AN INTEGRAL INVOLVING THE PRODUCT OF A BESSEL FUNCTION AND AN E-FUNCTION

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The formula *

$$4\int_0^\infty \lambda^{m-1} K_n(2\lambda) E(p\,;\alpha_r : q\,;\rho_s : x\lambda^{-2})\,d\lambda$$
$$= E(p+2\,;\alpha_r : q\,;\rho_s : x), \quad \text{.......(1)}$$

where $\alpha_{p+1} = \frac{1}{2}(m+n)$, $\alpha_{p+2} = \frac{1}{2}(m-n)$, $R(m \pm n) > 0$ and x is real and positive, was given by MacRobert (*Phil. Mag.*, Ser. 7, XXXI, p. 258). From it the formula (6) below will be deduced.

In (1) let it be assumed that $R(m\pm n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and let amp λ decrease by $\frac{1}{2}\pi$, amp x decreasing simultaneously by π, finally writing λ/i in place of λ and $xe^{-i\pi}$ in place of x: then, noting that

$$K_n(t) = i^n G_n(it), \quad \text{.......(2)}$$

we have

$$4i^{n-m}\int_0^\infty \lambda^{m-1} G_n(2\lambda) E(p\,;\alpha_r : q\,;\rho_s : x\lambda^{-2})\,d\lambda$$
$$= E(p+2\,;\alpha_r : q\,;\rho_s : xe^{-i\pi}). \quad \text{.......(3)}$$

Similarly, on increasing amp λ by $\frac{1}{2}\pi$ and amp x by π, we have

$$4i^{n+m}\int_0^\infty \lambda^{m-1} G_n(2\lambda e^{i\pi}) E(p\,;\alpha_r : q\,;\rho_s : x\lambda^{-2})\,d\lambda$$
$$= E(p+2\,;\alpha_r : q\,;\rho_s : xe^{i\pi}) \quad \text{.......(4)}$$

Hence, on applying the formula

$$\pi i J_n(t) = G_n(t) - i^{2n} G_n(te^{i\pi}), \quad \text{.......(5)}$$

it is found that

$$4i\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda) E(p\,;\alpha_r : q : \rho_s : x\lambda^{-2})\,d\lambda$$
$$= i^{m-n} E(p+2\,;\alpha_r : q\,;\rho_s : xe^{-i\pi})$$
$$- i^{-m+n} E(p+2\,;\alpha_r : q\,;\rho_s : xe^{i\pi}), \quad \text{.......(6)}$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and x is real and positive.

In particular, if $p \geqq q-1$, formula (6) can be written

$$2\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda) E(p\,;\alpha_r : q\,;\rho_s : x\lambda^{-2})\,d\lambda$$
$$= \sum_{r=1}^{p+1} \frac{\prod\limits_{s=1}^{p+2}{}' \Gamma(\alpha_s - \alpha_r)}{\prod\limits_{t=1}^{q} \Gamma(\rho_t - \alpha_r)} \Gamma(\alpha_r) \sin\left(\tfrac{1}{2}m - \tfrac{1}{2}n - \alpha_r\right)\pi\, x^{\alpha_r}$$
$$\times F\begin{Bmatrix} \alpha_r, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1\,;\ (-1)^{p-q+1}x \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_{p+2} + 1 \end{Bmatrix}, \quad \text{.......(7)}$$

* For the properties of the E-functions see MacRobert, *Functions of a Complex Variable*, third edition.

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and x is real and positive. It should be noted that the $(p+2)$th term on the right of (7) does not appear because $\alpha_{p+2}=\frac{1}{2}(m-n)$.

If $m=\beta+1$, $n=\beta-1$ and $\rho_q=\beta$, then

$$\alpha_{p+1}=\beta,\ \alpha_{p+2}=1$$

and

$$\alpha_r-\alpha_{p+1}+1=\alpha_r-\beta+1,$$

which cancels $\alpha_r-\rho_q+1$ on the right of (7).

Also

$$\alpha_r-\alpha_{p+2}+1=\alpha_r,$$

which cancels α_r on the right of (7).

Again, $\rho_q-\alpha_{p+1}=0$, so that

$$\frac{1}{\Gamma(\rho_q-\alpha_{p+1})}=0,$$

and therefore the last term on the right of (7) disappears.

Finally, noting that

$$\Gamma(\alpha_{p+2}-\alpha_r)\,\Gamma(\alpha_r)=\frac{\pi}{\sin(\alpha_r\pi)},$$

that

$$\sin(\tfrac{1}{2}m-\tfrac{1}{2}n-\alpha_r)\pi=\sin(\alpha_r\pi),$$

and that

$$\frac{\Gamma(\alpha_{p+1}-\alpha_r)}{\Gamma(\rho_q-\alpha_r)}=1,$$

we have, if $p\geqq q-1$,

$$2\int_0^\infty \lambda^\beta J_{\beta-1}(2\lambda)E(p\,;\alpha_r:q\,;\rho_s:x\lambda^{-2})\,d\lambda$$

$$=\sum_{r=1}^{p}\frac{\prod\limits_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)}{\prod\limits_{t=1}^{q-1}\Gamma(\rho_t-\alpha_r)}\,x^{\alpha_r}F\left(\begin{matrix}\alpha_r-\rho_1+1,\ \ldots,\ \alpha_r-\rho_{q-1}+1\,;\ (-1)^{p-q+1}x\\ \alpha_r-\alpha_1+1,\ \ldots*\ldots,\ \alpha_r-\alpha_p+1\end{matrix}\right), \qquad (8)$$

where $\rho_q=\beta$, $R(\beta)>0$, $R(\frac{1}{2}-\beta+2\alpha_r)>0$, $r=1, 2, \ldots, p$.

It should be noted that β does not appear on the right of (8).

This result was given, for the case $p=q+1$, by Meijer (*Proc. Akad. te Amsterdam*, XXXIX, 1936, p. 397).

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