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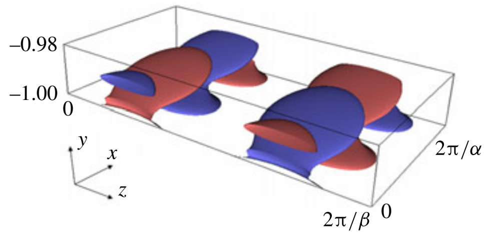
Exact coherent structures at extreme Reynolds number



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Exact coherent structures (ECS), unstable three-dimensional solutions of the Navier–Stokes equations, play a fundamental role in transitional and turbulent wall flows. Dempsey *et al.* (*J. Fluid Mech.*, vol. 791, 2016, pp. 97–121) demonstrate that at large Reynolds number reduced equations can be derived that simplify the computation and facilitate mechanistic understanding of these solutions. Their analysis shows that ECS in plane Poiseuille flow can be sustained by a novel inner–outer interaction between oblique near-wall Tollmien–Schlichting waves and interior streamwise vortices.

Key words: mathematical foundations, nonlinear instability, transition to turbulence

1. Introduction

The discovery of exact coherent states or structures (ECS), three-dimensional (3-D) invariant solutions of the Navier–Stokes (NS) equations (Nagata 1990), has led to a gradual but inexorable paradigm shift in research in transitional and turbulent wall flows. Although unstable, these solutions have been shown to provide a scaffold in phase space for moderate Reynolds number (*Re*) turbulent dynamics and an 'edge' that separates the laminar and turbulent basins of attraction – with direct implications for the prediction and control of transition. Waleffe (1997) conceived a nonlinear self-sustaining process (SSP) theory involving the interaction among streamwise (x) vortices, a spanwise (z) and wall-normal (y) varying streamwise shear flow and 3-D (Rayleigh wave) instability modes. This physically motivated conceptual framework enabled the *ab initio* computation of ECS at finite *Re* in linearly stable parallel shear flows.

Since the pioneering successes of Nagata and Waleffe, countless ECS in a variety of wall flows have been computed. The theoretical investigation by Dempsey *et al.* (2016) complements these strictly computational studies by providing a semi-analytical description of strongly nonlinear ECS that bifurcate from small-amplitude near-wall

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Tollmien–Schlichting (TS) waves in plane Poiseuille flow (PPF). Their elegant approach leverages a large-Re mathematical formalism for nonlinearly interacting 3-D wavy instabilities and 2-D streamwise vortices aptly termed vortex–wave interaction (VWI) theory. The VWI framework was developed by co-author P. Hall and collaborators independently of, but concurrently with, the ECS computations being performed by Nagata (1990); see in particular Hall & Smith (1991). Nevertheless, the complete interaction equations were not solved until Hall & Sherwin (2010) computed VWI solutions in a plane Couette flow (PCF) configuration. Only then was the intimate connection between VWI and SSP theories firmly established, with the VWI solutions obtained by Hall & Sherwin (2010) constituting the infinite-Re limit of the ECS computed by Nagata and Waleffe in PCF.

The asymptotic approach offers important advantages. By explicitly recognizing the emergence of multiple spatial and temporal scales, simplified partial differential equations (PDEs) are derived that enable a reduction in computational complexity. For example, Beaume (2012) and Blackburn, Hall & Sherwin (2013) show that the computation of certain ECS can be reduced to the coupled solution of two 2-D problems: a nonlinear problem for the x-averaged flow at unit effective Reynolds number; and a quasi-linear problem for inviscid wavy instabilities riding on the streamwise-averaged flow. In fact, a time-dependent version of these reduced PDEs is equivalent (for large Re) to the 'restricted nonlinear model' of turbulence in wall-bounded parallel shear flows recently proposed by Thomas *et al.* (2015). More significantly, the asymptotic approach lays bare the essential physics of the interactions that sustain the ECS. Here, the paper by Dempsey *et al.* (2016) is a tour-de-force, combining detailed mathematical analysis, careful numerics and deep physical insight to show in the context of PPF that ECS in the domain interior can be sustained by a near-wall viscous instability giving rise to TS waves.

2. Overview

A cornerstone of VWI theory is that at large Re weak $[O(Re^{-1})]$ vortices can act as an advection mechanism both for the rolls themselves and for the streamwise-averaged streamwise flow, creating streaks by inducing $O(1)$ spanwise modifications to the latter. Since the vortices are weak, viscous diffusion acts at leading order on the streamwise-averaged flow, which by design satisfies a 2-D/3-component PDE system. The streamwise vortices in this mean system must be maintained by the Reynolds stress divergence (RSD) induced by x-varying (3-D) fluctuations. In VWI and SSP theory, these 3-D motions are not turbulent fluctuations *per se* but wavy – that is, coherent – instability modes. For example, in the analysis by Dempsey *et al.* (2016) the wave-induced RSD in the domain interior drives a counter-rotating cellular mean flow when the size of the waves $\delta = \epsilon^6 \ll 1$, where the small parameter $\epsilon \equiv Re^{-1/7}$. An immediate consequence is that the waves satisfy quasi-linear equations about the $O(1)$ streaks both in the core and in the near-wall layer. It is this quasi-linearity that emerges in the large-Re limit that renders single-mode fluctuations (waves!) in the x direction admissible and necessary solution components of the ECS. A selection mechanism exists as the amplitude and wavelength of the x-varying disturbances must be precisely tuned to enable vortices and streaks to be driven that render those disturbances neutrally stable (Hall & Smith 1991; Beaume *et al.* 2015).

A second hallmark feature of VWI theory is that crucial wave processes occur within asymptotically thin internal or wall layers as $Re \to \infty$. Indeed, following the closely related analysis by Bennett, Hall & Smith (1991) for flows in curved channels, Dempsey *et al.* (2016) show that oblique TS waves arise as an instability of the

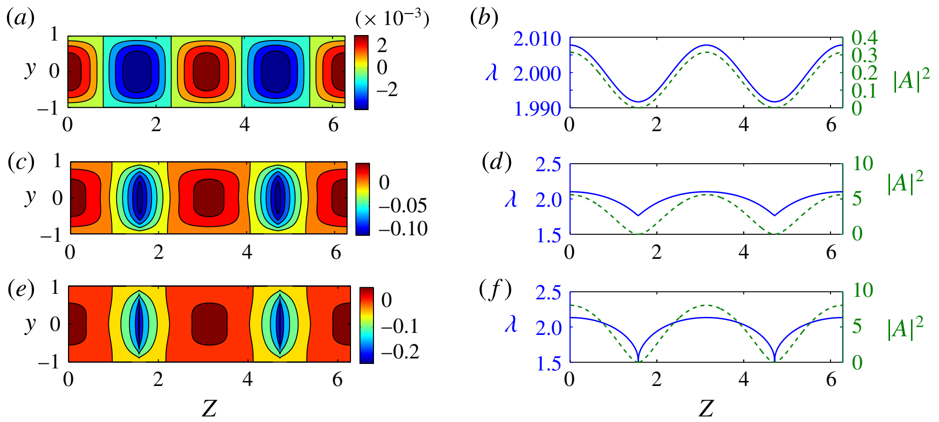

FIGURE 1. Morphology of vortex/TS wave ECS streak structure (*a*,*c*,*e*) and associated wall shear stress $\lambda(Z)$ and TS wave amplitude $|A(Z)|^2$ (*b*,*d*,*f*), where $Z \equiv \epsilon z$, with increasing magnitude of wave amplitude $\mathscr{A}^2 \equiv \int_0^{2\pi/\beta} |A(Z)|^2 \, \mathrm{d}Z$: (*a*,*b*) $\mathscr{A}^2 = 1$, (*c*,*d*) $\mathscr{A}^2 = 20$, (*e*,*f*) $\mathscr{A}^2 = 30$. Adapted from Dempsey *et al.* (2016).

streaky flow in a near-wall viscous layer of thickness $O(\epsilon^2)$. Within this layer the TS waves are governed by a system of unsteady 3-D linearized laminar boundary-layer equations with coefficients that depend on the streak-induced wall shear stress. Thus, the SSP maintaining the vortex/TS wave ECS may be summarized as follows: a streaky near-wall flow is unstable to oblique viscous TS waves whose inviscid extension into and nonlinear self-interaction within the core drives roll vortices that sustain the streaks – a mechanism that differs from the SSP articulated by Waleffe.

Employing the VWI equations, Dempsey *et al.* (2016) compute the asymptotic form of the vortex/TS wave ECS as a function of the spanwise wavenumber. The authors then compare their asymptotic solutions with ECS computed from the full NS equations at finite but large *Re*, demonstrating impressive agreement. As the wave amplitude is increased, the spanwise streak profile distorts from a gentle sinusoid to a strongly localized but still spanwise periodic structure (figure 1). In this regard, the authors' use of the term 'localized solution' differs from that commonly employed in modern dynamical systems studies to describe strictly isolated states in shear flows and convection. The issue is more than mere semantics. The physical mechanism responsible for the existence of localized ECS is long-wavelength modulational instability in bistable systems (Melnikov, Kreilos & Eckhardt 2014) rather than the nonlinear advective steepening that drives spanwise-periodic focusing in Dempsey *et al.*'s solutions. Nevertheless, the authors' speculation that this nonlinear steepening and the eventual loss of regularity of their VWI system could provide a mechanism for the formation of turbulent spots, widely associated with the breakdown of TS waves, is intriguing. Although the relevance of their solutions to transition may be questioned because $Re^{-1/7}$ must be small for the quantitative validity of their theory, it is difficult to imagine anything comparable to the level of mechanistic insight obtained by Dempsey *et al.* (2016) emerging from strictly computational studies.

3. Future

It would be of interest to use the VWI formulation employed by Dempsey *et al.* (2016) to investigate whether truly localized vortex/TS wave ECS are admitted by their reduced equations. More generally, asymptotically-simplified PDE models of shear flows should be useful for pattern formation studies of ECS in spatially-extended domains (Zhang *et al.* 2015). In particular, slow streamwise variability, crucial for studies of ECS in non-parallel flows, is naturally accommodated by VWI and related asymptotic theories (Hall & Smith 1991; Beaume *et al.* 2015). The discovery of turbulent superstructures, streamwise vortices and streaks extending in the downstream direction many multiples of the turbulent layer thickness, further highlights the potential utility of large-*Re* mathematical formalisms: a tantalizing possibility is that ECS associated with new SSPs may be operative across a hierarchy of spatio-temporal scales, including in the outer regions of turbulent wall flows at extremely large Reynolds number (Hwang & Cossu 2010).

Ultimately, there is a need for *a priori* reduced dynamical models of turbulent wall flows rather than a lexicon of ECS, no matter how complete. Here, too, the asymptotically reduced PDE framework may provide a path forward. For example, Julien & Knobloch (2007) have derived reduced PDE models of geophysical and astrophysical flows subjected to strong externally imposed restraints, enabling time-dependent simulations of turbulent flows in extreme parameter regimes that remain inaccessible to DNS. An outstanding open question is whether similar reduced models can be derived for large-*Re* wall flows that lack explicit external restraints but nevertheless clearly exhibit strongly anisotropic quasi-coherent flow structures.

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