

# Tensor Product Realizations of Simple Torsion Free Modules

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*Abstract.* Let  $\mathcal{G}$  be a finite dimensional simple Lie algebra over the complex numbers  $\mathbb{C}$ . Fernando reduced the classification of infinite dimensional simple  $\mathcal{G}$ -modules with a finite dimensional weight space to determining the simple torsion free  $\mathcal{G}$ -modules for  $\mathcal{G}$  of type  $A$  or  $C$ . These modules were determined by Mathieu and using his work we provide a more elementary construction realizing each one as a submodule of an easily constructed tensor product module.

## 0 Introduction

Let  $\mathcal{G}$  be a finite dimensional simple Lie algebra over the complex numbers  $\mathbb{C}$ , and let  $\mathcal{H}$  be a Cartan subalgebra of  $\mathcal{G}$ . A  $\mathcal{G}$ -module  $M$  is said to be a *weight module* if and only if  $M = \bigoplus_{\lambda \in \mathcal{H}^*} M_\lambda$ , where each weight space  $M_\lambda = \{v \in M \mid hv = \lambda(h)v (\forall h \in \mathcal{H})\}$  is finite dimensional. A weight module is *torsion free* provided all elements from  $\mathcal{G} \setminus \mathcal{H}$  act injectively on  $M$  and it has *degree 1* provided all the weight spaces are 1-dimensional.

Fernando [F] reduced the classification of all simple weight modules to determining the simple torsion free modules and showed that the only simple Lie algebras admitting torsion free modules are those of type  $A$  or  $C$ .

In [BL2], every simple torsion free module  $T(\vec{a})$  of degree 1 is explicitly constructed. Example 1.4 below presents this construction in the  $A_n$  case. The authors conjectured, see [L] for example, that all simple torsion free modules of arbitrary finite degree can be realized as submodules of a tensor product  $T(\vec{a}) \otimes L(\lambda)$  where  $L(\lambda)$  is a simple finite dimensional module.

In a recent paper Mathieu [M] has classified and provided a realization of all simple torsion free weight modules. Nevertheless, the proof of the conjecture would provide a more elementary and explicit realization of the simple torsion free modules than the realization given by Mathieu.

In this paper techniques employed by Mathieu are adapted to establish the conjecture. The proof of the conjecture for torsion free  $C_n$ -modules follows directly from a complete reducibility theorem [BHL] and Mathieu's work and will be briefly described at the end of this paper. The focus here is on the case of torsion free  $A_n$ -modules.

In the first section, we set down the notation and state the basic definitions and results required for this paper. The key concept of a  $\Sigma$ -injective  $A_n$ -family of modules

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is introduced in Section 2 together with some basic properties of such families. Section 3 is devoted to providing the Jordan-Holder composition factors for the tensor product modules  $\mathcal{T}^{(a)} = L(a\omega_1) \otimes L(\lambda)$  where  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  and  $\lambda$  is a dominant integral weight. Finally in Section 4, we combine the results from Sections 2 and 3 to prove the conjecture for torsion free  $A_n$ -modules.

### 1 Preliminaries

Let  $g\ell(n + 1, \mathbb{C})$  denote the Lie algebra of all  $(n + 1) \times (n + 1)$  complex matrices with commutator product and  $\{E_{i,j} \mid 1 \leq i, j \leq n + 1\}$  be the standard set of matrix units. Fix a realization of  $A_n$  as the Lie subalgebra of  $g\ell(n + 1, \mathbb{C})$  generated by  $\{E_{i,i+1}, E_{i+1,i} \mid 1 \leq i \leq n\}$  and fix its Cartan subalgebra to be  $\mathcal{H} = \text{span}\{h_i = E_{ii} - E_{i+1,i+1} \mid i = 1, \dots, n\}$ . Let  $\epsilon_i$  denote the projection of any  $(n + 1) \times (n + 1)$  matrix onto its  $(i, i)$ -th entry then a basis of simple roots for the root system  $\Delta$  of  $A_n$  is given by  $\Delta^{++} = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \dots, n\}$  and the corresponding positive roots  $\Delta^+ = \{\epsilon_i - \epsilon_j = \alpha_i + \dots + \alpha_{j-1} \mid i < j\}$ . For  $i \leq j$ ,  $E_{i,j+1}$  is a positive root element belonging to the positive root  $\epsilon_i - \epsilon_{j+1} = \alpha_i + \dots + \alpha_j$  and is denoted by  $X_{\alpha_i + \dots + \alpha_j}$ . Correspondingly,  $E_{j+1,i}$  belongs to the negative root  $-(\alpha_i + \dots + \alpha_j)$  and is denoted  $2_{\alpha_i + \dots + \alpha_j}$ . The rank  $n$   $\mathbb{Z}$ -lattice generated by  $\Delta^{++}$  will be denoted by  $Q$ . The weights  $\omega_i = \epsilon_1 + \dots + \epsilon_i - \frac{i}{n+1}(\epsilon_1 + \dots + \epsilon_{n+1})$  for  $i = 1, \dots, n$  provide the dual basis for  $\Delta^{++}$  with respect to the inner product determined by setting  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ . Finally, let  $\{h_i^\vee \mid i = 1, \dots, n\}$  be the basis of  $\mathcal{H}$  dual to  $\{\alpha_i\}$ —i.e.,  $\alpha_i(h_j^\vee) = \delta_{ij}$ .

Let  $\tilde{A}_n$  denote the Lie subalgebra of  $A_n$  generated by

$$(1.1) \quad \{E_{i,i+1}, E_{i+1,i} \mid 2 \leq i \leq n\}.$$

Clearly,  $\tilde{A}_n \simeq A_{n-1}$ .

Let  $U(A_n)$  denote the universal enveloping algebra of  $A_n$ ,  $U_+(A_n)$  the subalgebra of  $U(A_n)$  generated by the elements  $X_{\alpha_i}$ ,  $U_-(A_n)$  the subalgebra of  $U(A_n)$  generated by the elements  $Y_{\alpha_i}$ ,  $U(\mathcal{H})$  the universal enveloping algebra of the Cartan subalgebra  $\mathcal{H}$  and  $U_0$  the centralizer of  $\mathcal{H}$  in  $U(A_n)$ .

Let  $M$  be an  $A_n$  weight module. The set of all weights  $\lambda \in \mathcal{H}^*$  with  $M_\lambda \neq \{0\}$  is called the *support* of  $M$  and is denoted  $\text{Supp}(M)$ . An infinite dimensional weight module  $M$  is said to be *admissible* provided  $\text{Supp}(M)$  is contained in finitely many  $Q$  cosets and there exists a  $B \in \mathbb{Z}$  such that  $\dim M_\lambda \leq B$  for all  $\lambda \in \text{Supp}(M)$ . The *degree* of an admissible module is the maximum dimension of its weight spaces.

For each  $\xi \in \mathcal{H}^*$ , we denote by  $L(\xi)$  the simple  $A_n$ -module having highest weight  $\xi$  (with respect to  $\Delta^{++}$ ). Clearly  $L(\xi)$  is a weight module and its support is contained in  $\{\xi - \sum_{i=1}^n n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$ .

Following Mathieu define a *coherent  $A_n$ -family of degree  $d$*  to be a weight  $A_n$ -module  $\mathcal{M}$  such that

- (i)  $\dim \mathcal{M}_\lambda = d$  for all  $\lambda \in \mathcal{H}^*$  and
- (ii) for each  $u \in U_0$  the map  $\lambda \mapsto \text{trace}(u \downarrow \mathcal{M}_\lambda)$  is polynomial in  $\lambda$ .

A coherent  $A_n$ -family  $\mathcal{M}$  is said to be *irreducible* if and only if there exists a weight  $\lambda \in \mathcal{H}^*$  such that  $\mathcal{M}_\lambda$  is a simple  $U_0$ -module.

**Theorem 1.2**

- (i) [M, Lemma 3.3] Any admissible weight module  $M$  has a composition series of finite length.
- (ii) [M, Proposition 4.8] For every simple infinite dimensional admissible  $A_n$ -module  $M$  of degree  $d$  there exists a unique irreducible semisimple coherent  $A_n$ -family  $\mathcal{M}^{ss}$  of degree  $d$  which contains  $M$  as a submodule.
- (iii) [M, Proposition 5.4 and Theorem 10.2] For every irreducible coherent  $A_n$ -family  $\mathcal{M}$  of degree  $d$ , there is a  $\tau \in \mathcal{H}^*$  such that the submodule  $\mathcal{M}_{[\tau]} = \sum_{\nu \in [\tau]} \mathcal{M}_\nu$  is simple torsion free of degree  $d$ .

Let  $M$  be an infinite dimensional admissible simple module of degree  $d$ . Mathieu gives a general construction of a coherent family  $\mathcal{M}(M)$  which contains  $M$ . This is described briefly in Section 2. Theorem 1.2(i) allows us to form the “semisimplification”,  $\mathcal{M}^{ss}(M)$ , of  $\mathcal{M}(M)$ . These irreducible semisimple coherent families are labelled using  $\Lambda = \{ \sum_{i=1}^n a_i \omega_i \in \mathcal{H}^* \mid a_i \in \mathbb{Z}_{\geq 0} \text{ for } i = 2, \dots, n; a_1 \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \}$ .

**Theorem 1.3 [M, Prop. 8.5]** For each admissible irreducible semisimple coherent  $A_n$ -family  $\mathcal{M}^{ss}$  there exists a unique weight  $\text{wt}(\mathcal{M}) \in \Lambda$  such that the simple highest weight module  $L(\text{wt}(\mathcal{M}))$  is isomorphic to a submodule of  $\mathcal{M}^{ss}$ . This correspondence is bijective.

Of particular interest is the following construction of a coherent  $A_n$ -family of degree 1.

**Example 1.4** Fix  $a \in \mathbb{C}$ . Define

$$\mathcal{S}(a) = \text{span}_{\mathbb{C}} \left\{ x^{\vec{b}} = x_1^{b_1} \cdots x_{n+1}^{b_{n+1}} \mid b_1, \dots, b_{n+1} \in \mathbb{C} \text{ with } \sum_{i=1}^{n+1} b_i = a \right\}.$$

Then an  $A_n$ -module structure can be defined on  $\mathcal{S}(a)$  by embedding  $A_n$  into the Weyl algebra  $W_{n+1} = \langle x_i, \partial_i \mid i = 1, \dots, n + 1 \rangle$  with  $E_{ij} \mapsto x_i \partial_j$ . Here,  $W_{n+1}$  is the Lie algebra generated by  $x_i$ , and  $\partial_i$  where the action of  $x_i$  on  $\mathcal{S}(a)$  is multiplication by  $x_i$  and the action of  $\partial_i$  on  $\mathcal{S}(a)$  is partial differentiation with respect to  $x_i$ . It is easily verified that  $\mathcal{S}(a)$  is a coherent  $A_n$ -family of degree 1. Now fix  $a_1, \dots, a_{n+1} \in \mathbb{C} \setminus \mathbb{Z}$  with  $\sum_{i=1}^{n+1} a_i = a$ . Define  $\tau = \sum_{i=1}^n (a_i - a_{i+1}) \omega_i$  and let  $T(\vec{a}) \leq \mathcal{S}(a)$  be given by

$$T(\vec{a}) = \text{span}_{\mathbb{C}} \left\{ x^{\vec{b}} = x_1^{b_1} \cdots x_{n+1}^{b_{n+1}} \mid \sum_{i=1}^{n+1} b_i = a, a_i - b_i \in \mathbb{Z} \right\} = \mathcal{S}(a)_{[\tau]}.$$

Then  $T(\vec{a})$  is a simple torsion free  $A_n$ -module of degree 1, and as shown in [BL2] every such module may be realized in this manner. If  $a \notin \mathbb{Z}_{\geq 0}$ , then  $L(a\omega_1) < \mathcal{S}(a)$  is an admissible module with maximal vector  $x_1^a$  having weight  $a\omega_1 \in \Lambda$ . In fact

$$L(a\omega_1) \simeq \text{span}_{\mathbb{C}} \{ x_1^{a-\ell_1} x_2^{\ell_1-\ell_2} \cdots x_{n+1}^{\ell_n} \mid \ell_i \in \mathbb{Z}; \ell_1 \geq \ell_2 \geq \cdots \geq \ell_n \geq 0 \} < \mathcal{S}(a).$$

When  $a = N \in \mathbb{Z}_{\geq 0}$ ,  $L(N\omega_1)$  is a simple finite dimensional (hence not admissible) submodule of  $\mathcal{S}(N)$ . In this case

$$L(N\omega_1) \simeq \text{span}_{\mathbb{C}}\{x_1^{N-\ell_1} x_2^{\ell_1-\ell_2} \cdots x_{n+1}^{\ell_n} \mid \ell_i \in \mathbb{Z}; 0 \leq \ell_n \leq \cdots \leq \ell_1 \leq N\} < \mathcal{S}(N)$$

has a maximal vector  $x_1^N$  and the unique admissible submodule of  $\mathcal{S}^{ss}(N)$  with highest weight in  $\Lambda$  given by Theorem 1.3 is

$$L((−N − 2)\omega_1 + (N + 1)\omega_2) \simeq \langle x_1^{-1} x_2^{N+1} \rangle / L(N\omega_1).$$

The central character of  $\mathcal{S}(a)$  is  $\mathcal{X}_{a\omega_1}$  in either case.

If  $\mathcal{M}^{ss}(\xi) \longleftrightarrow \xi \in \Lambda$  is the bijection of Theorem 1.3 then the degree of  $\mathcal{M}^{ss}(\xi)$  is equal to the degree of  $L(\xi)$ . Mathieu partitions  $\Lambda$  into weights of three types:

- (i) if  $\xi(h_1) \notin \mathbb{Z}_{<0}$  then  $\xi$  is said to be *nonintegral*;
- (ii) if there exists an index  $i$  such that  $\xi(h_1 + \cdots + h_i) + i = 0$  then  $\xi$  is said to be *singular* and
- (iii) if  $\xi(h_1) \in \mathbb{Z}_{<0}$  and is not singular it is said to be *regular integral*.

The regular integral elements of  $\Lambda$  can be associated with dominant integral weights as follows. If  $\mu$  is a dominant integral weight then set  $\mu[0] = \mu$  and for  $k = 1, \dots, n$  define  $\mu[k] = \sigma_{\alpha_1 + \cdots + \alpha_k} \circ \cdots \circ \sigma_{\alpha_1} \cdot \lambda$  where  $\sigma_\gamma$  denotes the reflection of  $\mathcal{H}^*$  in the hyperplane perpendicular to  $\gamma$  for any  $\gamma \in \Delta^+$  and  $\cdot$  denotes the affine action of the Weyl group. For  $k = 1, \dots, n$  the weights  $\mu[k] \in \Lambda$ ,  $\mu[k]$  is linked to  $\mu$ ,  $\mu[k - 1] - \mu[k]$  is a positive integral multiple of  $\alpha_1 + \cdots + \alpha_k$  and each regular integral weight  $\xi$  is of the form  $\mu[k]$  for some dominant integral weight  $\mu$ .

For any weight  $\nu = \sum_{i=1}^n \nu_i \omega_i$  let  $\tilde{\nu}$  denote its restriction to  $\mathcal{H} \cap \tilde{A}_n$ , i.e.,  $\tilde{\nu} = \sum_{i=2}^n \nu_i \omega_i$ .

**Theorem 1.5 [M, Theorem 11.4]**

- (i) If  $\xi = \sum_{i=1}^n a_i \omega_i \in \Lambda$  is either nonintegral or singular then the degree of  $\mathcal{M}(\xi)$  is equal to the dimension of the simple  $\tilde{A}_n$ -module with highest weight  $\tilde{\xi}$ .
- (ii) For any dominant integral weight  $\mu$  the degree of  $\mathcal{M}^{ss}(\mu[n])$  is equal to the dimension of the simple  $\tilde{A}_n$ -module with highest weight  $\tilde{\mu}[n]$  and if  $1 \leq k < n$  the degree of  $\mathcal{M}^{ss}(\mu[k])$  plus the degree of  $\mathcal{M}^{ss}(\mu[k+1])$  is equal to the dimension of the simple  $\tilde{A}_n$ -module with highest weight  $\tilde{\mu}[k]$ .

**Proposition 1.6** Let  $\xi \in \Lambda$ . The coherent  $A_n$ -family  $\mathcal{M}^{ss}(\xi)$  has degree 1 if and only if  $\xi = a\omega_1$  for some  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  or  $\xi = -(N + 2)\omega_1 + (N + 1)\omega_2$  for some  $N \in \mathbb{Z}_{\geq 0}$ .

**Proof** If  $\xi$  has either of the two forms given then construct  $\mathcal{S}(a)$  as in Example 1.4. Use Theorem 1.2(ii) to obtain the semisimplification of  $\mathcal{S}(a)$ . Theorem 1.4 tells us that  $\mathcal{S}(a)^{ss} \simeq \mathcal{M}^{ss}(\xi)$  and hence the degree of  $\mathcal{M}^{ss}(\xi)$  is 1.

Conversely, suppose that the coherent  $A_n$ -family  $\mathcal{M}^{ss}(\xi)$  has degree 1. Applying Theorem 1.2(iii), let  $[\tau]$  be a  $Q$  coset such that  $\mathcal{M}^{ss}(\xi)_{[\tau]}$  is a torsion free submodule. Certainly this submodule is simple of degree 1 and so it is isomorphic to some  $T(\bar{a})$ . Set  $a = \sum_{i=1}^{n+1} a_i$ . Form  $\mathcal{S}(a)$  containing  $T(\bar{a})$  as in Example 1.4. Then by Theorem 1.2(ii),  $\mathcal{M}^{ss}(\xi) \simeq \mathcal{S}^{ss}(a)$  since both of these modules contain  $T(\bar{a})$ .  $\mathcal{S}^{ss}(a)$  contains  $L(a\omega_1)$  if  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  or  $L(-(N + 2)\omega_1 + (N + 1)\omega_2)$  if  $a = N \in \mathbb{Z}_{\geq 0}$ . By the uniqueness of Theorem 1.3,  $\xi = a\omega_1$  or  $\xi = -(N + 2)\omega_1 + (N + 1)\omega_2$ . ■

## 2 $\Sigma$ -Injective Coherent $A_n$ -Families

In this section, the focus is on a particular type of admissible coherent  $A_n$ -family. Its description starts by fixing a basis  $\Sigma = \{\alpha_1 + \dots + \alpha_k \mid k = 1, \dots, n\}$  of the root lattice  $Q$  which is a commuting basis in the sense that  $[Y_{\mu_1}, Y_{\mu_2}] = 0$  for all  $\mu_1, \mu_2 \in \Sigma$ .

**Definition 2.1** A  $\Sigma$ -injective coherent  $A_n$ -family of degree  $d$  is an  $A_n$  weight module  $\mathcal{M}$  such that

- (i)  $\dim \mathcal{M}_\zeta = d$  for each  $\zeta \in \mathcal{H}^*$ ,
- (ii)  $Y_\mu$  acts injectively on  $\mathcal{M}$  for all  $\mu \in \Sigma$ ,
- (iii) there exists a linear basis  $\mathcal{B} = \bigcup_{\zeta \in \mathcal{H}^*} \mathcal{B}_\zeta$  of  $\mathcal{M}$  where each  $\mathcal{B}_\zeta$  is a basis of the corresponding weight space  $\mathcal{M}_\zeta$  and  $\mathcal{B}_{\zeta-\mu} = Y_\mu \mathcal{B}_\zeta$  for each root  $\mu \in \Sigma$ , and
- (iv) for each element  $u \in U_0$ , there is a  $d \times d$  matrix of polynomials  $P^{(u)}(z_1, \dots, z_n) = [p_{ij}^{(u)}(z_1, \dots, z_n)]$  such that the matrix representation of the action of  $u$  on  $\mathcal{M}_\zeta$  with respect to  $\mathcal{B}$  is

$$[u \downarrow \mathcal{M}_\zeta]_{\mathcal{B}_\zeta} = [p_{ij}^{(u)}(c_1, \dots, c_n)] = P^{(u)}(c_1, \dots, c_n) \text{ when } \zeta = \sum_{i=1}^n c_i(\alpha_1 + \dots + \alpha_i).$$

To simplify notation we write  $p_{ij}^{(u)}(\zeta)$  for  $p_{ij}^{(u)}(c_1, \dots, c_n)$  when  $\zeta = \sum_{i=1}^n c_i(\alpha_1 + \dots + \alpha_i)$ . The  $p_{ij}^{(u)}(z_1, \dots, z_n)$ 's are called the *structure polynomials* of  $\mathcal{M}$ .

**General Assumption** Unless otherwise stated  $M$  is assumed to be an admissible  $A_n$ -module of degree  $d$  where  $\text{Supp}(M) \subset \kappa + Q$  for some  $\kappa \in \mathcal{H}^*$  and such that  $Y_\mu$  acts injectively on  $M$  for each  $\mu \in \Sigma$ . Fix  $\kappa$  so that  $M_\kappa$  has basis  $\mathcal{B}_\kappa = \{v_1, \dots, v_d\}$ .

The aim now is to show that there exists a unique  $\Sigma$ -injective  $A_n$ -family  $\mathcal{M}(M)$  of degree  $d$  which contains  $M$  as a submodule. For the existence of such a module we rely on the work of Mathieu.

Let  $Y_\Sigma$  denote the multiplicative subset of  $U$  generated by  $\{Y_\mu \mid \mu \in \Sigma\}$ . By [M, Lemma 4.2], the set  $Y_\Sigma$  satisfies the Ore conditions. Let  $U_\Sigma$  denote the localization of  $U$  with respect to  $Y_\Sigma$ . Set  $M'$  to be the  $U_\Sigma$ -module  $U_\Sigma \otimes_U M$ . Then  $M'$  is a weight module with a  $U$ -submodule isomorphic to  $M$  and  $\dim M'_\nu = d$  for all  $\nu \in \text{Supp}(M') = \kappa + Q$ .

For any  $n$ -tuple of integers  $(k_1, \dots, k_n)$  we define  $\Theta_{(k_1, \dots, k_n)}: U_\Sigma \rightarrow U_\Sigma$  to be the automorphism given by

$$\Theta_{(k_1, \dots, k_n)}(w) = Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n} w Y_{\alpha_1 + \dots + \alpha_n}^{-k_n} \dots Y_{\alpha_1}^{-k_1}.$$

Assume  $\kappa = \sum_{i=1}^n c_i(\alpha_1 + \dots + \alpha_i)$  is a weight of  $M$  with  $\mathcal{B}_\kappa = \{v_1, \dots, v_d\}$  a basis of the weight space  $M_\kappa$ . Fix an element  $u \in U_\Sigma$ . Since  $\Sigma$  is a basis of  $Q$  and the elements of  $Y_\Sigma$  act injectively on  $M'$ , for any weight  $\zeta \in \kappa + Q$  there exists a unique element  $Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n}$  in  $Y_\Sigma$  with  $k_i \in \mathbb{Z}$  such that  $\mathcal{B}_\zeta = \{Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n} v_i \mid i = 1, \dots, d\}$  is a basis for the weight space  $M'_\zeta$ .

Since the elements  $Y_\gamma$  for  $\gamma \in \Sigma$  are locally ad nilpotent on  $U_\Sigma$ , there exists an integer  $N$  such that  $\text{ad}(Y_\gamma)^{N+1}(u) = 0$  for all  $\gamma \in \Sigma$ . By [M, Lemma 4.3], we have that

$$Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} u Y_{\alpha_1+\cdots+\alpha_n}^{-k_n} \cdots Y_{\alpha_1}^{-k_1} \\ = \sum_{0 \leq i_1, \dots, i_n \leq N} \binom{k_1}{i_1} \cdots \binom{k_n}{i_n} (\text{ad } Y_{\alpha_1})^{i_1} \cdots (\text{ad } Y_{\alpha_1+\cdots+\alpha_n})^{i_n} (u) Y_{\alpha_1+\cdots+\alpha_n}^{-i_n} \cdots Y_{\alpha_1}^{-i_1},$$

and so  $\Theta_{(k_1, \dots, k_n)}$  given by

$$u \mapsto \sum_{i_1=0}^\infty \cdots \sum_{i_n=0}^\infty \binom{k_1}{i_1} \cdots \binom{k_n}{i_n} (\text{ad } Y_{\alpha_1})^{i_1} \cdots (\text{ad } Y_{\alpha_1+\cdots+\alpha_n})^{i_n} (u) Y_{\alpha_1+\cdots+\alpha_n}^{-i_n} \cdots Y_{\alpha_1}^{-i_1}$$

where

$$\binom{k_m}{i_m} = \frac{k_m(k_m - 1) \cdots (k_m - i_m + 1)}{i_m!}$$

is an automorphism for all integer values of  $k_1, \dots, k_n$ . Thus, for any  $c_1, \dots, c_n \in \mathbb{C}$ , the map  $\Theta_{(c_1, \dots, c_n)}: U_\Sigma \rightarrow U_\Sigma$  given by

$$u \mapsto \sum_{i_1=0}^\infty \cdots \sum_{i_n=0}^\infty \binom{c_1}{i_1} \cdots \binom{c_n}{i_n} (\text{ad } Y_{\alpha_1})^{i_1} \cdots (\text{ad } Y_{\alpha_1+\cdots+\alpha_n})^{i_n} (u) Y_{\alpha_1+\cdots+\alpha_n}^{-i_n} \cdots Y_{\alpha_1}^{-i_1}$$

is an automorphism.

The  $U_0$ -module structure of  $M'$  can be described through the use of the automorphisms  $\Theta_{(k_1, \dots, k_n)}$  with the  $k_i$ 's  $\in \mathbb{Z}$ . In fact, for each  $u \in U_0$ , each  $n$ -tuple  $(i_1, \dots, i_n)$  with  $0 \leq i_1, \dots, i_n \leq N$ ,  $N$  as above, and each  $j$ , select  $a_{j,k}^{i_1, \dots, i_n} \in \mathbb{C}$  such that

$$(\text{ad } Y_{\alpha_1})^{i_1} \cdots (\text{ad } Y_{\alpha_1+\cdots+\alpha_n})^{i_n} (u) Y_{\alpha_1+\cdots+\alpha_n}^{-i_n} \cdots Y_{\alpha_1}^{-i_1} v_j = \sum_{k=1}^d a_{j,k}^{i_1, \dots, i_n} v_k$$

where  $\mathcal{B}_\kappa = \{v_1, \dots, v_d\}$  is a basis of the weight space  $M'_\kappa$ . Let  $z_1, \dots, z_n$  denote  $n$  commuting variables and define polynomials

$$p_{jk}^{(u)}(z_1, \dots, z_n) = \sum_{0 \leq i_1, \dots, i_n \leq N} \binom{z_1 - c_1}{i_1} \cdots \binom{z_n - c_n}{i_n} a_{j,k}^{i_1, \dots, i_n}.$$

It follows that the action of  $u$  on  $\mathcal{B}_\zeta = \{Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} v_i \mid i = 1, \dots, d\}$  a basis  $M'_\zeta$  is given by

$$u Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} v_j = Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} \Theta_{(-k_1, \dots, -k_n)}(u) v_j \\ = Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} \sum_{k=1}^n p_{jk}^{(u)}(c_1 - k_1, \dots, c_n - k_n) v_k \\ = \sum_{k=1}^n p_{jk}^{(u)}(c_1 - k_1, \dots, c_n - k_n) Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1+\cdots+\alpha_n}^{k_n} v_k.$$

From this calculation one sees that the action of each  $u \in U_0$  on the weight spaces  $M'_\zeta$  with  $\zeta$  in the coset  $[\kappa] = \kappa + Q$  are “polynomially related” and the action of  $u$  on  $M'_\zeta$  is related to the action of  $\Theta_{(-k_1, \dots, -k_n)}(u)$  on  $M'_\kappa$  where  $\kappa - \zeta = \sum_{i=1}^n k_i(\alpha_1 + \dots + \alpha_i)$ . In particular, the action of  $U_0$  on  $M'_\zeta$  is the action of  $U_0$  on  $M'_\kappa$  twisted by  $\Theta_{(-k_1, \dots, -k_n)}$ .

To construct  $\mathcal{M}(M)$ : start with a copy,  $M'_{[\tau]}$ , of  $M'$  for each coset  $[\tau]$  of  $Q$  in  $\mathcal{H}^*$ ; form the direct sum indexed by these cosets

$$\mathcal{M}(M) = \bigoplus_{[\tau] \in \mathcal{H}^*/Q} M'_{[\tau]} \quad \text{with} \quad \kappa - \tau = \sum_{i=1}^n b_i(\alpha_1 + \dots + \alpha_i);$$

and define the action of each element  $u \in (U_\Sigma)_0$  on  $M'_{[\tau]}$  by twisting the action of  $u$  on  $M'$  through the use of the automorphism  $\Theta_{(-b_1, \dots, -b_n)}$ . Since by [M, Lemma 4.3]

$$\Theta_{(-k_1, \dots, -k_n)} \circ \Theta_{(-b_1, \dots, -b_n)} = \Theta_{(-b_1 - k_1, \dots, -b_n - k_n)}$$

for  $Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n} v_j \in M'_{[\tau]}$  we have

$$u Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n} v_j = \sum_{\ell=1}^n p_{j\ell}^{(u)}(c_1 - b_1 - k_1, \dots, c_n - b_n - k_n) Y_{\alpha_1}^{k_1} \dots Y_{\alpha_1 + \dots + \alpha_n}^{k_n} v_\ell.$$

Hence the action of  $u \in U_0$  on  $M'_{[\tau]}$  is determined by the structure polynomials.

As shown by Mathieu, the definition of  $M'_{[\tau]}$  is independent of the coset representative. Thus, the coherent family  $\mathcal{M}(M)$  of degree  $d$  containing  $M$ , as constructed by Mathieu, is in fact a  $\Sigma$ -injective family with structure polynomials  $p_{ij}^{(u)}(z_1, \dots, z_n)$ .

**Theorem 2.2**

- (i) *The structure polynomials of a  $\Sigma$ -injective coherent  $A_n$ -family  $\mathcal{M}$  of degree  $d$  containing  $M$  are uniquely determined by  $M$ , a fixed weight  $\kappa$  of  $M$  with  $\dim M_\kappa = d$  and a basis  $\mathcal{B}_\kappa = \{v_1, \dots, v_d\}$ .*
- (ii) *A  $\Sigma$ -injective coherent  $A_n$ -family  $\mathcal{M}$  of degree  $d$  is a  $U_\Sigma$ -module and is uniquely determined by its structure polynomials.*
- (iii) *There is a unique, up to isomorphism,  $\Sigma$ -injective coherent  $A_n$ -family  $\mathcal{M}$  containing  $M$  as a submodule.*

**Proof** (i) Let  $\mathcal{M}$  be any  $\Sigma$ -injective coherent  $A_n$ -family of degree  $d$  containing  $M$ . Then the values of the structure polynomials  $p_{ij}^{(u)}(z_1, \dots, z_n)$  are determined by  $M$  for  $(z_1, \dots, z_n) = (c_1 - k_1, \dots, c_n - k_n)$  with  $k_i \in \mathbb{Z}_{\geq 0}$ . By Zariski density, these polynomials are uniquely determined.

(ii) The fact that  $\mathcal{M}$  can be viewed as a  $U_\Sigma$ -module follows immediately from the injectivity of the elements  $Y_\mu$  for all  $\mu \in \Sigma$  and the assumption that all weight spaces have dimension  $d$ .

Since for any root  $\nu \in \Delta^+ \setminus \Sigma$  both  $Y_\nu$  and  $X_\nu$  can be expressed as commutators of appropriate elements  $Y_{\mu_1}$  and  $X_{\mu_2}$  with  $\mu_1, \mu_2 \in \Sigma$ , to complete the proof it suffices to show that for each  $\mu = \alpha_1 + \dots + \alpha_p \in \Sigma$  the actions of  $Y_\mu$  and  $X_\mu$  are determined by the structure polynomials. By our assumption on the bases of the weight

spaces the elements  $Y_\mu^{\pm 1}$  simply translate the basis elements. Let  $u = Y_\mu X_\mu \in U_0$  and  $\mathcal{B}_\kappa = \{v_j \mid j = 1, \dots, d\}$  be a fixed basis of  $M_\kappa$ . Then for any basis vector  $Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1 + \cdots + \alpha_n}^{k_n} v_j \in M'_{[\tau]}$  with  $\kappa - \tau = \sum_{i=1}^n b_i(\alpha_1 + \cdots + \alpha_i)$

$$\begin{aligned} & X_\mu Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1 + \cdots + \alpha_n}^{k_n} v_j \\ &= Y_\mu^{-1} Y_\mu X_\mu Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1 + \cdots + \alpha_n}^{k_n} v_j \\ &= Y_\mu^{-1} \sum_{\ell=1}^n p_{j\ell}^{(u)}(c_1 - b_1 - k_1, \dots, c_n - b_n - k_n) Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1 + \cdots + \alpha_n}^{k_n} v_\ell \\ &= \sum_{\ell=1}^n p_{j\ell}^{(u)}(c_1 - b_1 - k_1, \dots, c_n - b_n - k_n) Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_1 + \cdots + \alpha_p}^{k_p - 1} \cdots Y_{\alpha_1 + \cdots + \alpha_n}^{k_n} v_\ell \end{aligned}$$

where the  $p_{ij}^{(u)}$ 's are the structure polynomials belonging to  $u$ .

Part (iii) follows immediately from parts (i) and (ii). ■

At this point, one should note that there are two different unique coherent  $A_n$ -families associated with a simple module  $M$  satisfying the General Assumption: the irreducible semisimple coherent  $A_n$ -family  $\mathcal{M}^{ss}(M)$  and the  $\Sigma$ -injective coherent  $A_n$ -family  $\mathcal{M}(M)$ .

**Theorem 2.3** *Let  $\xi \in \Lambda$ . If  $\xi = a\omega_1$  for some  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  or  $\xi = -(N + 2)\omega_1 + (N + 1)\omega_2$  for some  $N \in \mathbb{Z}_{\geq 0}$  and  $\lambda$  is a dominant integral weight then*

$$\mathcal{M}(L(\xi) \otimes L(\lambda)) \simeq \mathcal{M}(L(\xi)) \otimes L(\lambda).$$

**Proof** Fix  $\xi$  as in the Theorem. Independent of the the choice of  $\xi$  both  $L(\xi)$  and  $L(\xi) \otimes L(\lambda)$  are  $\Sigma$ -injective  $A_n$ -modules and hence one can construct the  $\Sigma$ -injective coherent  $A_n$ -families  $\mathcal{M}(L(\xi)) \otimes L(\lambda)$  and  $\mathcal{M}(L(\xi) \otimes L(\lambda))$ . Since each of these have degree  $d = \dim L(\lambda)$  and contain a submodule isomorphic to  $L(\xi) \otimes L(\lambda)$ , it follows from Theorem 2.2 that  $\mathcal{M}(L(\xi) \otimes L(\lambda)) \simeq \mathcal{M}(L(\xi)) \otimes L(\lambda)$ . ■

**Theorem 2.4** *Let  $\xi = a\omega_1$  for some  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  or  $\xi = -(N + 2)\omega_1 + (N + 1)\omega_2$  for some  $N \in \mathbb{Z}_{\geq 0}$  and  $\lambda$  be a dominant integral weight. Then  $\mathcal{M}(L(\xi) \otimes L(\lambda))_{[\tau]}$  is torsion free if and only if  $\mathcal{M}(L(\xi))_{[\tau - \lambda]}$  is torsion free.*

**Proof** If  $\mathcal{M}(L(\xi))_{[\tau - \lambda]}$  is torsion free, then clearly  $\mathcal{M}(L(\xi))_{[\tau - \lambda]} \otimes L(\lambda) \simeq \mathcal{M}(L(\xi) \otimes L(\lambda))_{[\tau]}$  is torsion free.

Conversely, if  $\mathcal{M}(L(\xi) \otimes L(\lambda))_{[\tau]}$  is torsion free then it contains a simple torsion free submodule  $T$ . Therefore,

$$T \leq \mathcal{M}(L(\xi) \otimes L(\lambda))_{[\tau]} \simeq \mathcal{M}(L(\xi))_{[\tau - \lambda]} \otimes L(\lambda).$$

Since the degree of  $\mathcal{M}(L(\xi))$  is 1, the degree of  $\mathcal{M}(L(\xi))_{[\tau - \lambda]}$  is 1. Assume  $\mathcal{M}(L(\xi))_{[\tau - \lambda]}$  is not torsion free. Suppose further that  $\mathcal{M}(L(\xi))_{[\tau - \lambda]}$  is simple. By



[F], there is a root vector which is locally nilpotent on  $\mathcal{M}(L(\xi))_{[\tau-\lambda]}$  and hence on  $\mathcal{M}(L(\xi))_{[\tau-\lambda]} \otimes L(\lambda)$  which precludes it from containing a torsion free submodule. Thus, if  $\mathcal{M}(L(\xi))_{[\tau-\lambda]}$  is not torsion free, then  $\mathcal{M}(L(\xi))_{[\tau-\lambda]}$  is not simple. Since  $N = \mathcal{M}(L(\xi))_{[\tau-\lambda]}$  is admissible it has finite length by Theorem 1.2(i). Consider the composition series

$$0 = N_{s+1} \subset N_s \subset \dots \subset N_1 = N.$$

Certainly, if  $T \cap (N_j \otimes L(\lambda)) \neq \{0\}$  then by simplicity  $T \subseteq N_j \otimes L(\lambda)$ . Let  $j$  be the largest index with  $T \subseteq N_j \otimes L(\lambda)$ . Then  $T \oplus (N_{j+1} \otimes L(\lambda)) \subseteq N_j \otimes L(\lambda)$ . If there is a root vector which does not act injectively on  $N_j/N_{j+1}$ , then again by [F], it is locally nilpotent on  $N_j/N_{j+1}$  and hence on  $(N_j/N_{j+1}) \otimes L(\lambda) \simeq (N_j \otimes L(\lambda)) / (N_{j+1} \otimes L(\lambda))$ . However,  $T \cap N_{j+1} \otimes L(\lambda) = \{0\}$  and so  $(N_j \otimes L(\lambda)) / (N_{j+1} \otimes L(\lambda))$  contains an isomorphic copy of  $T$ , contrary to  $T$  being torsion free. On the other hand if all root vectors act injectively on  $N_j/N_{j+1}$  they also act injectively on  $N_j$  and hence each element of  $[\tau - \lambda]$  is a weight of  $N_j$ . This implies  $N_j = \mathcal{M}(L(\xi))_{[\tau-\lambda]}$  is torsion free as required. ■

**Theorem 2.5** *If*

$$0 = M_0 \subset M_1 \subset \dots \subset M_q = M$$

*is a sequence of admissible submodules of  $M$  maximal with respect to having strictly increasing degrees, and  $\mathcal{M}(M)_{[\tau]}$  is any torsion free submodule of  $\mathcal{M}(M)$  of degree  $d$ , then*

$$0 = \mathcal{M}(M_0)_{[\tau]} \subset \mathcal{M}(M_1)_{[\tau]} \subset \dots \subset \mathcal{M}(M_q)_{[\tau]} = \mathcal{M}(M)_{[\tau]}$$

*is a composition series of  $\mathcal{M}(M)_{[\tau]}$ .*

**Proof** Let  $\kappa$  be as in the General Assumption. Since each of  $M_1, \dots, M_q$  satisfies the General Assumption, we may construct

$$0 = (\mathcal{M}(M_0))_{[\tau]} \subset (\mathcal{M}(M_1))_{[\tau]} \subset \dots \subset (\mathcal{M}(M_q))_{[\tau]} = (\mathcal{M}(M))_{[\tau]}.$$

By Theorem 1.2(i)  $(\mathcal{M}(M))_{[\tau]}$  has a composition series. If this sequence is not a composition series, then for some  $i$  there is a submodule  $P$  such that  $(\mathcal{M}(M_i))_{[\tau]} \subset P \subset (\mathcal{M}(M_{i+1}))_{[\tau]}$ . Certainly, since these submodules are torsion free,  $d_i < d_P < d_{i+1} \leq d$  where  $d_i$ ,  $d_P$ , and  $d_{i+1}$  are the degrees of  $M_i$ ,  $P$ , and  $M_{i+1}$ , respectively. Necessarily,  $P$  satisfies the General Assumption and

$$0 = \mathcal{M}(M_0) \subset \dots \subset \mathcal{M}(M_i) \subset \mathcal{M}(P) \subset \mathcal{M}(M_{i+1}) \subset \dots \subset \mathcal{M}(M_q) = \mathcal{M}(M)$$

is a sequence of  $\Sigma$ -injective coherent  $A_n$ -families.

$$M_i = M \cap (\mathcal{M}(M_i))_{[\tau]} \subset M \cap (\mathcal{M}(P))_{[\tau]} \subset M \cap (\mathcal{M}(M_{i+1}))_{[\tau]} = M_{i+1}$$

and this contradicts the maximality condition on

$$0 = M_0 \subset M_1 \subset \dots \subset M_q = M. \quad \blacksquare$$

### 3 Admissible Highest Weight Modules

In this section, the focus is on the decomposition of the tensor product module

$$\mathcal{T}^{(a)} = L(a\omega_1) \otimes L(\lambda)$$

where  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  and  $\lambda$  is a dominant integral weight. The goal here is to show that for each  $\xi \in \Lambda$ ,  $L(\xi)$  occurs as a submodule of such a tensor product.

The key to the decomposition of  $\mathcal{T}^{(a)}$  is the branching of  $L(\lambda)$  into simple  $\tilde{A}_n$ -modules. The branching of  $L(\lambda)$  into  $A_{n-1}$ -modules is easily done using a Gel'fand-Zeitlin basis realization of  $L(\lambda)$ . The branching into  $\tilde{A}_n$ -modules can be found by using the diagram reversing automorphism before applying this technique. This suggests that a dominant integral weight  $\lambda = \sum_{i=1}^n \lambda_i \omega_i$  is associated with a partition  $\pi: 0 \leq \pi_1 \leq \pi_2 \leq \dots \leq \pi_n$  of  $|\pi| = \sum_{i=1}^n \pi_i$  where  $\pi_i = \sum_{j=1}^i \lambda_j$ . The desired branching into  $\tilde{A}_n$ -modules is easily described using a set of partitions

$$(3.1) \quad \Pi(\lambda) = \{p: 0 \leq p_1 \leq p_2 \leq \dots \leq p_n \mid 0 \leq p_1 \leq \pi_1 \leq p_2 \leq \pi_2 \leq \dots \leq p_n \leq \pi_n\}.$$

For each  $p \in \Pi(\lambda)$ , with  $|p| = \sum_{i=1}^n p_i$ , define

$$(3.2) \quad \lambda^{(p)} = (|p| + p_1 - |\pi|)\omega_1 + \sum_{i=2}^n (p_i - p_{i-1})\omega_i \quad \text{and} \quad \tilde{\lambda}^{(p)} = \sum_{i=2}^n (p_i - p_{i-1})\omega_i,$$

then,

$$L(\lambda) \simeq \bigoplus_{p \in \Pi(\lambda)} L(\tilde{\lambda}^{(p)})$$

as  $\tilde{A}_n$ -modules. For each  $p \in \Pi(\lambda)$ , fix a basis  $\{v_{pj} \mid j = 1, \dots, d_p\}$  of  $L(\tilde{\lambda}^{(p)})$  consisting of  $A_n$ -weight vectors in such a manner that  $v_{p1}$  is a highest weight vector for  $L(\tilde{\lambda}^{(p)})$  with respect to  $\tilde{A}_n$  and has  $A_n$  weight  $\lambda^{(p)}$ .

Let  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ . In this setting, a basis of  $\mathcal{T}^{(a)} = L(a\omega_1) \otimes L(\lambda)$  is

$$(3.3) \quad \mathcal{B} = \{x^{a, \vec{\ell}} \otimes v_{pj} \mid \ell_i \in \mathbb{Z}_{\geq 0}; \ell_1 \geq \dots \geq \ell_n \geq \ell_{n+1} = 0; p \in \Pi(\lambda); j = 1, \dots, d_p\}.$$

An alternate basis which takes advantage of the  $A_n$ -module structure of  $\mathcal{T}^{(a)}$  is now sought.

For each  $\vec{\ell} = (\ell_1, \dots, \ell_{n+1}) \in \mathbb{Z}^{n+1}$  with  $\ell_1 \geq \dots \geq \ell_{n+1} = 0$ , there exists a unique monomial  $u(\vec{\ell}) = C_{\vec{\ell}} Y_{\alpha_1}^{\ell_1 - \ell_2} \dots Y_{\alpha_1 + \dots + \alpha_n}^{\ell_n}$  with  $C_{\vec{\ell}} \in \mathbb{C}$  such that  $u(\vec{\ell})x_1^a = x^{a, \vec{\ell}}$ . Moreover, for each  $j = 1, \dots, d_p$ , there exists a (not necessarily unique) element  $u_{pj} \in U_-(\tilde{A}_n)$  such that  $u_{pj}v_{p1} = v_{pj}$ . Since  $ux_1^a = 0$  for any element  $u \in U_-(\tilde{A}_n)$ , it follows that for any  $u_{pj}, u'_{pj} \in U_-(\tilde{A}_n)$  with  $u_{pj}v_{p1} = u'_{pj}v_{p1} = v_{pj}$

$$u(\vec{\ell})u_{pj}(x_1^a \otimes v_{p1}) = u(\vec{\ell})(x_1^a \otimes v_{pj}) = u(\vec{\ell})u'_{pj}(x_1^a \otimes v_{p1}).$$

As is proven below a desirable basis is described using the elements

$$(x^{a,\vec{\ell}} \otimes v_{pj})^- = u(\vec{\ell})(x_1^a \otimes v_{pj}).$$

Observe that in the special case when  $\vec{\ell} = \vec{0}$  then  $(x_1^a \otimes v_{pj})^- = x_1^a \otimes v_{pj}$ .

For each  $p \in \Pi(\lambda)$ , define the subspace  $W_p$  to be

$$\text{span}_{\mathbb{C}}\{(x^{a,\vec{\ell}} \otimes v_{q,j})^- \mid \ell_i \in \mathbb{Z}, \ell_1 \geq \dots \geq \ell_{n+1} = 0; \\ q \in \Pi(\lambda); |q| > |p|; j = 1, \dots, d_q\}$$

and observe that  $W_q \subset W_p$  if and only if  $\lambda^{(q)}(h_1^\vee) > \lambda^{(p)}(h_1^\vee)$  since

$$\lambda^{(p)}(h_1^\vee) = \lambda^{(p)} \left( \frac{1}{n+1} \sum_{i=1}^n (n+1-i)h_i \right) = |p| - \frac{n}{n+1}|\pi|.$$

**Proposition 3.4** Let  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ .

(i) A linear basis for  $\mathcal{T}^{(a)}$  is given by

$$\mathcal{B}^- = \{(x^{a,\vec{\ell}} \otimes v_{pj})^- \mid \ell_i \in \mathbb{Z}_{\geq 0}; \ell_1 \geq \dots \geq \ell_{n+1} = 0; p \in \Pi(\lambda); j = 1, \dots, d_p\}.$$

(ii) For  $p \in \Pi(\lambda)$ ,  $W_p$  is a  $U$ -module with linear basis given by

$$\mathcal{B}_p^- = \{(x^{a,\vec{\ell}} \otimes v_{qj})^- \mid \ell_i \in \mathbb{Z}, \ell_1 \geq \dots \geq \ell_{n+1} = 0; \\ q \in \Pi(\lambda); |q| > |p|; j = 1, \dots, d_q\}.$$

(iii)  $U_-(x_1^a \otimes v_{p1}) + W_p$  is a  $U$ -module with a linear basis given by

$$\{(x^{a,\vec{\ell}} \otimes v_{pj})^- \mid \ell_i \in \mathbb{Z}, \ell_1 \geq \dots \geq \ell_{n+1} = 0; j = 1, \dots, d_p\} \cup \mathcal{B}_p^-.$$

**Proof** (i) Notice that if  $\nu = \lambda - \sum_{i=1}^n k_i \alpha_i$  is the weight of  $v_{pj}$ , and  $\mu = \alpha_1 + \dots + \alpha_m$ , then  $(\nu - \mu)(h_1^\vee) < \nu(h_1^\vee)$  and so  $Y_\mu v_{pj} \in \sum_{|q| < |p|} L(\tilde{\lambda}^{(q)})$ . Order the elements in  $\mathcal{B}$  and  $\mathcal{B}^-$  in such a way that  $x^{a,\vec{\ell}} \otimes v_{qj}$  occurs before  $x^{a,\vec{\ell}'} \otimes v_{pl}$ , respectively  $(x^{a,\vec{\ell}} \otimes v_{qj})^-$  occurs before  $(x^{a,\vec{\ell}'} \otimes v_{pl})^-$ , whenever  $|q| < |p|$ . Then if the elements in  $\mathcal{B}^-$  are expressed as linear combinations of the elements in  $\mathcal{B}$  the coefficient matrix is upper triangular with 1's on the main diagonal. Since  $\mathcal{B}$  is a basis of  $\mathcal{T}^{(a)}$  so is  $\mathcal{B}^-$ .

(ii) It is clear from part (i) and the definition of  $W_p$  that  $\mathcal{B}_p^-$  is a basis of  $W_p$ . Also, for any  $\mu \in \Delta^+$  and any basis vector  $v_{pj} \in L(\lambda)$  we have

$$X_\mu v_{pj} \in L(\tilde{\lambda}^{(p)}) \oplus \sum_{|q| > |p|} \oplus L(\tilde{\lambda}^{(q)}).$$

In fact, if  $\mu \in \Delta^+ \setminus \Sigma$  then  $X_\mu v_{pj} \in L(\tilde{\lambda}^{(p)})$  and if  $\mu \in \Sigma$  we have  $\mu(h_1^\vee) > 0$  and hence  $X_\mu v_{pj} \in \sum_{|q| > |p|} \oplus L(\tilde{\lambda}^{(q)})$ . Further, since  $X_\mu x_1^a = 0$  for any  $\mu \in \Delta^+$  we

conclude that  $X_\mu(x_1^a \otimes v_{pj}) \in W_p$ . Since  $W_p$  is generated, as a  $U_-$ -module, by the elements  $\{(x_1^a \otimes v_{q1})^- \mid |q| > |p|\}$  it follows that  $W_p$  is a  $U$ -module.

Part (iii) is proven in an analogous manner to part (ii). ■

Suppose  $v \in \mathcal{T}^{(a)}$  has weight  $\mu$ . Then  $v$  can be expressed as  $\sum_{\vec{\ell}} x^{a,\vec{\ell}} \otimes v_{\vec{\ell}}$  for unique choices of  $v_{\vec{\ell}} \in L(\lambda)$  having weights  $\mu$  minus the weight of  $x^{a,\vec{\ell}}$ . Moreover, when  $v$  is a maximal vector there is more to be said.

**Proposition 3.5** *If  $v^+ = \sum_{\vec{\ell}} x^{a,\vec{\ell}} \otimes v_{\vec{\ell}}$  is an  $A_n$ -maximal vector in  $\mathcal{T}^{(a)}$  then  $v_{\vec{0}}$  is a nonzero  $\tilde{A}_n$ -maximal vector in  $L(\lambda)$ . Conversely for each  $\tilde{A}_n$ -maximal vector  $v_{p1}$  in  $L(\lambda)$  there exists a unique  $A_n$ -maximal vector  $\sum_{\vec{\ell}} x^{a,\vec{\ell}} \otimes v_{\vec{\ell}}$  in  $\mathcal{T}^{(a)}$  such that  $v_{\vec{0}} = v_{p1}$ .*

**Proof** Assume, contrary to what we wish to prove, that  $v^+ = \sum_{\vec{\ell}} x^{a,\vec{\ell}} \otimes v_{\vec{\ell}}$  is an  $A_n$ -maximal vector in  $\mathcal{T}^{(a)}$  with  $v_{\vec{0}} = 0$ . Select an index  $\vec{\ell}$  such that  $\sum \ell_i$  is minimal among the indices with  $v_{\vec{\ell}} \neq 0$ . Without loss of generality, we may assume that  $\ell_q > 0$  and  $\ell_i = 0$  for  $i = q + 1, \dots, n + 1$ . Observe that  $E_{q,q+1}x^{a,\vec{\ell}} = \ell_q x^{a,\vec{\ell}} x_q / x_{q+1} \neq 0$ . Also since  $\sum \ell_i$  is assumed to be minimal, for any  $\vec{k}$  with  $v_{\vec{k}} \neq 0$ ,  $E_{q,q+1}x^{a,\vec{\ell}}$  is linearly independent of  $x^{a,\vec{k}}$  and assuming that  $E_{q,q+1}x^{a,\vec{k}} \neq 0$ , it is linearly independent of  $E_{q,q+1}x^{a,\vec{k}}$  when  $\vec{k} \neq \vec{\ell}$  since they have different weights. Since  $v^+$  is assumed to be maximal,

$$\begin{aligned} 0 &= E_{q,q+1}v^+ \\ &= \ell_q \frac{x^{a,\vec{\ell}} x_q}{x_{q+1}} \otimes v_{\vec{\ell}} + x^{a,\vec{\ell}} \otimes E_{q,q+1}v_{\vec{\ell}} + \sum_{\vec{k} \neq \vec{\ell}} (E_{q,q+1}x^{a,\vec{k}} \otimes v_{\vec{k}} + x^{a,\vec{k}} \otimes E_{q,q+1}v_{\vec{k}}). \end{aligned}$$

From the observations above, the first term of this sum is nonzero and is linearly independent of the other terms. This contradiction implies that we must have  $v_{\vec{0}} \neq 0$ .

The vector  $v_{\vec{0}}$  is claimed to be an  $\tilde{A}_n$ -maximal vector. Certainly, for any index  $i \geq 2$ ,

$$\begin{aligned} 0 &= E_{i,i+1}v^+ \\ &= x_1^a \otimes E_{i,i+1}v_{\vec{0}} + \sum_{\vec{k} \neq \vec{0}} (E_{i,i+1}x^{a,\vec{k}} \otimes v_{\vec{k}} + x^{a,\vec{k}} \otimes E_{i,i+1}v_{\vec{k}}). \end{aligned}$$

If  $E_{i,i+1}v_{\vec{0}} \neq 0$  then a linear independence argument shows that this equation is not possible and so  $E_{i,i+1}v_{\vec{0}} = 0$ . Hence,  $v_{\vec{0}}$  is a maximal vector with respect to  $\tilde{A}_n$  as claimed.

To prove the second assertion it suffices to show that  $\mathcal{T}^{(a)}$  contains a maximal vector of weight  $a\omega_1 + \lambda^{(p)}$ . To this end select any integer  $N \geq \sum_{i=1}^n k_i$  where  $\lambda = \sum_{i=1}^n k_i \omega_i$ . Applying the Pieri Formula we have that

$$\mathcal{T}^{(N)} = L(N\omega_1) \otimes L(\lambda) \simeq \oplus \sum_{p \in \Pi(\lambda)} L(N\omega_1 + \lambda^{(p)}).$$

Therefore for each  $p \in \Pi(\lambda)$ , there exists a nonzero maximal weight vector in  $\mathcal{T}^{(N)}$  of the form  $v^+ = \sum_{\vec{\ell}} x^{N, \vec{\ell}} \otimes v_{\vec{\ell}}$  having weight  $N\omega_1 + \lambda^{(p)}$ . For any positive root vector  $E_{ij}$  one has that  $i < j$  and so

$$0 = E_{ij}v^+ = \sum_{\vec{\ell}} (x_i \partial_j x^{N, \vec{\ell}}) \otimes v_{\vec{\ell}} + \sum_{\vec{\ell}} x^{N, \vec{\ell}} \otimes (E_{ij}v_{\vec{\ell}})$$

from which it follows that the coefficients of the simple tensors do not involve  $N$ . Clearly, the vectors  $v_{\vec{\ell}}$  in this vector  $v^+$  are independent of the value of  $N$ . It follows then that for any  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  the vector  $\sum_{\vec{\ell}} x^{a, \vec{\ell}} \otimes v_{\vec{\ell}} \in \mathcal{T}^{(a)}$  is nonzero, has weight  $a\omega_1 + \lambda^{(p)}$  and is maximal. ■

**Remark 3.6** According to Proposition 3.5, for each  $p \in \Pi(\lambda)$  there exists a unique maximal vector of weight  $a\omega_1 + \lambda^{(p)}$  in  $\mathcal{T}^{(a)}$  having the form

$$x_1^a \otimes v_{p1} + \sum_{\vec{\ell} \neq 0} x^{a, \vec{\ell}} \otimes v_{\vec{\ell}}.$$

This vector is denoted by  $(x^a \otimes v_{p1})^+$ .

**Lemma 3.7** *The degree of the  $A_n$ -module  $(U_-(x_1^a \otimes v_{p1}) + W_p)/W_p$  equals the dimension  $d_p$  of the  $\tilde{A}_n$ -module  $L(\tilde{\lambda}^{(p)})$ .*

**Proof** By Proposition 3.4(iii)  $V = (U_-(x_1^a \otimes v_{p1}) + W_p)/W_p$  has a linear basis given by  $\{(x^{a, \vec{\ell}} \otimes v_{pj})^- + W_p \mid \ell_i \in \mathbb{Z}_{\geq 0}, \ell_1 \geq \ell_2 \geq \dots \geq \ell_{n+1} = 0; j = 1, \dots, d_p\}$ . Since  $L(a\omega_1)$  has degree 1 it follows that the degree of  $V$  is less than or equal to  $d_p$ , the dimension of the  $\tilde{A}_n$ -module  $L(\tilde{\lambda}^{(p)})$ . It remains to be shown that  $V$  has a weight space having dimension equal to  $d_p$ .

Assume that the  $A_n$  weight of the vector  $v_{pj}$  is given by  $\lambda^{(p)} - \sum_{i=2}^n m_{ji}\alpha_i$  and define  $B$  to be the maximum of the coefficients  $m_{ji}$  for  $j = 1, \dots, d_p$  and  $i = 2, \dots, n$ . For each  $j = 1, \dots, d_p$  we define the  $n$  tuple  $\vec{\ell}(j) = (nB, (n-1)B - m_{j2}, \dots, B - m_{jn})$ . It is readily verified that for each  $j = 1, \dots, d_p$  the vector  $(x^{a, \vec{\ell}(j)} \otimes v_{pj})^- + W_p$  is in the  $a\omega_1 + \lambda^{(p)} - \sum_{i=1}^n (n-i+1)B\alpha_i$  weight space. Since these vectors are linearly independent, this weight space has dimension  $d_p$  as claimed. ■

**Lemma 3.8** *Let  $\lambda = \sum_{i=1}^n \lambda_i \omega_i$  be a dominant integral weight and  $a \in \mathbb{C}$ . For any  $p \in \Pi(\lambda)$  there exists at most one other element  $q \in \Pi(\lambda)$  such that  $a\omega_1 + \lambda^{(p)}$  is linked to  $a\omega_1 + \lambda^{(q)}$ . Further, in this case, there exists a dominant integral weight  $\mu$  such that  $\{a\omega_1 + \lambda^{(p)}, a\omega_1 + \lambda^{(q)}\} = \{\mu[k-1], \mu[k]\}$  for some index  $k \in \mathbb{Z}_{>0}$ .*

**Proof** Let  $|\pi| = \sum_{i=1}^n (n-i+1)\lambda_i$  and assume that  $a\omega_1 + \lambda^{(p)}$  and  $a\omega_1 + \lambda^{(q)}$  are linked. This means that their  $\epsilon$  coordinates are permutations of one another, a fact which remains valid when any multiple of  $\sum_{i=1}^{n+1} \epsilon_i$  is added to each. According to (3.2), the  $\epsilon$  coordinates of

$$a\omega_1 + \lambda^{(p)} + \delta - \frac{1}{n+1} \left( a + |p| + p_1 - |\pi| + 1 + \sum_{i=1}^n (p_i - p_{i-1} + 1) \right) \sum_{i=1}^{n+1} \epsilon_i$$

and

$$a\omega_1 + \lambda^{(q)} + \delta - \frac{1}{n+1} \left( a + |q| + q_1 - |\pi| + 1 + \sum_{i=1}^n (q_i - q_{i-1} + 1) \right) \sum_{i=1}^{n+1} \epsilon_i$$

are respectively,

$$(a + |p| + p_n + n - |\pi|, p_n - p_1 + n - 1, p_n - p_2 + n - 2, \dots, p_n - p_{n-1} + 1, 0)$$

and

$$(a + |q| + q_n + n - |\pi|, q_n - q_1 + n - 1, q_n - q_2 + n - 2, \dots, q_n - q_{n-1} + 1, 0).$$

By assumption these two tuples are permutations of each other and their last  $n$  coordinates form a decreasing sequence of integers. Since the sum of their components are  $a + (n + 1)p_n + \frac{n(n+1)}{2} - |\pi|$  and  $a + (n + 1)q_n + \frac{n(n+1)}{2} - |\pi|$  respectively it follows that  $p_n = q_n$ . For any indices  $i < j$  since  $p_i \leq \pi_i \leq \dots \leq \pi_{j-1} \leq q_j$  it follows that  $p_n - p_i + n - i \neq q_n - q_j + n - j$ . Therefore if  $q \neq p$ , there exists an index  $k$  such that  $p_i = q_i$  for  $i \neq k$ ,  $a + |p| + p_n - |\pi| = q_n - q_k + n - k$  and  $p_n - p_k + n - k = a + |q| + q_n - |\pi|$ . This establishes the uniqueness.

Assume that  $p_k < q_k$  and define the dominant integral weight

$$\begin{aligned} \mu = & (p_2 - p_1)\omega_1 + \dots + (p_k - p_{k-1})\omega_{k-1} + (q_k - p_k - 1)\omega_k \\ & + (p_{k+1} - q_k)\omega_{k+1} + (p_{k+2} - p_{k+1})\omega_{k+2} + \dots + (p_n - p_{n-1})\omega_n. \end{aligned}$$

By direct computation we find that  $\mu[k - 1] = a\omega_1 + \lambda^{(p)}$  and  $\mu[k] = a\omega_1 + \lambda^{(q)}$ . ■

**Lemma 3.9** *If  $\tau \in \mathcal{H}^*$  such that  $L(\tau)$  is an admissible  $\Sigma$ -injective module with central character  $\chi_\mu$  for some dominant integral weight  $\mu$ , then  $\tau = \mu[k]$  for some  $1 \leq k \leq n$ .*

**Proof** According to Mathieu, since  $L(\tau)$  is admissible with central  $\chi_\mu$ ,  $\tau = \sum_{i=1}^n a_i \omega_i$  with  $a_i \in \mathbb{Z}$  having exactly one  $a_k \in \mathbb{Z}_{<0}$ . If  $a_1 = m \geq 0$  and  $v^+$  is the maximal vector of  $L(\tau)$  then by  $\Sigma$ -injectivity  $Y_{\alpha_1}^{m+1} v^+ \neq 0$  is a maximal vector, contrary to the simplicity of  $L(\tau)$ . Therefore,  $\tau \in \Lambda$  and as noted in Section 1, this implies that  $\tau = \mu[k]$  for some  $1 \leq k \leq n$ . ■

**Theorem 3.10** *Let  $\mu$  be a dominant integral weight and assume that  $V$  is a  $\Sigma$ -injective  $A_n$ -module generated by a highest weight vector  $v_0$  of weight  $\mu[k]$  and degree equal to the dimension of the  $\tilde{A}_n$ -module  $L(\tilde{\mu}[k])$ . Then*

- (i) *if  $k = n$ , then  $V \simeq L(\mu[n])$ ;*
- (ii) *if  $k < n$ , then  $V$  contains the submodule  $L(\mu[k + 1])$  and is indecomposable with composition factors  $L(\mu[k + 1])$  and  $L(\mu[k]) \simeq V/L(\mu[k + 1])$ .*

**Proof** It follows from Lemma 3.9 that if  $V$  is not simple it must contain a submodule isomorphic to  $L(\mu[l])$  for some index  $l > k$ . If  $k = n$  this is impossible and hence in this case  $V$  must be simple—i.e., equivalent to  $L(\mu[n])$ .

Assume now that  $0 \leq k < n$ . By Lemma 1.5(ii), we have that the degree of  $L(\mu[k])$  is strictly less than the dimension of the  $\tilde{A}_n$ -module  $L(\tilde{\mu}[k])$ . Therefore  $V$  is not equivalent to  $L(\mu[k])$ —i.e. is not simple. Since  $V$  is generated by a highest weight vector,  $V$  is a homomorphic image of the Verma module  $M(\mu[k])$  having highest weight  $\mu[k]$ . Also by Theorem 3.9  $V$  must contain a submodule isomorphic to  $L(\mu[l])$  for some  $l > k$ .

If  $l \geq k + 2$  then since  $\mu[k + 1] > \mu[k + 2]$  in the Bruhat ordering, the BGG resolution implies that  $V$  must contain highest weight vectors of weight  $\mu[k + 1]$  and  $\mu[k + 2]$ . However, the sum of the degrees of  $L(\mu[k])$ ,  $L(\mu[k + 1])$  and  $L(\mu[k + 2])$  is strictly larger than the degree of  $V$ . It follows that  $V$  must have a highest weight vector, say  $v_1$ , of weight  $\mu[k + 1]$ . Therefore  $V$  is indecomposable and further  $V/Uv_1 \simeq L(\mu[k])$ ,  $Uv_1 \simeq L(\mu[k + 1])$  and hence  $L(\mu[k + 1]) \leq V$ . ■

The composition factors of the tensor product module  $\mathcal{T}^{(a)}$  are now given in the following theorem.

**Theorem 3.11** *Let*

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \dots \subset \mathcal{T}_q = \mathcal{T}^{(a)}$$

*be a composition series of  $\mathcal{T}^{(a)}$ . Then the composition factors for  $\mathcal{T}^{(a)}$  are  $L(\nu)$  with*

$$\begin{aligned} \nu \in \Omega = \{a\omega_1 + \lambda^{(p)} \mid p \in \Pi(\lambda)\} \\ \cup \{\mu[k + 1] \mid 0 \leq k < n, \text{ whenever } a\omega_1 + \lambda^{(p)} = \mu[k] \\ \text{for some } p \in \Pi(\lambda) \text{ and some dominant integral weight } \mu\}. \end{aligned}$$

*Moreover, in the corresponding sequence of  $\Sigma$ -injective coherent  $A_n$ -families*

$$0 = \mathcal{M}(\mathcal{T}_0) \subseteq \mathcal{M}(\mathcal{T}_1) \subseteq \dots \subseteq \mathcal{M}(\mathcal{T}_q) = \mathcal{M}(\mathcal{T}^{(a)})$$

*equality holds between  $\mathcal{M}(\mathcal{T}_i)$  and  $\mathcal{M}(\mathcal{T}_{i+1})$  if and only if  $\mathcal{T}_{i+1}/\mathcal{T}_i \simeq L(a\omega_1 + \lambda^{(p)})$  where  $a\omega_1 + \lambda^{(p)}$  is dominant integral.*

**Proof** For each  $p \in \Pi(\lambda)$  such that  $a\omega_1 + \lambda^{(p)}$  is a nonintegral or a singular weight  $(x_1^a \otimes v_{p1})^+$  is a highest weight vector in  $\mathcal{T}^{(a)}$  and by Theorem 1.5 the submodule generated by this vector has degree equal to the degree of  $L(a\omega_1 + \lambda^{(p)})$ —i.e.,  $U(x_1^a \otimes v_{p1})^+ \simeq L(a\omega_1 + \lambda^{(p)})$ . On the other hand assume that  $p \in \Pi(\lambda)$  where  $a\omega_1 + \lambda^{(p)} = \mu[k]$  for some dominant integral weight  $\mu$ . By Lemma 3.7 together with Theorem 1.5(ii) there exists  $q \in \Pi(\lambda)$  such that  $a\omega_1 + \lambda^{(q)} = \mu[k - 1]$  or  $\mu[k + 1]$ . Without loss of generality assume that  $a\omega_1 + \lambda^{(q)} = \mu[k + 1]$ . Since  $W_p$  is the direct sum of generalized eigenspaces belonging to central characters  $\chi_{a\omega_1 + \lambda^{(q)}}$  where  $|q| > |p|$ ,  $U(x_1^a \otimes v_{p1})^+ \cap W_p = 0$ . Hence, by Lemma 3.7, the degree of  $U(x_1^a \otimes v_{p1})^+$  is equal to the dimension of  $\tilde{A}_n$ -module  $L(\tilde{\mu}[k])$ . By Theorem 3.10, the submodule generated by the highest weight vector  $(x_1^a \otimes v_{p1})^+$  is an indecomposable module with composition factors given by  $L(\mu[k])$  and  $L(\mu[k + 1])$ . Further the highest weight vector  $(x_1^a \otimes v_{q1})^+$  is contained in this submodule. In this case the vector  $x_1^a \otimes v_{q1}$  does

not belong to  $U(x_1^a \otimes v_{p1})^+$  and therefore  $U(x_1^a \otimes v_{q1}) + W_q$  is a highest weight module with highest weight  $\mu[k + 1]$  in  $\mathcal{T}^{(a)}/W_q$  having degree equal to the dimension of the  $\tilde{A}_n$ -module  $L(\tilde{\mu}[k + 1])$ . By Theorem 3.10,  $U(x_1^a \otimes v_{q1}) + W_q$  is isomorphic to  $L(\mu[n])$  if  $k + 1 = n$  and if  $k + 1 < n$ , then it has composition factors  $L(\mu[k + 1])$  and  $L(\mu[k + 2])$ .

By the first part of the proof,

$$\mathcal{T}_{i+1}/\mathcal{T}_i \simeq L(\tau) = L(a\omega_1 + \lambda^{(\ell)}) \quad \text{or} \quad L(\mu[k + 1]).$$

If  $\tau$  is not dominant integral, then  $\mathcal{M}^{ss}(\mathcal{T}_i) \subset \mathcal{M}^{ss}(\mathcal{T}_{i+1})$ , since the larger module contains a copy of  $L(\tau)$  not contained in the smaller one. If  $\tau$  is dominant integral, then  $\tau = a\omega_1 + \lambda^{(\ell)}$  since  $\mu[k + 1]$  is not and moreover the degree of  $\mathcal{T}_i$  equals the degree of  $\mathcal{T}_{i+1}$ . Thus since  $\mathcal{M}(\mathcal{T}_i) \subseteq \mathcal{M}(\mathcal{T}_{i+1})$ , it must be the case that  $\mathcal{M}(\mathcal{T}_i) = \mathcal{M}(\mathcal{T}_{i+1})$ . ■

**Corollary 3.12** *For any weight  $\xi \in \Lambda$  there exists a dominant weight  $\lambda$  and a scalar  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  such that  $L(\xi)$  is isomorphic to a submodule of  $L(a\omega_1) \otimes L(\lambda)$ .*

**Proof** If  $\xi = \sum_{i=1}^n a_i \omega_i \in \Lambda$  is either nonintegral or singular then Theorem 1.5 implies that the degree of  $L(\xi)$  is equal to the dimension of the simple  $\tilde{A}_n$ -module  $L(\tilde{\xi})$ . Let  $\lambda = \xi - a_1 \omega_1$ . Let  $v^+$  be a maximal vector of  $L(\lambda)$  and  $x_1^a \otimes v^+$  be the highest maximal vector in  $L(a_1 \omega_1) \otimes L(\lambda)$ . By Lemma 3.7,  $U(x_1^a \otimes v^+)$  has degree equal to the degree of the  $\tilde{A}_n$ -module  $L(\tilde{\xi})$ .  $\Sigma$ -injectivity implies that  $U(x_1^a \otimes v^+) \simeq L(\xi)$  and hence in this case  $L(\xi)$  is isomorphic to a submodule of  $L(a\omega_1) \otimes L(\lambda)$ .

If  $\xi$  is a regular integral weight in  $\Lambda$  then let  $\mu$  denote the unique dominant integral weight such that  $\xi = \mu[k]$  for some  $k = 1, \dots, n$ . Assume first that  $k > 1$ . Then  $\mu[k - 1] = \sum_{i=1}^n b_i \omega_i$  where  $b_1 \in \mathbb{Z}_{< 0}$  and  $b_2, \dots, b_n \in \mathbb{Z}_{\geq 0}$ . Let  $V$  be the submodule generated by the highest maximal vector of  $L(b_1 \omega_1) \otimes L(\mu[k - 1] - b_1 \omega_1)$ . By Theorem 3.10(ii),  $L(\xi)$  is a submodule of  $V$ . Now suppose that  $k = 1$ . By Lemma 3.8,  $L(\xi)$  is a submodule of  $L(-\omega_1) \otimes L(\mu + \omega_1)$  since the submodule generated by the highest maximal vector, which has weight  $\mu = \mu[0]$ , is not simple. ■

The following example is presented to illustrate the concepts of this section.

**Example 3.13** Consider the  $A_3$  module  $\mathcal{T}^{(a)} = L(a\omega_1) \otimes L(\lambda)$  where  $a = -2$  and  $\lambda = \omega_2$ . A basis for  $L(\omega_2)$  is given by

$$\{x_i \wedge x_j \mid 1 \leq i < j \leq 4\}$$

and a basis for  $L(-2\omega_1)$  is given by

$$\{x_1^{-2-\ell_1} x_2^{\ell_1-\ell_2} x_3^{\ell_2-\ell_3} x_4^{\ell_3} \mid \ell_i \in \mathbb{Z}_{\geq 0}; \ell_1 \geq \ell_2 \geq \ell_3\}.$$

The partition  $\pi$  associated with  $\lambda = \omega_2$  is  $\{0, 0, 1, 1\}$  and  $\Pi(\omega_2) = \{\{0, 1, 1\}, \{0, 0, 1\}\}$ . The corresponding weights are  $\lambda^{\{0,1,1\}} = \omega_2$ ,  $\lambda^{\{0,0,1\}} = -\omega_1 + \omega_3$ . Observe that  $x_1 \wedge x_2$  and  $x_2 \wedge x_3$  are highest weight vectors in  $L(\omega_2)$  with respect to



the subalgebra  $\tilde{A}_3$  and have weights  $\lambda^{\{0,1,1\}} = \omega_2, \lambda^{\{0,0,1\}} = -\omega_1 + \omega_3$  respectively. The associated highest weight vectors in  $\mathcal{J}^{(-2)}$  are

$$\begin{aligned} (x_1^{-2} \otimes (x_1 \wedge x_2))^+ &= x_1^{-2} \otimes (x_1 \wedge x_2) \quad \text{and} \\ (x^{-2} \otimes (x_2 \wedge x_3))^+ &= x_1^{-2} \otimes (x_2 \wedge x_3) - x_1^{-3}x_2 \otimes (x_1 \wedge x_3) + x_1^{-3}x_3 \otimes (x_1 \wedge x_2) \end{aligned}$$

having weights  $-2\omega_1 + \lambda^{\{0,1,1\}} = -2\omega_1 + \omega_2$  and  $-2\omega_1 + \lambda^{\{0,0,1\}} = -3\omega_1 + \omega_3$  respectively.

Note that  $-2\omega_1 + \omega_2 = \mu[1]$  and  $-3\omega_1 + \omega_3 = \mu[2]$  are regular integral weights where  $\mu$  is the dominant integral weight 0. According to Theorem 3.11, therefore, the composition factors for  $\mathcal{J}^{(-2)}$  are the simple highest weight modules with highest weights  $-2\omega_1 + \omega_2, -3\omega_1 + \omega_3, -3\omega_1 + \omega_3$  and  $-4\omega_1$ . Define the following submodules

$$\begin{aligned} T_0 &= U \cdot (x_1^{-2} \otimes (x_2 \wedge x_3)) \\ T_1 &= U \cdot (x_1^{-2} \otimes (x_1 \wedge x_2)) \\ T_2 &= U \cdot (x_1^{-3}x_4 \otimes (x_2 \wedge x_3) - x_1^{-4}x_3x_4 \otimes (x_1 \wedge x_2) + x_1^{-4}x_2x_4 \otimes (x_1 \wedge x_3)) \\ T_3 &= U \cdot (x_1^{-2} \otimes (x_2 \wedge x_3) - x_1^{-3}x_2 \otimes (x_1 \wedge x_3) + x_1^{-3}x_3 \otimes (x_1 \wedge x_2)). \end{aligned}$$

It is readily verified that  $\mathcal{J}^{(-2)} = T_0 \supset T_1 \supset T_3$ . A Jordan Holder series for  $\mathcal{J}^{(-2)}$  is given by

$$\mathcal{J}^{(-2)} = T_0 \supset T_1 + T_2 \supset T_2 \supset T_3$$

with composition factors  $T_0/(T_1+T_2) \simeq L(-3\omega_1 + \omega_3), (T_1+T_2)/T_2 \simeq L(-2\omega_1 + \omega_2), T_2/T_3 \simeq L(-4\omega_1),$  and  $T_3 \simeq L(-3\omega_1 + \omega_3)$  as expected.

### 4 Simple Torsion Free $A_n$ -Modules

In this section, the previous results are combined to determine the composition factors for the tensor product of a torsion free  $A_n$ -module of degree 1 having central character  $\chi_{a\omega_1}$  with  $a \notin \mathbb{Z}_{\geq 0}$  and a simple finite dimensional module. Finally it is shown that every simple torsion free  $A_n$ -module of finite degree is isomorphic to a submodule of such a tensor product module.

Fix any simple torsion free  $A_n$ -module  $M$  of degree 1 having central character  $\chi_{a\omega_1}$  with  $a \notin \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ . Then by [BL2],  $M$  is of the form  $T(\vec{a})$  as constructed in Example 1.4 and by Proposition 1.6, there exists a coset  $[\tau] \in \mathcal{H}^*/Q$  such that  $M \simeq \mathcal{M}(L(a\omega_1))_{[\tau]}$ .

**Theorem 4.1** *Let  $M$  be a simple torsion free  $A_n$ -module of degree 1 as above,  $L(\lambda)$  be any simple finite dimensional  $A_n$ -module, and  $\Omega$  be as in Theorem 3.11. Then the composition factors for the tensor product module  $M \otimes L(\lambda)$  are  $\mathcal{M}(L(\nu))_{[\tau+\lambda]}$  where  $\nu \in \Omega$  and  $\nu$  is not dominant integral.*

**Proof** Since  $M \simeq \mathcal{M}(L(a\omega_1))_{[\tau]}$ ,

$$M \otimes L(\lambda) \simeq \mathcal{M}(L(a\omega_1))_{[\tau]} \otimes L(\lambda) \simeq \mathcal{M}(L(a\omega_1) \otimes L(\lambda))_{[\tau+\lambda]}.$$

Let  $\mathcal{T}^{(a)} = L(a\omega_1) \otimes L(\lambda)$  and

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \dots \subset \mathcal{T}_q = \mathcal{T}^{(a)}$$

be a composition series of  $\mathcal{T}^{(a)}$ . Modify this sequence to obtain a sequence of admissible submodules maximal with respect to having strictly increasing degrees. By Theorem 2.5, this sequence is transferred to a composition series of  $M \otimes L(\lambda)$ . ■

**Remark** According to Proposition 1.6 there are two types of weights  $\xi \in \Lambda$  associated with degree 1 coherent  $A_n$ -families, namely  $\xi = a\omega_1$  for  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  and  $\xi = -(N+2)\omega_1 + (N+1)\omega_2$  for  $N \in \mathbb{Z}_{\geq 0}$ . For completeness we state without proof that if  $\lambda$  is a dominant integral weight and  $M = \mathcal{M}(L(-(N+2)\omega_2 - (N+1)\omega_2))_{[\tau-\lambda]}$  is a simple torsion free  $A_n$ -module of degree 1 having central character  $\chi_{N\omega_1}$  for some  $N \in \mathbb{Z}_{\geq 0}$  then the composition factors of  $M \otimes L(\lambda)$  are  $\mathcal{M}(L(\xi))_{[\tau]}$  where

$$\begin{aligned} \xi \in & (\{N\omega_1 + \lambda^{(p)} \mid p \in \Pi(\lambda)\} \cap \Lambda) \\ & \cup \{\mu[k+1] \mid \exists p \in \Pi(\lambda) \text{ and } \mu \text{ dominant integral} \\ & \text{such that } \mu[k] = N\omega_1 + \lambda^{(p)} \text{ for some } 0 \leq k < n\}. \end{aligned}$$

**Theorem 4.2** Every simple torsion free module of finite degree is isomorphic to a submodule of  $T(\vec{a}) \otimes L(\lambda)$  for some choice of a simple finite dimensional  $A_n$ -module,  $L(\lambda)$ , and some choice of a simple torsion free module,  $T(\vec{a})$ , of degree 1.

**Proof** Let  $T$  be a simple torsion free module of degree  $d$  and  $\mathcal{M}(T)$  be the unique  $\Sigma$ -injective coherent family of degree  $d$  containing  $T$ , i.e., for some  $\tau \in \mathcal{H}^*$ ,  $T = \mathcal{M}(T)_{[\tau]}$ . From this form the irreducible semisimple coherent family  $\mathcal{M}^{ss}(T)$  containing  $T$ . According to Theorem 1.3, there is a unique simple highest weight module  $L(\xi)$  of degree  $d$  with  $\xi \in \Lambda$  such that  $L(\xi)$  is a submodule of  $\mathcal{M}^{ss}(T)$ . The unique  $\Sigma$ -injective coherent family  $\mathcal{M}(L(\xi))$  of degree  $d$  containing  $L(\xi)$  has the property that its semisimple form contains  $T$  and so  $T = \mathcal{M}(L(\xi))_{[\tau]}$ . This means that  $\mathcal{M}(L(\xi)) \simeq \mathcal{M}(T)$  by Theorem 2.2(iii).

By Corollary 3.12, there is a dominant integral weight  $\lambda$  and an  $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$  such that  $L(\xi)$  is isomorphic to a submodule of  $L(a\omega_1) \otimes L(\lambda)$ . Therefore,

$$T \simeq \mathcal{M}(L(\xi))_{[\tau]} \leq \mathcal{M}(L(a\omega_1) \otimes L(\lambda))_{[\tau]} \simeq \mathcal{M}(L(a\omega_1))_{[\tau-\lambda]} \otimes L(\lambda)$$

and by Theorem 2.4,  $\mathcal{M}(L(a\omega_1))_{[\tau-\lambda]}$  is torsion free. ■

**Remark** As noted in the introduction, the proof of the conjecture for torsion free  $C_n$ -modules is much more transparent. In fact, if  $T$  is a simple torsion free  $C_n$ -module of degree  $d$  then by [M, Theorem 4.5], there exists an admissible highest weight  $\xi$  and a coset  $[\tau] \in \mathcal{H}^*/Q$  such that  $T \simeq \mathcal{M}(L(\xi))_{[\tau]}$ . Moreover, Mathieu tells us that if  $\omega = -1/2\omega_n$  then  $\lambda := \xi - \omega$  is a dominant integral weight. According to [BHL],  $L(\omega) \otimes L(\lambda)$  is completely reducible. Certainly, it is admissible and contains a submodule isomorphic to  $L(\xi)$ . It follows then that

$$T \simeq \mathcal{M}(L(\xi))_{[\tau]} \leq \mathcal{M}(L(\omega) \otimes L(\lambda))_{[\tau]} \simeq \mathcal{M}(L(\omega))_{[\tau-\lambda]} \otimes L(\lambda)$$

where  $\mathcal{M}(L(\omega))_{[\tau-\lambda]}$  is a simple torsion free module of degree 1.

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