BASIC SEQUENCES IN F-SPACES AND THEIR APPLICATIONS

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1. Introduction

The aim of this paper is to establish a conjecture of Shapiro (10) that an F-space (complete metric linear space) with the Hahn-Banach Extension Property is locally convex. This result was proved by Shapiro for F-spaces with Schauder bases; other similar results have been obtained by Ribe (8). The method used in this paper is to establish the existence of basic sequences in most F-spaces.

It was originally stated by Banach that every B-space contains a basic sequence, and proofs have been given by Bessaga and Pelczynski (1), (2), Gelbaum (4) and Day (3). In (1) Bessaga and Pelczynski give a general method of construction in locally convex F-spaces, but we shall show in Section 3 that this construction can be modified to apply in any F-space $(X, \tau)$ on which there is a weaker vector topology $\rho$ such that $\tau$ has a base of $\rho$-closed neighbourhoods. The basic result of the paper is Theorem 3.2, and this is a natural generalisation of a locally convex version due to Bessaga and Mazur and given (essentially) in Pelczynski (6), (7).

In Section 4 we study the problem of existence of a basic sequence in an arbitrary F-space, and show that in fact repeated applications of Theorem 3.2 give a basic sequence in any F-space with a non-minimal topology. Since the only example we know of a minimal F-space is the space $a^\omega$ of all sequences (which has a basis) it seems likely that every F-space contains a basic sequence.

The results of Section 5 do not depend on Section 4; in this section are gathered together the applications of the existence theory of Section 3. We show that if $(X, \tau)$ is an F-space and $\rho \leq \tau$ is a topology defining the same closed linear subspaces as $\tau$, then $\rho$ and $\tau$ define the same bounded sets—a result familiar in locally convex theory. The Shapiro conjecture follows immediately. The final theorem is a generalisation of the Eberlein-Smulian theorem employing techniques developed by Pelczynski (7).

The author would like to thank Professor J. H. Shapiro for several helpful comments and supplying a copy of (8), and also the referee for pointing out some serious mistakes in the first draft of the paper.
An F-semi-norm  \( n \) on a vector space  \( X \) is a non-negative real-valued function defined on  \( X \) such that

(i) \( n(x + y) \leq n(x) + n(y) \).

(ii) \( n(tx) \leq |t| n(x) \quad |t| \leq 1 \),

(iii) \( \lim_{t \to 0} n(tx) = 0 \quad x \in X \).

If in addition  \( n(x) = 0 \) implies that  \( x = 0 \) then we call  \( n \) an F-norm. Any vector topology on  \( X \) may be defined by a collection of F-semi-norms; any metrisable topology may be defined by one F-norm. From this point, unless specifically stated, all vector topologies are assumed to be Hausdorff.

Now suppose  \((X, \rho)\) is a topological vector space and  \( \tau \) is a vector topology on  \( X \); we shall say that  \( \tau \) is \( \rho \)-polar if  \( \tau \) has a base of neighbourhoods which are \( \rho \)-closed.

**Proposition 2.1.** If  \( \tau \) is \( \rho \)-polar then  \( \tau \) may be defined by a collection of F-semi-norms  \((\eta_x: x \in A)\) of the form

\[ \eta_x(x) = \sup \{\lambda(x): \lambda \in \Lambda_x\} \]

where each  \( \Lambda_x \) is a collection of \( \rho \)-continuous F-semi-norms. If  \( \tau \) is metrisable then  \( \tau \) may be defined by one such F-norm.

**Proof.** Let  \((\gamma_x: x \in A)\) be a collection of F-semi-norms defining  \( \tau \) such that every \( \tau \)-neighbourhood of 0 contains a set \( \{x: \gamma_x(x) \leq \varepsilon\} \) for some  \( x \in A \) and  \( \varepsilon > 0 \); let  \( \Delta \) be the collection of all \( \rho \)-continuous F-semi-norms. We define  \( \Lambda_x \) to be the collection of F-semi-norms of the form

\[ \lambda^\delta(x) = \inf (\delta(y) + \gamma_x(z): y + z = x). \]

(Thus  \( \Lambda_x = \{\lambda^\delta: \delta \in \Delta_x\} \).) As  \( \lambda^\delta \leq \delta \) each  \( \lambda^\delta \) is \( \rho \)-continuous and an F-semi-norm \( (\lambda^\delta \leq \delta \) implies condition (iii) in particular). Now define

\[ \eta_x(x) = \sup (\lambda^\delta(x): \delta \in \Delta). \]

Clearly  \( \eta_x \leq \gamma_x \) and so is an F-semi-norm. Now if  \( U \) is a \( \tau \)-neighbourhood of 0 we may find  \( \delta \) and  \( \varepsilon > 0 \) such that if  \( x_0 \in \{x: \gamma_x(x) \leq \varepsilon\} \) (closure in  \( \rho \) then  \( x_0 \in U \). Suppose now  \( x_0 \in \{x: \eta_x(x) < \varepsilon\} \); then it is easy to show that  \( x_0 \in \{x: \gamma_x(x) \leq \varepsilon\} \) and so  \( (\eta_x: x \in A) \) defines  \( \tau \).

If  \( \tau \) is metrisable,  \( A \) may be taken to be a singleton and therefore  \( \tau \) may be defined by a single F-norm of the required type.

**Proposition 2.2.** Suppose  \((X, \tau)\) is an F-space (complete metric linear space) and suppose  \( \rho < \tau \) is a vector topology on  \( X \). Then

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(i) If the net \( x_a \to 0(\rho) \) but \( x_a \not\to 0(\tau) \), then there are vector topologies \( \alpha, \beta \) such that

(a) \( \rho \leq \alpha < \beta \leq \tau \);
(b) \( \beta \) is metrisable and \( \alpha \)-polar;
(c) \( x_a \to 0(\alpha) \) but \( x_a \not\to 0(\beta) \).

(ii) If \( U \) is a \( \tau \)-neighbourhood of 0 but not a \( \rho \)-neighbourhood then there are vector topologies \( \alpha, \beta \) satisfying (a), (b) and (c) such that \( U \) is a \( \beta \)-neighbourhood of 0 but not an \( \alpha \)-neighbourhood of 0.

(iii) If \( \tau \) is locally bounded then there is a topology \( \alpha \) such that \( \alpha < \tau \) but \( \tau \) is \( \alpha \)-polar.

Proof. (i) Let \( \alpha \) be the largest vector topology such that \( \rho \leq \alpha \leq \tau \) and \( x_a \to 0(\alpha) \) (it is easy to see that there is such a topology). Let \( \beta \) be the vector topology with a base of neighbourhoods consisting of the \( \alpha \)-closures of \( \tau \)-neighbourhoods of 0. Since \( \alpha \leq \tau \) it follows that \( \alpha \leq \beta \leq \tau \). If \( \alpha = \beta \) then the identity map \( i: (X, \alpha) \to (X, \tau) \) is almost continuous and so by the Closed Graph Theorem (cf. Kelley (5), p. 213) \( \alpha = \tau \) contrary to hypothesis on the net \( (x_a) \). Therefore \( \alpha < \beta \); clearly also since \( \tau \) is metrisable so is \( \beta \), and \( x_a \not\to 0(\beta) \).

(ii) (We are grateful to J. H. Shapiro for the following simplification of the original proof.) By an application of Zorn’s Lemma it may be shown that there is a maximal vector topology \( \alpha \) such that \( \rho \leq \alpha \leq \tau \) and \( U \) is not an \( \alpha \)-neighbourhood (we do not assert that \( \alpha \) is the largest such topology). Then proceed as in (i).

(iii) Follows from (ii) by considering a single bounded neighbourhood (\( \beta = \tau \)).

Two vector topologies on \( X \) will be called compatible if they define the same closed subspaces.

Proposition 2.3. Let \( \tau \) and \( \rho \) be compatible topologies on \( X \); they define the same continuous linear functionals.

Proof. \( f \) is \( \tau \)- or \( \rho \)-continuous according as its null space is \( \tau \)- or \( \rho \)-closed.

A sequence \( (x_n) \) in a topological vector space \( X \) is called a basis if every \( x \in X \) has a unique expansion in the form

\[
x = \sum_{i=1}^{\infty} t_{i}x_{i}.
\]

In this case we may define linear functionals \( f_{n} \) such that

\[
f_{n}(x) = t_{n}
\]

and linear operators \( S_{n} \) by

\[
S_{n}(x) = \sum_{i=1}^{n} t_{i}x_{i} = \sum_{i=1}^{n} f_{i}(x)x_{i}.
\]

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If $X$ is an $F$-space then it is well known (cf. (10), (12)) that each $f_n$ is necessarily continuous and the family $\{S_n\}$ is equicontinuous.

Suppose now that $X$ is metrisable but not necessarily complete; we shall call a sequence $(x_n)$ in $X$ a basic sequence if it is a basis for its closed linear span in the completion of $X$. We shall call $(x_n)$ a semi-basic sequence if we simply have $x_n \notin \overline{\text{lin}} \{x_{n+1}, x_{n+2}, \ldots\}$ for every $n$.

We now give a useful and elementary criterion for a sequence $(x_n)$ to be basic or semi-basic. Let $(x_n)$ be linearly independent and let $E$ be the linear span of $(x_n)$; then for $x \in E$

$$x = \sum_{i=1}^{\infty} t_i x_i$$

uniquely where $(t_i)$ is finitely non-zero. Define

$$f_n(x) = t_n$$

and

$$S_n x = \sum_{i=1}^{n} f_i(x)x_i,$$

where $S_n : E \to E$ is linear.

**Lemma 2.4.** (i) $(x_n)$ is semi-basic if and only if each $S_n$ is continuous or equivalently each $f_n$ is continuous.

(ii) $(x_n)$ is basic if and only if the family $\{S_n\}$ is equicontinuous.

**Proof.** (i) If $(x_n)$ is semi-basic, let $N_k$ be the null space of $f_k$; then $N_k$ is a maximal linear subspace of $E$. Then $N_1 = \overline{\text{lin}} \{x_i : i \geq 2\}$ and since $x_1 \notin N_1$, $N_1$ is closed and $f_1$ is continuous; while if $k \geq 2$,

$$N_k = \overline{\text{lin}} \{x_i : i \neq k\} = \overline{\text{lin}} \{x_i : i < k\} + \overline{\text{lin}} \{x_i : i > k\}.$$ 

Hence

$$N_k = \overline{\text{lin}} \{x_i : i < k\} + \overline{\text{lin}} \{x_i : i > k\},$$

since the former space is finite-dimensional. Suppose $x_k \in N_k$; then

$$x_k = \sum_{i=1}^{k-1} t_i x_i + y,$$

where $y \in \overline{\text{lin}} \{x_i : i > k\}$. Since $x_k \notin \overline{\text{lin}} \{x_i : i > k\}$ we conclude that there is a first index $l$ such that $t_l \neq 0$. Then we obtain $x_l \in \overline{\text{lin}} \{x_{l+1}, x_{l+2}, \ldots\}$ and a contradiction. Hence $x_k \notin N_k$ and by the maximality of $N_k$, $N_k$ is closed and $f_k$ is continuous.

The converse is trivial.

(ii) (Cf. Shapiro (12), Proposition C.)

It follows from the definition of basic sequence that if $(x_n)$ is basic then the family $\{S_n\}$ is equicontinuous (consider $(x_n)$ as a basis of its closed linear span in the completion of $X$). Conversely, $S_n(x) \to x$ for $x \in E$ and if the family is
equicontinuous \( S_n(x) \to x \) for \( x \in \bar{E} \) (closure in the completion of \( X \)), and it easily follows that \( (x_n) \) is a basis for \( \bar{E} \).

3. Construction of basic sequences

**Lemma 3.1.** Let \( E \) be a finite-dimensional space and suppose \( V \) is a closed balanced subset of \( E \). If \( V \) intersects every one-dimensional subspace of \( E \) in a bounded set then \( V \) is bounded.

**Proof.** We may suppose \( E \) is normed; suppose \( x_n \in V \) and \( \| x_n \| \to \infty \). Then by selecting a subsequence we may suppose \( \| x_n^{-1}x_n \to z \) where \( \| z \| = 1 \). Then for any \( N \) there is an \( m \) such that for \( n \geq m, \) \( \| x_n \| \geq N \) and

\[
\| x_n^{-1}x_n \in \| z \|^{-1} V \cap N^{-1} V.
\]

Therefore \( z \in N^{-1} V \) for all \( N \) and hence \( V = \text{lin} \{ z \} \).

**Theorem 3.2.** Suppose \( (X, \tau) \) is a metric linear space and \( \rho \) is a vector topology on \( X \) such that \( \tau \) is \( \rho \)-polar. Suppose \( (x_n) \) is a net such that \( x_n \to 0(\rho) \) but \( x_n \not \to 0(\tau) \); suppose \( z_1 \neq 0 \in X \). Then there is a sequence \( (a(k): k \geq 2) \) such that

\[
a(k + 1) > a(k)
\]

for all \( k \geq 2 \) and the sequence \( (z_n)_{n=1}^\infty \) is a basic sequence where \( z_n = x_{a(n)}n \geq 2 \).

**Proof.** We may suppose (Proposition 2.1) that \( (X, \tau) \) is normed by an \( F \)-norm \( \| . \| \) such that

\[
\| x \| = \sup (\lambda(x): \lambda \in \Lambda),
\]

where \( \Lambda \) is a collection of \( \rho \)-continuous \( F \)-norms. Let \( \theta > 0 \) be chosen such that

(i) \( \| z_1 \| \geq 4\theta \).

(ii) For all \( a, 3a \geq a \) such that \( \| x_a \| \geq 4\theta \).

Let \( V = \{ x: \| x \| \leq \theta \} \); then \( V \cap \text{lin} \{ z_1 \} \) is compact (since \( \| z_1 \| \geq 4\theta \)). We shall construct the sequence \( (a(n): n \geq 2) \) by induction so that if

\[
E_n = \text{lin} (z_1, x_{a(2)}, ..., x_{a(n)})
\]

then \( E_n \cap V \) is compact.

Suppose \( \{ a(2), ..., a(n) \} \) have been chosen (this set can be empty at the first step, the selection of \( a(2) \)) and let \( E_n = \text{lin} (z_1, x_{a(2)}, ..., x_{a(n)}) \). By the inductive hypothesis \( V \cap E_n \) is compact.

For \( 1 \leq k \leq 2^{n+3} \) let

\[
W_k^n = \{ x: \| x \| = k \cdot 2^{-(n+3)} \} \cap E_n.
\]

Each \( W_k^n \) is compact and so we may choose finite subsets \( U_k^n \) so that for \( w \in W_k^n \) there exists \( u \in U_k^n \) with

\[
\| w - u \| \leq 2^{-(n+3)} \theta.
\]

Let \( U^n = \bigcup_{k=1}^{2^{n+3}} U_k^n \), and for \( u \in U^n \) choose \( \lambda_u \in \Lambda \) so that

\[
\lambda_u(u) \geq \| u \| - 2^{-(n+3)} \theta.
\]
Then choose \( b > a(n) \) so that if \( c \geq b \) then
\[
\lambda_d(x_c) \leq 2^{-(n+3)} \theta
\]  
(2)

for \( u \in U^n \) (possible since \( U^n \) is finite and \( x_n \to 0(\rho) \)).

Choose a subnet \((x_d : d \in D)\) of \((x_c : c \geq b)\) such that \( \| x_d \| \geq 4 \theta \), and suppose for every such \( x_d \) the set \( V \cap \text{lin}(E_n, x_d) \) is unbounded. By Lemma 3.1, for every \( d \) there exists \( t_d x_d + u_d \neq 0 \) where \( u_d \in E_n \) such that the linear span of \((t_d x_d + u_d)\) is contained in \( V \). Clearly \( u_d \neq 0 \) and so we may normalize in such a way that \( \| u_d \| = \theta \) (since \( V \cap E_n \) is compact). Then
\[
\| t_d x_d \| \leq \| t_d x_d + u_d \| + \| u_d \|
\]
\[
\leq 2 \theta
\]

so that \( |t_d| \leq 1 \). Hence since \( x_d \to 0(\rho) \), \( t_d x_d \to 0 \) in \( (\rho) \). By selection again of a subnet we may suppose \( u_d \to u \) in \( E_n \) (since \( V \cap E_n \) is compact) and \( \| u \| = \theta \). Then for any \( t \in \mathbb{R} \)
\[
\| tu \| \leq \lim inf \| t(t_d x_d + u_d) \|
\]
\[
\leq \theta
\]

so that \( \text{lin} \{ t \} \subset V \cap E_n \), a contradiction.

Hence we may choose \( a(n+1) \geq b \) such that \( \| x_{a(n+1)} \| \geq 4 \theta \) and \( V \cap E_{n+1} \) is compact. This completes the construction of \( a(n) \); now let \( z_n = x_{a(n)} \), \( n \geq 2 \). It remains to establish that by using (1) and (2) \((z_n)\) is a basic sequence.

For convenience we shall replace \( \| \cdot \| \) by an equivalent \( F\)-norm \( \| \cdot \|^* \) given by
\[
\| x \|^* = \min (\| x \|, \theta).
\]

We next show that if \( t_1, \ldots, t_{n+1} \) is a scalar sequence
\[
\left\| \sum_{i=1}^{n+1} t_i z_i \right\|^* \geq \left\| \sum_{i=1}^{n} t_i z_i \right\|^* - 2^{-(n+1)} \theta.
\]  
(3)

Choose the greatest integer \( k \) such that
\[
\left\| \sum_{i=1}^{n} t_i z_i \right\|^* \geq k \cdot 2^{-(n+3)} \theta.
\]

Then \( 0 \leq k \leq 2^{n+3} \); if \( k = 0 \) there is nothing to prove. If \( k \geq 1 \) then we may choose a scalar \( s \) with \( |s| \leq 1 \) such that
\[
\left\| \sum_{i=1}^{n} st_i z_i \right\| = k \cdot 2^{-(n+3)} \theta.
\]

Then choose \( u \in U^*_k \) so that
\[
\left\| u - \sum_{i=1}^{n} st_i z_i \right\| \leq 2^{-(n+3)} \theta.
\]
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If $|st_{n+1}| \leq 1$ then

$$
\| u + st_{n+1}z_{n+1} \| \geq \lambda_u(u) - \lambda_u(z_{n+1})
\geq (k-2)2^{-(n+3)\theta}
$$

by (1) and (2). If $|st_{n+1}| \geq 1$ then

$$
\| u + st_{n+1}z_{n+1} \| \geq \| z_{n+1} \| - \| u \|
\geq 3\theta \geq (k-2)2^{-(n+3)\theta}.
$$

Hence

$$
\left\| s \sum_{i=1}^{n+1} t_iz_i \right\| \geq (k-2)2^{-(n+3)\theta} - 2^{-(n+3)\theta}
= (k-3)2^{-(n+3)\theta}
\geq \left\| \sum_{i=1}^{n} t_iz_i \right\| - 2^{-(n+1)\theta}.
$$

Hence since $|s| \leq 1$

$$
\left\| \sum_{i=1}^{n+1} t_iz_i \right\| \geq \left\| \sum_{i=1}^{n} t_iz_i \right\| - 2^{-(n+1)\theta}
$$
and (3) follows.

From (3) it is clear that $(z_n)$ is linearly independent for if

$$
\sum_{i=1}^{n_1} t_iz_i \geq \theta
$$
then

$$
\left\| \sum_{i=1}^{n+1} t_iz_i \right\| \geq \frac{1}{2}\theta;
$$
thus if $\sum_{i=1}^{n+1} t_iz_i = 0$, then for every $s$, $s \left\| \sum_{i=1}^{n} t_iz_i \right\| \leq \theta$

and so since $V \cap E_n$ is compact, $\sum_{i=1}^{n} t_iz_i = 0$. Let $E$ be the linear span of $\{z_n\}$
and define $S_k$ by

$$
S_k \left( \sum_{i=1}^{\infty} t_iz_i \right) = \sum_{i=1}^{k} t_iz_i
$$
where $(t_i)$ is finitely non-zero. Then by (3)

$$
\| S_{n+k}x \| \geq \| S_nx \| - 2^{-n}\theta
$$
and therefore for $x \in E$ and $n \geq 1$

$$
\| x \| \geq \| S_nx \| - 2^{-n}\theta.
$$
Suppose $\| x_m \| \to 0$ but $\| S_kx_m \| \nrightarrow 0$; then since $V \cap E_k$ is compact we may, by
selecting a subsequence and multiplying by a bounded sequence of scalars, suppose that $\| S_kx_m \| = \theta$. Thus $\| x_m \| \geq \frac{1}{2}\theta > 0$, and we have a contradiction. Thus each $S_k$ is continuous.

To establish equicontinuity of $\{S_m: m \geq 1\}$ we must show that if $p(m)$ is any sequence and $x_m \to 0$ then $S_{p(m)}x_m \to 0$. Suppose not; then we may suppose

for all $m$. Then

$$
\| x_m \| \geq \gamma - 2^{-p(m)}\theta
$$
and as \( \| x_m \| \to 0 \) we conclude that \( p(m) \) is bounded. But then we may select a constant subsequence and this contradicts the continuity of each \( S_n \). Thus by Lemma 2.4 we have established the theorem.

**Corollary 3.3.** Under the assumptions of Theorem 3.2 suppose \( \mu \) is a pseudo-metrisable topology on \( X \) such that \( \mu \leq \rho \). Then \( (z_n) \) may be chosen so that \( z_n \to 0(\mu) \).

An examination of the proof of Theorem 3.2 reveals that we can insist that \( \eta(z_n) \to 0 \) for any single \( \rho \)-continuous \( F \)-semi-norm.

**Corollary 3.4.** Suppose that \( (X, \tau) \) is an \( F \)-space and that \( \rho \) is a vector topology on \( X \) with \( \rho < \tau \). Suppose \( x_\alpha \to 0(\rho) \) but \( x_\alpha \to 0(\tau) \), and that \( z_1 \in X \). Then there is a sequence \( a(k) \) so that \( a(k+1) > a(k) \) \( k \geq 2 \) and such that the sequence \( (z_n) \) is a semi-basic sequence where \( z_n = x_\alpha(n)n \geq 2 \).

**Proof.** Proposition 2.2 combined with Theorem 3.2 establishes that we may choose \( (z_n) \) to be a basic sequence in a weaker topology than \( \tau \). This clearly implies that \( (z_n) \) is at least a semi-basic sequence in \( (X, \tau) \).

### 4. Existence of basic sequences

In this section we consider the question of whether an \( F \)-space need possess a basic sequence. The results we obtain will not be used in Section 5, and this section may be omitted. We shall call a topological vector space \((E, \tau)\) minimal if for every Hausdorff vector topology \( \rho \leq \tau \) we have \( \rho = \tau \). It is well known that \( \omega \) is minimal if we restrict to locally convex topologies.

**Proposition 4.1.** \( \omega \) is a minimal \( F \)-space.

**Proof.** Suppose \( \rho \) is a weaker vector topology on \( \omega \) and \( x_\alpha \to 0(\rho) \) but \( \| x_\alpha \| \geq \theta \) (where \( \| \cdot \| \) is an \( F \)-norm determining the topology of \( \omega \)). Then there is a sequence \( (z_n) \), with \( \| z_n \| \geq \theta \), which is a basic sequence for some weaker Hausdorff vector topology on \( \omega \) (Proof of 3.4). Let \( E \) be the closed linear span of \( (z_n) \) in the original topology, then \( E \cong \omega \). However, the dual functionals of \( (z_n) \) induce on \( E \) a weaker Hausdorff locally convex topology. It follows that \( z_n \to 0 \) contrary to assumption.

We do not know any other examples of minimal \( F \)-spaces; their existence is crucial to the problem of basic sequences in view of the following theorem.

**Theorem 4.2.** Every non-minimal \( F \)-space contains a basic sequence.

Before proceeding to the proof of Theorem 4.2 we first prove a stability theorem for basic sequences similar to a locally convex version given by Weill (13) (cf. also Shapiro (11), p. 1085). A sequence in a topological vector space is regular if it is bounded away from zero.
Lemma 4.3. Suppose $X$ is an $F$-space and $(x_n)$ is a regular basic sequence. Suppose $\sum \| u_n \| < \infty$, and let $y_n = x_n + u_n$. If whenever
\[
\sum_{n=1}^{\infty} t_n y_n = 0
\]
then $t_n = 0$, then $(y_n)$ is also a basic sequence.

Proof. Define a map $S: l_{\infty} \to X$ by
\[
S(t) = \sum_{n=1}^{\infty} t_n u_n.
\]
Since $\sum \| u_n \| < \infty$, $S$ is well defined and $S$ is continuous by the Banach-Steinhaus Theorem. Now suppose $(t^{(n)})$ is a sequence in $l_{\infty}$ such that
\[
\lim_{n \to \infty} t^{(n)} = 0
\]
and
\[
\sup_n \| t^{(n)} \|_{\infty} < \infty
\]
Then it is easy to verify that $\| S(t^{(n)}) \| \to 0$.

Let $E$ be the closed linear span of $\{x_n\}$ and suppose $f_n \in E'$ is the bi-orthogonal sequence. For $x \in E$, $\lim_{n \to \infty} f_n(x) = 0$, since $(x_n)$ is regular. We define $R: E \to c_0$ by $R(x) = (f_n(x))$; $R$ is continuous by the Closed Graph Theorem. Hence the map $T: E \to X$ defined by $T = I + SR$ is also continuous. Since $T$ takes the form
\[
T(x) = \sum_{n=1}^{\infty} f_n(x) y_n.
\]
$T$ is injective. Now suppose $(z_n) \subset E$ is a sequence such that $\| T(z_n) \| \to 0$; suppose $\| z_n \| > \varepsilon > 0$. We suppose at first
\[
\lim_{n \to \infty} \| R(z_n) \|_{\infty} < \infty.
\]
Then by selecting a subsequence we may suppose $R(z_n) \to t$ co-ordinatewise in $l_{\infty}$ and hence
\[
S(R(z_n)) \to S(t) \text{ in } X.
\]
Now
\[
z_n = T(z_n) - S(R(z_n)) \to -S(t).
\]
Therefore $S(t) \in E$ and
\[
R(z_n) + RS(t) \to 0 \text{ in } l_{\infty}.
\]
i.e.
\[
t + RS(t) = 0
\]
\[
S(t) + SRS(t) = 0
\]
\[
T(S(t)) = 0
\]
\[
S(t) = 0
\]
and so
\[
\lim_{n \to \infty} z_n = 0
\]
contrary to assumption. It follows that no subsequence of \( \| Rz_n \|_\infty \) is bounded.

If, on the contrary, \( \| Rz_n \|_\infty \to \infty \), then we may consider \( \| Rz_n \|_\infty^{-1} z_n \) and obtain a similar contradiction. We establish that for such a sequence \( \| Rz_n \|_\infty^{-1} z_n \to 0 \) and hence \( \| Rz_n \|_\infty Rz_n \to 0 \) in \( l_\infty \), which is a contradiction. Hence \( T \) is an isomorphism on to its image, and as \( Tx_n = y_n, (y_n) \) is a basic sequence.

**Proof of Theorem 4.2.** Let \( U_n \) be a base of neighbourhoods of 0 in \( (X, \tau) \).

We may assume, without loss of generality, that \( U_1 \) is not a neighbourhood of 0 in some weaker vector topology. By Proposition 2.2 there are vector topologies \( \alpha, \beta \) in \( X \) such that \( \alpha < \beta \leq \tau \), \( \beta \) is metrisable and \( \alpha \)-polar and \( U_1 \) is a \( \beta \)-neighbourhood. Then by Theorem 3.2 there is a basic sequence \( (w^{(1)}_k) \) in \( (X, \beta) \). Then let \( E_1 \) be the \( \tau \)-closed linear hull of the sequence \( (w^{(1)}_k) \) and let \( F_1 \) be the linear span; let \( \gamma_1 = \beta \). Then by induction we construct sequences \( (h^{(n)}_k), E_n, F_n, \gamma_n \) such that \( F_n = \text{lin} \{ w^{(n)}_k: k = 1, 2, \ldots \} \), \( E_n \) is the \( \tau \)-closure of \( F_n \) and \( \gamma_n \) is a metrisable vector topology on \( E_n \) such that \( (w^{(n)}_k: k = 1, 2, \ldots) \) is a basis of \( (E_n, \gamma_n) \). Furthermore

(i) \( (w^{(n)}_k) \) is block basic with respect to \( (w^{(n-1)}_k) \) for \( n \geq 2 \), i.e. \( w^{(n)}_k \) takes the form

\[
 w^{(n)}_k = \sum_{j=1}^{p_k} c_{kj} w^{(n-1)}_j,
\]

where \( p_0 = 0 < p_1 < p_2, \ldots \). Thus \( F_n \subset F_{n-1} \) for \( n \geq 2 \) and \( E_n \subset E_{n-1} \) \( n \geq 2 \).

(ii) The topology \( \gamma_n \) on \( E_n \) is finer than \( \gamma_{n-1} \) restricted to \( E_n \) for \( n \geq 2 \), and coarser than \( \tau \).

(iii) \( U_n \cap E_n \) is a \( \gamma_n \)-neighbourhood of 0.

We now describe the inductive construction; suppose \( (w^{(n)}_k), E_n, F_n \) and \( \gamma_n \) have been chosen. If \( U_{n+1} \cap E_n \) is a \( \gamma_n \)-neighbourhood of 0 then let \( \gamma_{n+1} = \gamma_n \) and \( w^{(n+1)}_k = w^{(n)}_k \) for all \( k \). Otherwise by Proposition 2.2 we may find topologies \( \alpha \) and \( \gamma_{n+1} \) on \( E_n \) such that \( \gamma_n \leq \alpha \leq \gamma_{n+1} \leq \tau \), \( \gamma_{n+1} \) is \( \alpha \)-polar and metrisable and \( U_{n+1} \cap E_n \) is a \( \gamma_{n+1} \)-neighbourhood of 0 but not an \( \alpha \)-neighbourhood.

Since \( F_n \) is \( \tau \)-dense in \( E_n \), \( F_n \) is also \( \gamma_{n+1} \)-dense and hence \( \alpha \leq \gamma_{n+1} \) on \( F_n \).

Thus by Corollary 3.3 we may determine a \( \gamma_{n+1} \)-regular basic sequence \( (z_k) \) in \( F_n \) such that \( z_k \to 0(\gamma_n) \). Thus

\[
 z_k = \sum_{i=1}^{q(k)} c_{ki} w^{(n)}_i,
\]

where \( \lim_{k \to \infty} c_{ki} = 0 \) for each \( i \) (since the co-ordinate functionals for \( (w^{(n)}_k) \) are \( \gamma_n \)-continuous). It follows easily that we may find a subsequence \( (y_k) \) and a block basic sequence \( (w^{(n+1)}_k) \) such that \( \sum_k \| y_k - w^{(n+1)}_k \|_{n+1} < \infty \) where \( \| . \|_{n+1} \) is an \( F \)-norm determining \( \gamma_{n+1} \). If

\[
 \sum_{k=1}^{\infty} t_k w^{(n+1)}_k = 0 \quad (\gamma_{n+1})
\]


then

\[ \sum_{k=1}^{\infty} t_k w_k^{(n+1)} = 0 \quad (y_n) \]

and thus since the co-ordinate functionals for \( w_k^{(n)} \) are \( \gamma_n \)-continuous \( t_k = 0 \) for all \( k \). Thus \((w_k^{(n+1)})\) is a \( \gamma_{n+1} \)-basic sequence, and we proceed by letting \( F_{n+1} = \text{lin} \{w_k^{(n)}\} \), \( E_{n+1} = F_{n+1} \) (in \( \tau \)). This completes the inductive construction.

Finally take the "diagonal sequence"

\[ v_n = w_n^{(n)}. \]

Then for each \( n \), \((v_k: k \geq n)\) is block basic with respect to \((w_k^{(n)})\). In particular \((v_k)\) is block basic with respect to \((w_k^{(1)})\) and hence there are \( \gamma_1 \)-continuous linear functionals \((f_k)\) defined on \( \text{lin} \{v_k\} \) such that \( f_i(v_j) = \delta_{ij} \). These are then also \( \tau \)-continuous and extend to the closed linear span \( H \) of \( \{v_k\} \). Now suppose \( x \in H \); we show

\[ \sum_{i=1}^{\infty} f_i(x)v_i = x. \]

For any \( n \), \((v_k: k \geq n)\) is a basic sequence in \((E_n, \gamma_n)\); let

\[ R_n(x) = x - \sum_{i=1}^{n-1} f_i(x)v_i. \]

Then \( R_n(x)\) is in the \( \gamma_n \)-closure of \( \{v_k: k \geq n\} \). Thus \( R_n(x)\) is in \( E_n \) and in the \( \gamma_n \)-closure of \( \text{lin} \{v_k: k \geq n\} \). Therefore

\[ R_n(x) = \sum_{i=n}^{\infty} f_i(x)v_i \quad (y_n) \]

and so for some \( N \) and all \( m \geq N \),

\[ R_n(x) - \sum_{i=n}^{m} f_i(x)v_i \in U_n, \]

and

\[ x - \sum_{i=1}^{m} f_i(x)v_i \in U_n. \]

Thus \( x = \sum_{i=1}^{\infty} f_i(x)v_i \) for \( x \in H \), and \((v_i)\) is a basic sequence.

If \( E \) is a minimal \( F \)-space, then \( E \) may still possess a basic sequence (see Proposition 4.1). The author does not know if every \( F \)-space must possess a basic sequence.

**Theorem 4.4.** Let \((X, \tau)\) be an \( F \)-space; the following are equivalent:

(i) \( X \) contains no basic sequence.

(ii) Every closed subspace of \( X \) with a separating dual is finite-dimensional.
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Proof. Clearly (ii)⇒(i) so we have to show (i)⇒(ii). If $E$ is a subspace of $X$ with a separating dual, then the weak topology $\sigma$ on $E$ is weaker than $\tau$. If $E$ is infinite-dimensional, then by Theorem 4.2 $\sigma = \tau$. But in this case $E \cong \omega$, and so has a basis. Therefore, $E$ is finite-dimensional.

5. Applications

We now can apply basic sequences or rather semi-basic sequences to derive many results familiar in locally convex theory.

Theorem 5.1.

(i) Let $(X, \tau)$ be an F-space and suppose $\rho \leq \tau$ is a vector topology on $X$ compatible with $\tau$. Then every $\rho$-bounded set is $\tau$-bounded.

(ii) Suppose $X$ is a vector space and $\rho \leq \tau$ are two vector topologies on $X$ such that $\rho$ and $\tau$ are compatible and $\tau$ is $\rho$-polar. Then any $\rho$-bounded set is $\tau$-bounded.

Proof. (i) It is enough to show that if $x_n \to 0(\rho)$ and $c_n$ is a sequence of scalars such that $c_n \to 0$ then $c_n x_n \to 0(\tau)$. Suppose $x_n \to 0(\rho)$; then choose $x_0 \neq 0$. For $c_n \to 0$, $c_n \neq 0$,

$$c_n(x_n + x_0) \to 0(\rho).$$

Suppose $c_n(x_n + x_0) \to 0(\tau)$; then by Corollary 3.4, there is a semi-basic sequence $(z_n)$ with $z_1 = x_0$ and

$$z_n = c_{m_n}(x_{m_n} + x_0) \quad (n \geq 2),$$

where $(m_n)$ is an increasing sequence of integers. Then

$$c_{m_n}^{-1}z_n \to x_0(\rho)$$

and hence $x_0$ is in the $\rho$-closure of $\text{lin}\{z_n: n \geq 2\}$. Thus $x_0$ is also in the $\tau$-closure of $\text{lin}\{z_n: n \geq 2\}$, contradicting the fact that $(z_n)$ is a semi-basic sequence. Thus since $c_n x_0 \to 0$, $c_n x_n \to 0(\tau)$.

The proof of (ii) is somewhat similar; let $\eta$ be a $\rho$-lower-semi-continuous $\tau$-continuous $F$-semi-norm and let $N = \{x: \eta(x) = 0\}$. Then $X/N$ is metrisable under $\eta$ and may be given the quotient topology $\hat{\rho}$ of $\rho$ ($N$ is $\rho$-closed). Every $\eta$-closed subspace of $X/N$ is $\hat{\rho}$-closed and so an argument similar to (i) may be employed.

Corollary 5.2. Suppose $(X, \tau)$ is an F-space and $\rho \leq \tau$ is a metrisable vector topology compatible with $\tau$. Then $\rho = \tau$.

Corollary 5.3. Let $(X, \tau)$ be an F-space with the Hahn-Banach Extension Property. Then $X$ is locally convex.

Proof. Let $\sigma$ be the weak topology on $N$; then $\sigma \leq \tau$ and $\sigma$ and $\tau$ are compatible by the HBEP. For suppose $Y$ is a $\tau$-closed subspace and $x \notin Y$; then
by HBEP there is a continuous linear functional $\phi$ such that $\phi(Y) = 0$ and $\phi(x) = 1$. Let $\mu$ be the associated Mackey topology; then (see Shapiro (10), Proposition 3) $\sigma \leq \mu \leq \tau$ and $\mu$ is metrisable. Hence by Corollary 5.2 $\mu = \tau$ and $\tau$ is locally convex.

**Corollary 5.4.** Suppose $(X, \tau)$ is an $F$-space and $\rho \leq \tau$ is a vector topology compatible with $\tau$. Then $\tau$ is $\rho$-polar.

**Proof.** Let $\gamma$ be the topology induced by the $\rho$-closures of $\tau$-neighbourhoods of 0; then $\rho \leq \gamma \leq \tau$ and $\gamma$ is metrisable. Hence by 5.2, $\gamma = \tau$.

**Theorem 5.5.** Let $(X, \tau)$ be an $F$-space and let $(x_n)$ be a basis of $X$ in a compatible topology $\rho \leq \tau$. Then $(x_n)$ is a basis of $X$.

**Proof.** By the previous corollary we may assume that $\tau$ is defined by a $\rho$-lower-semi-continuous $F$-norm $\| \cdot \|$ (see Proposition 2.1). Each $x \in X$ may be expanded in the form

$$x = \sum_{i=1}^{\infty} f_i(x_i)$$

(the linear functionals $f_n$ are not necessarily $\rho$-continuous). Now for each $x \in X$, the sequence $\left( \sum_{i=1}^{n} f_i(x_i) \right)$ is $\rho$- and therefore $\tau$-bounded (Theorem 5.1) and so we may define

$$\| x \|_* = \sup \left\| \sum_{i=1}^{n} f_i(x_i) \right\|.$$

Then $\lim_{t \to 0} \| tx \|_* = 0$ since $\lim_{t \to 0} t y = 0$ uniformly for $y$ in a bounded set; hence $\| \cdot \|_*$ is an $F$-norm on $X$. Clearly also $\| x \|_* \geq \| x \|$ by the $\rho$-lower-semi-continuity of $\| \cdot \|$.

It remains to establish that $(X, \| \cdot \|_*)$ is complete and then by the Closed-Graph Theorem it will follow that $\| \cdot \|_*$ and $\| \cdot \|$ are equivalent. Let $(y_n)$ be a $\| \cdot \|_*$-Cauchy sequence; then since $\| y_n - y_m \| \leq \| y_n - y_m \|_*$ for all $m, n$, $(y_n)$ is $\tau$-convergent to $y$ say. Further, it can be seen that the sequences

$$\left( \sum_{i=1}^{m} f_i(y_n) x_i \right)$$

are $\tau$-convergent uniformly in $m$; clearly $\lim_{n \to \infty} f_i(y_n) = t_i$ exists and

$$\lim_{n \to \infty} \sum_{i=1}^{m} f_i(y_n) x_i = \sum_{i=1}^{m} t_i x_i$$

uniformly in $m$ for the topology $\tau$. Thus working in the weaker topology $\rho$

$$\lim_{m \to \infty} \sum_{i=1}^{m} t_i x_i = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=1}^{m} f_i(y_n) x_i = y.$$
(The limits are interchangeable by uniform convergence.) Therefore it follows that
\[ \lim_{n \to \infty} \sum_{i=1}^{m} f_i(y_n) x_i = \sum_{i=1}^{m} f_i(y) x_i (r) \]
uniformly in \( m \) and that \( \| y - y_n \| \to 0 \). Hence \( \| . \| \) and \( \| . \|^* \) are equivalent, and by an application of Lemma 2.4, \((x_n)\) is a basic sequence in \((X, \| . \|)\). By the compatibility of \( \rho, (x_n) \) is a basis of \( X \).

Shapiro (12) proves that the Weak Basis Theorem fails in any non-locally convex locally bounded \( F \)-space. With regard to this theorem we establish that a weaker version of the Weak Basis Theorem holds always.

**Proposition 5.6.** Let \((x_n)\) be a weak basis of \((X, \tau)\), where \((X, \tau)\) is an \( F \)-space with a separating dual. Then the associated linear functionals \( \{\gamma_i\} \) are continuous.

**Proof.** Let \( \sigma, (x_n) \) be the (metrisable) Mackey topology. Then \((X, \sigma, (x_n))\) is barrelled, for if \( C \) is a \( \mu \)-barrel then \( C \) is \( \tau \)-closed and by the Baire Category Theorem we may show \( C \) has \( \tau \)-interior. It follows easily that \( C \) is a \( \tau \)-neighbourhood of \( 0 \) and thus a \( \mu \)-neighbourhood ((10), Proposition 3).

Now let \( \| . \|_n \) be a sequence of semi-norms defining \( \mu \) and let
\[ \| x \|_n^* = \sup_m \left\| \sum_{i=1}^{m} f_i(x) x_i \right\|_n \]
(finite, since \( \mu \) and \( \sigma \) have the same bounded sets). Let \( \mu^* \) be the topology induced by the sequence \( \| . \|_n^* \) and let \( \hat{X} \) be the \( \mu^* \)-completion of \( X \). Consider the identity map \( i: (X, \mu) \to (\hat{X}, \mu^*) \). Suppose \( z_n \in X, z_n \to z (\mu) \) and \( z_n \to z (\mu^*) \).

Then \( \left\{ \sum_{i=1}^{m} f_i(z_n) x_i \right\}_{n=1}^{\infty} \) is uniformly \( \mu \)-Cauchy for \( m = 1, 2, \ldots \); thus in the topology \( \sigma \leq \mu \)
\[ \lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=1}^{m} f_i(z_n) x_i = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=1}^{m} f_i(z_n) x_i \]
and we conclude
\[ \lim_{n \to \infty} f_i(z_n) = t_i \] exists for each \( i \)
and
\[ \lim_{n \to \infty} z_n = z = \sum_{i=1}^{\infty} t_i x_i \] in \( \sigma \).

Thus \( f_i(z) = t_i \) and therefore
\[ \lim_{n \to \infty} \sum_{i=1}^{m} f_i(z_n - z) x_i = 0 \] \( \mu \)-uniformly in \( m \).

Hence \( z_n \to z \) in \((X, \mu^*)\) and \( i \) has Closed Graph. By the Closed Graph Theorem ((9), p. 116), since \((\hat{X}, \mu^*)\) is complete and metric, \( \mu \geq \mu^* \) and it follows easily that each \( f_n \) is \( \mu \) and hence \( \tau \)-continuous.

The idea of the next theorem is due to Pelczynski (7).
Theorem 5.7. Let \((X, \tau)\) be an \(F\)-space and suppose \(\rho \leq \tau\) is a compatible vector topology. Let \(K\) be a subset of \(X\); then the following are equivalent

(i) \(K\) is \(\rho\)-compact,
(ii) \(K\) is \(\rho\)-sequentially compact,
(iii) \(K\) is \(\rho\)-countably compact.

Proof. (i)\(\Rightarrow\)(iii) and (ii)\(\Rightarrow\)(iii) are well known. Let \(\|\cdot\|\) be an \(F\)-norm determining \(\tau\); by Corollary 5.4 we may suppose \(\|\cdot\|\) is \(\rho\)-lower-semi-continuous.

(iii)\(\Rightarrow\)(i). It is easy to see that \(K\) is \(\rho\)-precompact; we show that \(K\) is also \(\rho\)-complete. Let \(\langle X, \rho \rangle\) be the \(\rho\)-completion of \(X\) and let \(Y \subset \langle X, \rho \rangle\) be the vector space of all \(y \in X\) such that there is a \(\rho\)-bounded net \(x_a \in X\) such that \(x_a \rightarrow y\). By Theorem 5.1 a \(\rho\)-bounded net is \(\tau\)-bounded. Let \(B_1 = \{x \in X: \|x\| \geq \lambda\}\); then for \(y \in Y\) we define

\[
\|y\|_* = \inf \{\lambda: y \in B_\lambda, \text{closure in } \rho\}.
\]

Let \(y \in Y\) and suppose \(x_a\) is a \(\tau\)-bounded net converging to \(y\) in \(\rho\); then

\[
\|y\|_* \leq \sup_{x} \|x\| < \infty
\]

and

\[
\lim_{t \to 0} \|ty\|_* = \lim_{t \to 0} \sup_{x} \|tx\| = 0
\]

since the net \(\{x_a\}\) is bounded (cf. Theorem 5.5). It follows without difficulty that \(\|\cdot\|_*\) is an \(F\)-semi-norm on \(Y\), and that \(\|\cdot\|_*\) is \(\rho\)-lower-semi-continuous; also from the definition, \(\|x\| = \|x\|_*\) for \(x \in X\), since each \(B_1\) is \(\rho\)-closed. Next if \(y \in Y\) and \(\|y\|_* = 0\) then for each \(\lambda > 0\) and \(V\) a neighbourhood of \(0\) in \(\langle X, \rho \rangle\) we may find \(x_{\lambda, V} \in X\) such that \(x_{\lambda, V} - y \in V\) and \(\|x_{\lambda, V}\| \leq \lambda\). The set \(\{(\lambda, V): \lambda > 0, V \text{ a } \rho\text{-neighbourhood of } 0\}\) is directed in the obvious way [(\(\lambda, V\) \(\geq\) \((\lambda', V')\) if and only if \(\lambda \leq \lambda'\) and \(V \subset V'\)]; then the net \(x_{\lambda, V}\) converges to \(0\) in \((X, \tau)\) and \(x_{\lambda, V} \rightarrow 0\) in \((X, \rho)\). However \(x_{\lambda, V} \rightarrow y\) in \(\langle X, \rho \rangle\) and so \(y = 0\)

Thus \(Y\) is a metrisable vector space under \(\|\cdot\|_*\) and \(\|\cdot\|_*\) is \(\rho\)-lower-semi-continuous.

Now suppose \(x_a \in K\) is a \(\rho\)-Cauchy net; then \(x_a \rightarrow y\) in \((\langle X, \rho \rangle, y)\) and \(y \in Y\). Suppose at first \(\|x_a - y\|_* \rightarrow 0\); then by the completeness of \((X, \tau)\) \(y \in X\), and there is a sequence \((a(n))\) such that \(x_{a(n)} \rightarrow y(\tau)\). Thus \(y\) is the sole \(\rho\)-cluster point of \(\{x_{a(n)}\}\) in \(X\); since \(K\) is countably compact, \(y \in K\), and \(x_a \rightarrow y\) in \((K, \rho)\).

Now suppose \(\|x_a - y\|_* \rightarrow 0\) and that \(y \notin X\); since \(y \neq 0\) we may suppose \(x_a \notin V\) for all \(a\), where \(V\) is a \(\rho\)-neighbourhood of \(0\). Then by Theorem 3.2 there is a basic sequence \((z_n)\) in \((Y, \|\cdot\|_*)\) such that:

(i) \(z_1 = y\).
(ii) \(z_n = w_n - y, n \geq 2\) where \(w_n = x_{a(n)}\) for some increasing sequence.
(iii) \(\inf \|z_n\|_* > 0\).
Let $Z$ be the closed linear span of $\{z_n\}_{n=1}^{\infty}$ and let $W$ be the closed linear span of $\{w_n\}_{n=1}^{\infty}$. Since $z_1 \notin X$ and $W \subseteq X$, $W$ is a closed subspace of co-dimension one in $Z$. Let $\phi$ be the continuous linear functional on $(Z, \|\cdot\|_*)$ such that $\phi(z_1) = 1$ and $\phi(W) = 0$; we define $A: Z \to Z$ by $A(z) = z - \phi(z)z_1$. Then for $n \geq 2$

$$A z_n = A w_n - A z_1$$

Similarly define $B: Z \to Z$ by

$$B \left( \sum_{i=1}^{\infty} t_i z_i \right) = \sum_{i=2}^{\infty} t_i z_i.$$ 

Then

$$B w_n = B(z_1 + z_n)$$

It follows that $B A z_n = z_n$, $n \geq 2$ and hence that $A$ is an isomorphism of $\overline{\text{lin}} \{z_n: n \geq 2\}$ on to its image. In particular $(w_n: n \geq 2)$ is a basic sequence in $(X, \|\cdot\|)$. However $w_n \in K$ for $n \geq 2$, and so $(w_n)$ possesses a $\rho$-cluster point. Now suppose $w_0$ is a $\rho$-cluster point; then $w_0$ is in the $\tau$-closed linear span of $(w_n)$ by compatibility. It follows that

$$w_0 = \sum_{i=2}^{\infty} \psi_i(w_0) w_i,$$

where $\psi_i$ is the dual sequence of $\tau$-continuous linear functionals on $W$. Each $\psi_i$ is also $\rho$-continuous by compatibility and hence

$$\psi_i(w_0) = 0 \quad i \geq 2.$$ 

Therefore $w_0 = 0$. This contradicts the original choice of $x_n \notin V$, where $V$ is a $\rho$-neighbourhood of 0. Thus we have a contradiction.

Finally suppose $\|x_\rho - y\| \to 0$ and $y \in X$; determine the basic sequence $(z_n: n \geq 2)$ satisfying (ii)-(iii). In this case if $w_0$ is a $\rho$-cluster point of $(w_n: n \geq 2)$ then $w_0 - y$ is a $\rho$-cluster point of $(z_n: n \geq 2)$. Since $w_0 - y \in X$ and $z_n \in X$ we conclude that $w_0 - y$ is in the $\tau$-closed linear span of $\{z_n: n \geq 2\}$ by compatibility and it follows as usual that $w_0 - y = 0$. Hence $y \in K$. We conclude that any $\rho$-Cauchy net converges in $K$ and so $K$ is complete and therefore compact.

(iii)⇒(ii). Let $(x_n)$ be a sequence in $K$ and let $x_0$ be a $\rho$-cluster point. Then there is a net $(z_\alpha)$ in $K$ such that each $z_\alpha$ is some $x_n$ and $z_\alpha \to x_0$ $(\rho)$. If $z_\alpha \to x_0$ in $\tau$ then there is nothing to prove, as it will follow that some subsequence of $(x_n)$ converges to $x_0$. Otherwise we may find a basic sequence $(u_n)$ of the form $u_n = z_{\alpha(n)} - x_0$. Let $w$ be a $\rho$-cluster point of $(z_{\alpha(n)})$ in $K$; then clearly $w - x_0 \in \overline{\text{lin}} \{u_n\}$ and since $\tau$ and $\rho$ are compatible it follows as in (iii)⇒(i) that $w - x_0 = 0$. Hence $x_0$ is the sole cluster point of $(z_{\alpha(n)})$ and so $z_{\alpha(n)} \to x_0$. However $z_{\alpha(n)}$ is simply a subsequence of $(x_n)$ $(\alpha(n) \to \infty$ since the $z_{\alpha(n)}$ are distinct).
BASIC SEQUENCES IN $F$-SPACES

[ADDED IN PROOF: The problem of determining conditions under which the Hahn-Banach Extension Property is equivalent to local convexity was originally posed by Duren, Romberg and Shields (14) p.59.]

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