Sixth Meeting, 10th May 1901.

J. W. BUTTERS, Esq., M.A., B.Sc., President, in the Chair.

The General Equation of a Geodesic on a Surface of Revolution applied to the Sphere.

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1. In attempting some work on geodesics on a spheroid, I was led to work out the geodesic on a sphere, and it may be interesting to see how the usual Spherical Trigonometry results arise from the general equation of a geodesic on a surface of revolution.

2. The general equation of a geodesic on a surface of revolution is

$$r^2\frac{d\phi}{ds} = \text{constant},$$

where r is the distance of a point on geodesic from the axis, and ϕ the angle of azimuth, or angle between the plane through the axis and this point and a fixed plane through the axis,

$\therefore$ on a sphere, taking θ as colatitude, ϕ as longitude, a radius of sphere, the equation becomes $a^2\sin^2\theta\frac{d\phi}{ds} = \text{constant} = a^2/k$, say,

$$\therefore \quad k\sin^2\theta\frac{d\phi}{ds} = 1,$$

$$\text{or} \quad a^2d\theta^2 + a^2\sin^2\theta d\phi^2 = k^2\sin^4\theta d\phi^2,$$

$$i.e. \quad d\phi^2\sin^2\theta\left(\frac{k^2}{a^2}\sin^2\theta - 1\right) = d\theta^2.$$

The direct way to evaluate the length of a geodesic would be to integrate this latter equation between θ and ϕ, then find a relation between s and θ from the equation

$$ds^2 = k^2\sin^2\theta d\theta^2\Big/\left(\frac{k^2}{a^2}\sin^2\theta - 1\right),$$

https://doi.org/10.1017/S0013091500032624 Published online by Cambridge University Press

taking in both as constants initial values θ_0, ϕ_0, and then eliminating k. This gives s in terms of θ and ϕ of points at either end of geodesic. This method is done in § 4, but I give first a more direct method of integration.

3.
$$\left(\frac{d\theta}{d\phi}\right)^2 = \sin^2\theta\left(\frac{k^2}{a^2}\sin^2\theta - 1\right),$$

$$\therefore \quad \frac{k^2}{a^2} = \frac{1}{\sin^4\theta}\left(\frac{d\theta}{d\phi}\right)^2 + \frac{1}{\sin^2\theta}.$$

Differentiate to get rid of k: after reduction, the equation takes the form
$$\sin\theta\frac{d^2\theta}{d\phi^2} - 2\cos\theta\left(\frac{d\theta}{d\phi}\right)^2 - \cos\theta\sin^2\theta = 0.$$

The solution is to be periodic in ϕ, $\therefore$ assume a solution of form

$$\sin\phi\frac{d\theta}{d\phi} = Q\cos\phi + R, \text{ where } Q, R \text{ are functions of } \theta.$$

$$\therefore \quad \sin\phi\frac{d^2\theta}{d\phi^2} + \cos\phi\frac{d\theta}{d\phi} = -Q\sin\phi + \cos\phi\frac{dQ}{d\theta}\cdot\frac{d\theta}{d\phi} + \frac{dR}{d\theta}\cdot\frac{d\theta}{d\phi};$$

$$\therefore \quad \sin\phi\frac{d^2\theta}{d\phi^2} - \frac{d\theta}{d\phi}\left\{\cos\phi\frac{dQ}{d\theta} + \frac{dR}{d\theta} - \cos\phi\right\} + Q\sin\phi = 0.$$

Our equation to be solved, if it agrees with this, is

$$\sin\theta\sin\phi\frac{d^2\theta}{d\phi^2} - 2\cos\theta\frac{d\theta}{d\phi}\{Q\cos\phi + R\} - \sin\phi\cos\theta\sin^2\theta = 0,$$

i.e.
$$\sin\phi\frac{d^2\theta}{d\phi^2} - \frac{d\theta}{d\phi}\{2Q\cot\theta\cos\phi + 2R\cot\theta\} - \sin\phi\sin\theta\cos\theta = 0,$$

and is to be same as

$$\sin\phi\frac{d^2\theta}{d\phi^2} - \frac{d\theta}{d\phi}\left\{\left(\frac{dQ}{d\theta} - 1\right)\cos\phi + \frac{dR}{d\theta}\right\} + Q\sin\phi = 0;$$

$$\therefore \quad Q\sin\phi = -\sin\phi\sin\theta\cos\theta, \quad \therefore \quad Q = -\sin\theta\cos\theta,$$

$$\text{and} \quad \frac{dQ}{d\theta} - 1 = 2Q\cot\theta, \quad \text{and} \quad \frac{dR}{d\theta} = 2R\cot\theta.$$

$$Q = -\sin\theta\cos\theta, \quad \therefore \quad \frac{dQ}{d\theta} = -\cos 2\theta, \quad \therefore \quad \frac{dQ}{d\theta} - 1 = -2\cos^2\theta = 2Q\cot\theta,$$

agreeing.

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If $\frac{dR}{d\theta} = 2R\cot\theta$, $R = c\sin^2\theta$, where c is a constant;

$\therefore$ the solution of the equation is

$$\sin\phi\frac{d\theta}{d\phi} = -\sin\theta\cos\theta\cos\phi + c\sin^2\theta.$$

Measure for convenience ϕ from starting point of geodesic,

$$\therefore \quad \text{when } \phi = 0, \; \theta = \theta_0 \quad \therefore \quad 0 = -\sin\theta_0\cos\theta_0 + c\sin^2\theta_0,$$

$$\therefore \quad c = \cot\theta_0\,;$$

$\therefore$ the solution is $\sin\theta_0\sin\phi\frac{d\theta}{d\phi} = \sin\theta[\cos\theta_0\sin\theta - \sin\theta_0\cos\theta\cos\phi]$

$$= P\sin\theta, \text{ say.}$$

Now $\left(\frac{ds}{d\phi}\right)^2 = a^2\sin^2\theta + a^2\left(\frac{d\theta}{d\phi}\right)^2$; $\therefore$ if $s = a\sigma$,

$$\sin^2\theta_0\sin^2\phi\left(\frac{d\sigma}{d\phi}\right)^2 = \sin^2\theta_0\sin^2\theta\sin^2\phi + P^2\sin^2\theta\,;$$

$$\therefore \quad \frac{d\sigma}{d\phi} = \frac{\sin\theta(P^2 + \sin^2\theta_0\sin^2\phi)}{\sin\theta_0\sin\phi\sqrt{P^2 + \sin^2\theta_0\sin^2\phi}}\,;$$

but $P^2\sin\theta + \sin^2\theta_0\sin\theta\sin^2\phi = P\sin\theta_0\sin\phi\frac{d\theta}{d\phi} + \sin^2\theta_0\sin\theta\sin^2\phi$;

$$\therefore \quad \frac{d\sigma}{d\phi} = \frac{P\frac{d\theta}{d\phi} + \sin\theta_0\sin\theta\sin\phi}{\sqrt{P^2 + \sin^2\theta_0\sin^2\phi}}.$$

Now $P^2 + \sin^2\theta_0\sin^2\phi + (\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi)^2 = 1$,

and $\frac{d}{d\phi}\{\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi\} = -\sin\theta_0\sin\theta\sin\phi - P\frac{d\theta}{d\phi}$;

$\therefore$ integrating, $\sigma = \cos^{-1}\{\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi\} + \text{constant}$,

and putting $\sigma = 0$ when $\theta = \theta_0$, $\phi = \phi_0$, we get this constant $= 0$,

$$\therefore \quad \cos\frac{s}{a} = \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi\,;$$

in general, the result will be

$$\cos\frac{s - s_0}{a} = \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\phi - \phi_0),$$

which is the Spherical Trigonometry result.

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4. Take now the method mentioned in § 2.

$$d\theta^2 = d\phi^2\sin^2\theta\left(\frac{k^2}{a^2}\sin^2\theta - 1\right) = d\phi^2\sin^2\theta(m^2\sin^2\theta - 1), \text{ where } m = \frac{k}{a}.$$

$$\therefore \quad d\phi = \frac{d\theta\operatorname{cosec}\theta}{\sqrt{m^2 - \operatorname{cosec}^2\theta}} = -\frac{dv}{\sqrt{m^2 - 1 - v^2}}, \text{ where } v = \cot\theta,$$

$$\therefore \quad \phi = \cos^{-1}\frac{v}{\sqrt{m^2 - 1}} + \text{constant}.$$

$$\therefore \quad \phi - \phi_0 = \cos^{-1}\frac{\cot\theta}{\sqrt{m^2 - 1}} - \cos^{-1}\frac{\cot\theta_0}{\sqrt{m^2 - 1}}.$$

$$ds^2 = k^2\sin^2\theta d\theta^2 \Big/ \left(\frac{k^2}{a^2}\sin^2\theta - 1\right);$$

$$\therefore \quad d\sigma = \frac{m\sin\theta\, d\theta}{\sqrt{m^2\sin^2\theta - 1}} = -\frac{md\xi}{\sqrt{m^2 - 1 - m^2\xi^2}}, \text{ where } \xi = \cos\theta,$$

$$\therefore \quad \sigma = \cos^{-1}\frac{m\xi}{\sqrt{m^2 - 1}} + \text{constant};$$

$$\therefore \quad \sigma - \sigma_0 = \cos^{-1}\frac{m\cos\theta}{\sqrt{m^2 - 1}} - \cos^{-1}\frac{m\cos\theta_0}{\sqrt{m^2 - 1}}.$$

$$\therefore \cos(\sigma - \sigma_0) = \frac{m^2\cos\theta\cos\theta_0}{m^2 - 1} + \frac{1}{m^2 - 1}\sqrt{(m^2\sin^2\theta - 1)(m^2\sin^2\theta_0 - 1)},$$

$$\text{and } \cos(\phi - \phi_0) = \frac{\cot\theta\cot\theta_0}{m^2 - 1} + \frac{1}{m^2 - 1}\sqrt{(m^2 - \operatorname{cosec}^2\theta)(m^2 - \operatorname{cosec}^2\theta_0)}.$$

$$\therefore \cos(\sigma - \sigma_0) = \cos\frac{s - s_0}{a} = \sin\theta\sin\theta_0\cos(\phi - \phi_0) + \cos\theta\cos\theta_0, \text{ as before.}$$

5. We can also find s in terms of $\phi - \phi_0$, and the inclinations of the tangents at the extremities of the arc to the meridian.

Let ζ be inclination at point where colatitude $= \theta$, and length of arc s, and a at point θ_0, s_0, then equation of the geodesic may be written

$$a^2\sin^2\theta\frac{d\phi}{ds} = \text{constant},$$

$$\textit{i.e.} \quad a\sin\theta\sin\zeta = \text{constant} = a\sin\theta_0\sin\alpha = a^2/k;$$

$$\therefore \quad \sin\theta\sin\zeta = \sin\theta_0\sin\alpha \text{ and } m\sin\theta_0\sin\alpha = 1.$$

Get equations for ϕ, s in terms of ζ, integrate, and then eliminate θ_0.

$$\sin\theta = \sin\theta_0 \sin\alpha \operatorname{cosec}\zeta\,;$$

$$\therefore \quad \cos\theta\frac{d\theta}{d\phi} = -\sin\theta_0\sin\alpha\cot\zeta\operatorname{cosec}\zeta\frac{d\zeta}{d\phi}\,;$$

$$\therefore \quad \sin^2\theta\cos^2\theta(m^2\sin^2\theta - 1) = \cos^2\theta\left(\frac{d\theta}{d\phi}\right)^2$$

$$= \sin^2\theta_0\sin^2\alpha\cot^2\zeta\operatorname{cosec}^2\zeta\left(\frac{d\zeta}{d\phi}\right)^2.$$

Substitute for $\sin\theta$, $\cos\theta$ in terms of ζ, and the equation becomes

$$\sin^2\zeta\left(\frac{d\zeta}{d\phi}\right)^2 = \sin^2\zeta - \sin^2\theta_0\sin^2\alpha\,;$$

$$\therefore \; d\phi = \frac{\sin\zeta\, d\zeta}{\sqrt{\sin^2\zeta - \sin^2\theta_0\sin^2\alpha}} = -\frac{d\eta}{\sqrt{1-\sin^2\theta_0\sin^2\alpha - \eta^2}}, \text{ where } \eta = \cos\zeta,$$

$$\therefore \quad \phi - \phi_0 = \cos^{-1}\frac{\cos\zeta}{\sqrt{1-\sin^2\theta_0\sin\alpha}} - \cos^{-1}\frac{\cos^2\alpha}{\sqrt{1-\sin^2\theta_0\sin^2\alpha}}$$

$$= \cos^{-1}\frac{m\cos\zeta}{\sqrt{m^2-1}} - \cos^{-1}\frac{m\cos\alpha}{\sqrt{m^2-1}}.$$

Now $d\sigma^2 = \sin^2\theta d\phi^2 + d\theta^2$,

$$\therefore \quad \left(\frac{d\sigma}{d\zeta}\right)^2 = \sin^2\theta\,.\,\frac{\sin^2\zeta}{\sin^2\zeta - \sin^2\theta_0\sin^2\alpha} + \frac{\sin^2\theta_0\sin^2\alpha\cot^2\zeta\operatorname{cosec}^2\zeta}{\cos^2\theta}$$

$$= \frac{\sin^2\theta_0\sin^2\alpha}{\sin^2\zeta(\sin^2\zeta - \sin^2\theta_0\sin^2\alpha)}, \quad \text{on reduction,}$$

$$\therefore \quad d\sigma = \frac{d\zeta}{\sin\zeta\sqrt{m^2\sin^2\zeta - 1}} = -\frac{dv}{\sqrt{m^2-1-v^2}}, \text{ where } v = \cot\zeta,$$

$$\therefore \quad \sigma - \sigma_0 = \cos^{-1}\frac{\cot\zeta}{\sqrt{m^2-1}} - \cos^{-1}\frac{\cot\alpha}{\sqrt{m^2-1}},$$

$$\therefore \cos(\sigma-\sigma_0) = \cos\frac{s-s_0}{a} = \frac{\cot\zeta\cot\alpha}{m^2-1} + \frac{1}{m^2-1}\sqrt{(m^2-\operatorname{cosec}^2\zeta)(m^2-\operatorname{cosec}^2\alpha)}.$$

$$\text{And} \quad \cos(\phi-\phi_0) = \frac{m^2\cos\zeta\cos\alpha}{m^2-1} + \frac{1}{m^2-1}\sqrt{(m^2\sin^2\zeta-1)(m^2\sin^2\alpha-1)},$$

$\therefore \quad \cos(\phi-\phi_0) = \cos\alpha\cos\zeta + \sin\alpha\sin\zeta\cos\frac{s-s_0}{a}$, the known result in Spherical Trigonometry from triangle with side $s - s_0$, and angles $\phi - \phi_0$, $\pi - \alpha$, ζ or $\phi - \phi_0$, α, $\pi - \zeta$.

https://doi.org/10.1017/S0013091500032624 Published online by Cambridge University Press