, d) = 0$, i.e., if the only bisector in X is the empty set θ.

REMARK 3.1. It is obvious that rigidity implies the property of being bisector-empty. The simple example of four points $\{(0, 0), (2, 0), (2, 1), (0, 1)\}$ in (E^2, e) shows that the converse is false.

We shall need some well known facts from dimension theory.

THEOREM 3.1. (J. Nagata). *A metrizable space X has the inductive dimension* $\mathrm{Ind}(X) \le n$ *if and only if there exists a σ-locally finite open basis L for X such that the dimension* $\mathrm{Ind}[\mathrm{Bd}(U)]$ *of the boundary* $\mathrm{Bd}(U)$ *of each element $U \in L$ is* $\le n - 1$.

For the proof see [5] *page* 18.

REMARK 3.2. Since on the class of separable metric spaces all the three dimension functions $\mathrm{ind}(X)$, $\mathrm{Ind}(X)$ and $\dim(X)$ coincide and since we do not consider other spaces in the sequel, we shall denote this function by $\dim(X)$.

COROLLARY 3.1. *If a separable metric space X has a basis L such that the boundary* $\mathrm{Bd}(V)$ *of each $V \in L$ has dimension $\le n - 1$ then* $\dim(X) \le n$.

Proof. Since X is separable, L contains a countable subfamily L^* which is also a basis for X. Thus L^* is a σ-locally finite basis and the conclusion follows.

From this corollary and Lemma 2.1., we obtain the necessary connection between the metric concept of bisector and the topological concept of dimension.

THEOREM 3.2. *If in a compact metric space* (X, d) *every bisector has dimension $\le n - 1$ then* $\dim(X) \le n$. $(n = 0, 1, \ldots)$.

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Proof. Taking as a basis L the family of all finite intersections of half-spaces as considered in Lemma 2.1., we observe that the boundary of each element of L is a subset of a finite union of bisectors (see Remark 2.1.) and hence has dimension $\le n-1$. Corollary 3.1. completes our proof.

THEOREM 3.3. *A nonempty separable metrizable space X is zero-dimensional if and only if there is $d \in M(X)$ such that (X, d) is metrically rigid.*

Proof. (See [2]).

COROLLARY 3.2. *For a nonempty compact metrizable space X we have:*

$$b(X)=0 \quad \text{iff} \quad \dim(X)=0$$

Proof. If $b(X)=0$ then there is $d \in M(X)$ with $b(X, d)=0$ implying that the metric space (X, d) is bisector-empty. Theorem 3.2. implies that $\dim(X)=0$. On the other hand if $\dim(X)=0$ Theorem 3.3. implies that there exists a rigid metric $d \in M(X)$ on X and this in turn implies that (X, d) is bisector-empty and a fortiori $b(X)=0$.

COROLLARY 3.3. *For a compact metrizable space X we have*

$$b(X)=1 \rightarrow \dim(X)=1.$$

Proof. Since $b(X)=1$ there is $d \in M(X)$ with $b(x, d)=1$ implying that for every nonempty bisector Y in (X, d) we have $b(Y, d)=0$, and a fortiori $b(Y)=0$. Theorem 3.2. implies that $\dim(X)\le 1$. But if $\dim(X)$ were zero then Corollary 3.3. would imply $b(X)=0$ contrary to our hypothesis. Thus we have $\dim(X)=1$.

COROLLARY 3.4. *For a compact metrizable space X we have*

$$b(X)\le n \rightarrow \dim(X)\le n \quad \textit{for} \quad n=0,1,\ldots.$$

Proof. The implication has been proven for $n=0$ and 1. We proceed by induction. Assume it is true for $n-1$. Since $b(X)\le n$ there is $d \in M(X)$ with $b(X, d)\le n$, implying that for every nonempty bisector Y in (X, d) we have $b(Y, d)\le n-1$, and a fortiori $b(Y)\le n-1$, which by the induction hypothesis implies $\dim(Y)\le n-1$. The conclusion that $\dim(X)\le n$ follows from Theorem 3.2., thus completing our proof.

From what we have proved so far follows that $b(X)$ majorizes $\dim(X)$ on the class of compact metric spaces. To obtain a result in the opposite direction we need:

LEMMA 3.1. *For a nonempty separable metrizable space X we have*

$$\dim(X)\le n \rightarrow b(X)\le 2n+1 \quad \textit{for} \quad n=0,1,\ldots.$$

Proof. This follows readily from the fact that X can be topologically embedded in E^{2n+1} and the fact that $b(E^{2n+1})\le 2n+1$.

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COROLLARY 3.5. *For a compact metrizable space X we have*

$$b(X)=\infty \to \dim(X)=\infty.$$

Proof. If $\dim(X)$ were finite, say k then Lemma 3.1. would imply that $b(X) \leq 2k+1$ contrary to the hypothesis.

The proof of Theorem 2.1. now follows as the conjunction of results obtained by Corollaries 3.1.–3.5.

Finally for the Euclidean cube I^n we obtain a sharp result:

THEOREM 3.4. $b(I^n)=n$ *for* $n=1,2,\ldots.$

Proof. Since $I^n \subset E^n$ and since $b(E^n) \leq n$ we have $b(I^n) \leq n$. But since I^n is compact and $\dim(I^n)=n$ our Theorem 2.1. implies that $b(I_n)=n$.

We are now in the position to extend this sharp result to Euclidean spaces E^n.

COROLLARY 3.6. *For the Euclidean space E^n we have $b(E^n)=n$ for $n=1,2,\ldots.$*

Proof. We already know that $b(E^n) \leq n$. From the fact that $I^n \subset E^n$ and Theorem 3.4. we conclude the opposite inequality which completes our proof.

As the reader might have noticed, the hypothesis of compactness in Theorem 2.1. is needed only because it is involved in Lemma 2.1. The validity of this theorem may therefore be extended to all separable metric spaces X with the property that for every $d \in M(X)$ the family of half-spaces in (X, d) form a subbasis for topology of X.

We conjecture that in fact $b(X)$ coincides with the dimension of X on the class of separable metric spaces.

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