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IDEMPOTENTS IN THE GROUPOID OF ALL SP CLASSES OF LATTICES

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1. **Introduction.** In [5], Mal'cev generalized the group theoretical results of H. Neumann (see [6] Chapter 2) to produce the notion of the product, $\mathcal{A} \cdot \mathcal{B}$, of two subclasses of a given variety of algebras, $\mathcal{K}$. Following the group theorectic example, members of $\mathcal{A} \cdot \mathcal{B}$ were called extensions of algebras in $\mathcal{A}$ by algebras in $\mathcal{B}$. When $\mathcal{K} = \mathcal{L}$, the variety of all lattices, this product has been investigated for example by Lender [4] and at the Oberwolfach meeting in 1976, Shevrin posed the following conjecture:

Are $\mathcal{L}$ and $\mathcal{T}$ the only varieties of lattices idempotent under this product? ($\mathcal{T}$ is the variety of all lattices satisfying $x = y$)

The purpose of this note is to answer this conjecture affirmatively.

2. **Preliminaries.** If $\mathcal{A}$ and $\mathcal{B}$ are abstract classes of lattices, their Mal'cev product is defined by: $C \in \mathcal{A} \cdot \mathcal{B}$ iff for some $\theta \in \mathrm{Con}(C)$, $C/\theta \in \mathcal{B}$ and for all $x \in C$, $[x]_\theta \in \mathcal{A}$. ($[x]_\theta$ is the congruence class of x modulo θ.) A prevariety of lattices is a subclass of $\mathcal{L}$ closed under **S** and **P**, and as shown in [5] the Mal'cev product of prevarieties is again such. We should also note that any non-trivial prevariety contains all distributive lattices.

We also need a construction in lattices defined originally in [1]. If A is a lattice and $I = [u, v]$ is a closed interval in A, then $A[I] = (A \backslash I) \cup (I \times \mathbf{2})$ is a lattice with the product order relation on $I \times \mathbf{2}$ and the original (and/or first projection order relation otherwise). There is a natural epimorphism $\kappa_I : A[I] \twoheadrightarrow \mathcal{A}$. We define $\mathbf{Int}\, \mathcal{A} = \{A[I] : A \in \mathcal{A} \text{ and } I = [u, v] \leq A\}$.

We also need some facts about free lattices. For $A \in \mathcal{L}$, $(a, b, c, d) \in A^4$ satisfies Whitman's condition iff $(W): a \wedge b \leq c \vee d$ implies $\{a, b, c, d\} \cap [a \wedge b, c \vee d] \neq \phi$. This condition comes from the well-known solution to the word problem for free lattices given in Whitman [7]. The form of this theorem needed here is in Jónsson [3].

(2.1) THEOREM. *Let L be a lattice generated by a subset $X \subseteq L$; then L is freely generated by X if and only if L satisfies (W) and for all finite subsets $Y, Z \subseteq X$, $\wedge Y \leq \vee Z$ iff $Y \cap Z \neq \phi$.*

The following result from [2] is also needed.

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(2.2) THEOREM. *For each lattice A in $\mathscr{L}$ there exists a sequence of lattices $(A_n)_{n\in N}$ and epimorphisms $\rho_n : A_{n+1} \twoheadrightarrow A_n$ such that*

(1) $$A_0 = A \quad \text{and} \quad A_{n+1} \in \mathbf{SP}\,\mathbf{Int}\{A_n\}$$

(2) $$A_\infty = \varprojlim (A_n, \rho_n) \text{ satisfies } (W).$$

3. **The results.** While some of the results stated below obviously hold under weaker assumptions we will assume all classes of lattices considered are prevarieties.

(3.1) LEMMA. *For prevarieties $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A}$ non-trivial,* $\mathbf{Int}\,\mathscr{B} \subseteq \mathscr{A}\cdot\mathscr{B}$.

Proof. The congruence classes of $\kappa_I : B[I] \twoheadrightarrow B$ are isomorphic to either **1** or **2** both of which belong to $\mathscr{A}$.

(3.2) COROLLARY. *If $\mathscr{A}$ is a non-trivial prevariety that is idempotent then* $\mathbf{Int}\,(\mathscr{A}) \subseteq \mathscr{A}$.

(3.3) THEOREM. *Any idempotent non-trivial prevariety $\mathscr{A}$ contains $FL(X)$, the free lattice on X generators, for each set X.*

Proofs. As $\mathscr{A}$ is non-trivial, we have that for any set X, $FD(X)$, the free distributive lattice on X generators, belongs to $\mathscr{A}$. Now using $A_0 = FD(X)$ in (2.2) we have by the lemma, $A_n \in \mathscr{A}$ for all $n \in N$ and therefore also $A_\infty \in \mathscr{A}$. Now if $\rho_\infty : A_\infty \twoheadrightarrow A_0 = FD(X)$ is the canonical epimorphism, then any set of representatives $\bar{X}$ from $\{\rho_\infty^{-1}(x) : x \in X\}$ must satisfy the second property of (2.1). Since we also have A_∞ satisfying (W), we have by (2.1), $FL(X) \simeq \langle \bar{X} \rangle \in \mathscr{A}$.

(3.4) COROLLARY. *If $\mathscr{V}$ is a variety of lattices that is idempotent then $\mathscr{V} = \mathscr{T}$ or $\mathscr{V} = \mathscr{L}$.*

Proof. If $\mathscr{V} \neq \mathscr{T}$ then since $\mathscr{V}$ is a prevariety, we have by the theorem $FL(X) \in \mathscr{V}$ for all sets X. Since $\mathscr{V}$ is also closed under $\mathbf{H}$, this forces $\mathscr{V} = \mathscr{L}$.

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