ON HAUSDORFF DIMENSION FOR ATTRACTORS OF
ITERATED FUNCTION SYSTEMS

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Abstract

A conjecture on the Hausdorff dimension for Markov attractors of disjoint hyperbolic iterated
function systems was given by Ellis and Branton. This paper proves the conjecture and generalizes
the result to more general cases.


1. Introduction

In [2] Ellis and Branton have discussed the Hausdorff dimension of attractors of
disjoint hyperbolic iterated function systems. The main results of [2] are

THEOREM A. (Ellis and Branton) Let $A$ be the attractor of a disjoint hyperbolic
iterated function system $(X; T_1, \ldots, T_n)$. Suppose that

$$s_i d(x, y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x, y) \quad \forall x, y \in X, \quad 1 \leq i \leq n$$

for some constants $0 < s_i \leq \bar{s}_i < 1$. Then

$$l \leq \dim(A) \leq u,$$

where $\sum_{i=1}^n s_i^l = 1$ and $\sum_{i=1}^n \bar{s}_i^u = 1$. 

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216
THEOREM B. (Ellis and Branton) Assume that \((X; T_1, \ldots, T_n)\) is the same as in Theorem A, and \(M\) is a primitive Markov transition matrix, \(A_M\) is the Markov attractor of the iterated function system associated with \(M\). Then

\[ \dim(A_M) \leq u, \]

where

\[ \|MS^u\| = 1, \]

and

\[ S = \begin{pmatrix} \tilde{s}_1 & 0 \\ \vdots & \ddots \\ 0 & \tilde{s}_n \end{pmatrix}. \]

For the lower bound of \(\dim(A_M)\), they gave the following conjecture:

CONJECTURE. Under the same assumptions as Theorem B, we have

\[ \dim(A_M) \geq l, \]

where

\[ \|MS^l\| = 1, \]

and

\[ S = \begin{pmatrix} s_1 & 0 \\ \vdots & \ddots \\ 0 & s_n \end{pmatrix}. \]

Ellis and Branton showed that the conjecture is true in some special cases. In this paper we shall prove that the conjecture is true in the general case and extend the result in the case when \(M\) is not primitive and when not all \(T_i\)'s are contractions. The main result of this paper is

THEOREM. Suppose that \(M\) is a Markov transition matrix with at least one non-zero eigenvalue, and that \((X, T_1, \ldots, T_n)\) is a disjoint iterated function system such that

\[ s_id(x, y) \leq d(T_ix, T_iy) \leq \tilde{s}_id(x, y) \]

and

\[ 0 < s_i \leq \tilde{s}_i < 1. \]
Then we have
\[ l \leq \dim(A_M) \leq u, \]
where
\[ \|MS\| = 1 \quad \text{and} \quad \|M\tilde{S}^n\| = 1, \]
and \( A_M, S, \tilde{S} \) are as before.

In Section 2 we review the notions of iterated function systems and some results we will use. In Section 3 we will give the proof of the conjecture in the case \( M \) is irreducible. In the Section 4 we will generalize the result to the case \( M \) is reducible. In the last section we consider the situation when not all the \( T_i \)'s are contractions.

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2. Preliminaries

**Definition 1.** An iterated function system \((X; T_1, \ldots, T_n)\) is a compact metric space, \(X\), together with continuous maps \(T_i: X \mapsto X\).

We say \((X; T_1, \ldots, T_n)\) is hyperbolic if there exists a constant \(0 < s < 1\) such that
\[ d(T_ix, T_iy) \leq sd(x, y) \quad \forall x, y \in X, \quad 1 \leq i \leq n. \]

For a hyperbolic iterated function system \((X; T_1, \ldots, T_n)\), a subset \(A\) of \(X\) is called the attractor of the system if
(i) \( \emptyset \neq A \) is closed;
(ii) \( T_i(A) \subset A \), for \( 1 \leq i \leq n \);
(iii) \( A \) is minimal with respect to (i) and (ii).

Hutchinson proved that the attractor \(A\) exists for every hyperbolic iterated function system and \( A = \bigcup_{i=1}^{n} T_i(A) \). In addition \( \forall a \in A \), there exists a sequence \(i_1, i_2, \ldots\) such that
\[ \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x = a. \]

for all \( x \in X \) (see [5]).

If the attractor \(A\) of a hyperbolic iterated function system \((X; T_1, \ldots, T_n)\) satisfies \( T_i(A) \cap T_j(A) = \emptyset \) when \( i \neq j \), then the system is called disjoint.
EXAMPLE. Let 
\[ \Sigma^+_n = \{(i_1, i_2, \ldots) \mid 1 \leq i \leq n\} \]
and define maps \( \sigma_i : \Sigma^+_n \mapsto \Sigma^+_n \) by
\[ \sigma_i(i_1, i_2, \ldots) = (i, i_1, i_2, \ldots), \quad 1 \leq i \leq n. \]
If we define a metric, \( d \), on \( \Sigma^+_n \) by
\[ d(i, j) = 2^{-k} \quad \text{when} \quad i_1 = j_1, \ldots, i_k = j_k; \quad i_{k+1} \neq j_{k+1}, \]
where \( i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots) \). Then \( ((\Sigma^+_n, d); \sigma_1, \ldots, \sigma_n) \) is a disjoint hyperbolic iterated function system satisfying
\[ d(\sigma_i(i), \sigma_i(j)) = \frac{1}{2} d(i, j) \quad 1 \leq i \leq n, \]
with \( A = (\Sigma^+_n, d) \) as its attractor. By Theorem A we know that
\[ \dim((\Sigma^+_n, d)) = \frac{\log n}{\log 2}. \]

DEFINITION 2. An \( n \times n \) matrix \( M \) is called a Markov transition matrix if all of its entries are 1 or 0.

We say a sequence (finite or infinite) \( i_1, i_2, \ldots \) is \( M \)-admissible, if
\[ M_{i_j i_{j+1}} = 1 \]
for all \( j = 1, 2, \ldots \), and where \( i_j \in \{1, 2, \ldots, n\} \).

Let 
\[ \Sigma^+_M = \{(i_1, i_2, \ldots) \mid (i_1, i_2, \ldots) \text{ is } M\text{-admissible}\}. \]
Then, under the metric \( d \) defined above, \( \Sigma^+_M \) is a closed, therefore compact, subspace of \( (\Sigma^+_n, d) \).

DEFINITION 3. A non-negative square matrix \( M \) (all entries of \( M \) are non-negative, written \( M \geq 0 \)) is called primitive if \( M^k > 0 \) (all entries > 0) for some positive integer \( k \).

DEFINITION 4. An \( n \times n \) matrix \( M \) is called reducible if there is a permutation that puts it into the form
\[ \tilde{M} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \]
where \( M_{11} \) and \( M_{22} \) are square matrices. Otherwise \( M \) is called irreducible.
Obviously, a primitive matrix is irreducible. For an irreducible non-negative matrix $M$, we have the following Frobenius Theorem (see [4]).

**Theorem.** (Frobenius) An irreducible non-negative matrix $M$ always has a positive eigenvalue $\lambda$. The moduli of all the other eigenvalues do not exceed $\lambda$. And there is an eigenvector associated to $\lambda$ with all positive coordinates.

In this paper we use $\|M\|$ to denote the maximal modulus of eigenvalues of $M$.

**Definition 5.** Let $(X; T_1, \ldots, T_n)$ be a hyperbolic iterated function system with attractor $A$, and let $M$ be a Markov transition matrix. We say that a point $a \in A$ is $M$-attractive, if there exists an $M$-admissible sequence $i_1, i_2, \ldots$ such that

$$a = \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x$$

for all $x \in X$. The set of all $M$-attractive points of $A$, denoted as $A_M$, is called the Markov attractor of the system associated with $M$.

Let $B_i = A_M \cap T_i(A) = \{a | a = \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x, (i, i_2, \ldots, i_m, \ldots) \in \Sigma_M^+\}$. Then we have

$$B_i = \bigcup_{M_{ij}=1} T_i(B_j).$$

For $((\Sigma^+_n, d); \sigma_1, \ldots, \sigma_n), (\Sigma^+_M, d)$ is the Markov attractor. If $M$ is a primitive Markov transition matrix, it is shown in [2] that

$$\dim((\Sigma^+_M, d)) = \frac{\log \|M\|}{\log 2}.$$

**3. Proof of the conjecture**

In this section we give a proof of the conjecture when $M$ is irreducible. That is, we will prove

**Theorem 1.** Suppose that $(X; T_1, \ldots, T_n)$ is a disjoint hyperbolic iterated function system satisfying

$$s_i d(x, y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x, y) \quad \forall x, y \in X, \quad 1 \leq i \leq n,$$
where $0 < s_i \leq \hat{s}_i < 1$, $M$ is an irreducible Markov transition matrix, and $A_M$ is the Markov attractor associated with $M$. Then

$$l \leq \dim(A_M) \leq u$$

where

$$\|MS'\| = 1 \quad \text{and} \quad \|M\hat{\sigma}\| = 1$$

and

$$S = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & s_n & 0 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} \hat{s}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \hat{s}_n & 0 \end{pmatrix}.$$

In order to prove Theorem 1, we need to consider $\Sigma_n^+$. Now we define another metric $d'$ on $\Sigma_n^+$ by

$$d'(i, j) = \begin{cases} (s_{i_1} \ldots s_{i_k})^l, & i_1 = j_1, \ldots, i_k = j_k; \ i_{k+1} \neq j_{k+1} \\ 0, & i_1 = j_1, i_2 = j_2, \ldots \\ 1, & i_1 \neq j_1 \end{cases}$$

where $i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots)$.

It is easy to see that $d'$ is a metric on $\Sigma_n^+$. Clearly, $((\Sigma_n^+, d'); \sigma_1, \ldots, \sigma_n)$ is also a disjoint hyperbolic iterated function system with attractor $(\Sigma_n^+, d')$. In fact, $(\Sigma_n^+, d')$ is a self-similar set with $\dim(\Sigma_n^+, d') = k$, where $k$ satisfies

$$s_1^{l_1} + s_2^{l_2} + \cdots + s_n^{l_k} = 1.$$

For a Markov transition matrix $M$, $(\Sigma_M^+, d')$ is also the Markov attractor of $((\Sigma_n^+, d'); \sigma_1, \ldots, \sigma_n)$. We want to prove

**PROPOSITION 1.** Suppose that $M$ is an irreducible Markov transition matrix. Then

$$\dim((\Sigma_M^+, d')) = 1.$$
where $d'[i_1, i_2, \ldots, i_k]$ is the diameter of the set $[i_1, i_2, \ldots, i_k]$. Clearly,

$$d'([i_1, i_2, \ldots, i_k]) \to 0 \quad \text{as} \quad k \to \infty.$$ 

So, if $\beta_k(1) < \infty$ for any $k$, then $\dim(\Sigma^+_M, d') \leq 1$. Now

$$\beta_k(1) = \sum_{i_1, \ldots, i_k \text{ admissible}} d'([i_1, \ldots, i_k]) = \sum_{i_1, \ldots, i_k \text{ admissible}} (s_{i_1} \ldots s_{i_k})' \leq c \sum_{i_1, \ldots, i_k \text{ admissible}} s'_{i_1} (M S')_{i_1 i_2} \ldots (M S')_{i_k-1 i_k} v_{i_k}$$

$$= c \sum_{i_1} s'_{i_1} \sum_{i_k} (M S')_{i_1 i_k} v_{i_k} = c \sum_{i_1} s'_{i_1} v_{i_1} \leq c \sum_{i=1}^n s'_{i} < \infty,$$

where $c = 1/\min_i \{v_i\}$.

Now we show that the set

$$\{ \beta(1) | \beta = \{B_j\} \text{ is a cover of } \Sigma^+_M \}$$

has a positive lower bound. Hence the 1-dimensional Hausdorff measure of $(\Sigma^+_M, d')$ is positive, therefore $\dim((\Sigma^+_M)) \geq 1$.

Since $(\Sigma^+_M, d')$ is compact, we need only consider finite covers. Suppose that $\beta = \{B_j, 1 \leq j \leq m\}$ is a cover of $\Sigma^+_M$. For any $B_j$, there exists $x, y \in B_j$, such that

$$d'(B_j) = d'(x, y) = (s_{i_1} s_{i_2} \ldots s_{i_j})'.$$

Hence for any $z \in B_j$, we have

$$d'(x, z) \leq d'(x, y),$$

so $z \in [i_1, \ldots, i_k]$, and we get

$$B_j \subset [i_1, \ldots, i_k].$$

It is reasonable to assume that each $B$ is a block $[i_1, \ldots, i_k]$ for some $i_1, \ldots, i_k$.

Let $t$ be the maximal length of a block in the cover $\beta = \{B_1, \ldots, B_m\}$. For each $B_j = [i_1, \ldots, i_k]$, consider the cover $\alpha_t(B_j)$ of $B_j$ by blocks of length $t$. Then

$$\alpha_t(B_j)(1) = \sum_{\nu_{j+1}, \ldots, \nu_t} d'([i_1, \ldots, i_k, \nu_{j+1}, \ldots, \nu_t])$$

$$= \sum_{\nu_{j+1}, \ldots, \nu_t} (s_{i_1} \ldots s_{i_k} s_{\nu_{j+1}} \ldots s_{\nu_t})', $$
where the sum \( \sum' \) is over admissible sequences with \( M_{i_{v_j+1}v_{i+1}} = 1 \). Thus

\[
\alpha_i(B_j)(1) = (s_{i_1} \ldots s_{i_{v_j}})^t \sum_{u_{v_j+1}, \ldots, u_v} (M_{i_{v_j+1}u_{v_j+1}} s_{u_{v_j+1}}^t) \cdots (M_{u_{i-1}v_i} s_{u_i}^t) v_i
\]

\[
\leq c(s_{i_1} \ldots s_{i_{v_j}})^t \sum_{u_{v_j+1}, \ldots, u_v} (M_{i_{v_j+1}u_{v_j+1}} s_{u_{v_j+1}}^t) \cdots (M_{u_{i-1}v_i} s_{u_i}^t) v_i
\]

\[
= c(s_{i_1} \ldots s_{i_{v_j}})^t \sum_v (M S^t)^{i_{v_j+1}-k} v_v
\]

\[
= c(s_{i_1} \ldots s_{i_{v_j}})^t v_{i_{v_j}} \leq c d'(B_i) .
\]

Hence

\[
\beta(1) = \sum_j d'(B_j) \geq c^{-1} \sum_j \alpha_i(B_j)(1)
\]

\[
= c^{-1} \sum_j \sum' (s_{i_1} \ldots s_{i_{v_j}} s_{u_{v_j+1}} \ldots s_{u_v})^t
\]

\[
\geq c^{-1} \sum_j \sum' (s_{i_1} \ldots s_{i_{v_j}} s_{u_{v_j+1}} \ldots s_{u_v})^t v_i v_j
\]

\[
\geq c^{-1} \sum_{i_{v_j+1}, \ldots, u_v} (s_{i_1} \ldots s_{i_{v_j}})^t v_i v_j
\]

\[
= c^{-1} \sum_{i, j} (M S^t)^{i_j} v_j
\]

\[
= c^{-1} \sum_i v_i = c^{-1} .
\]

Now we have proved Proposition 1.

If we use \( \bar{s}_i \) instead of \( s_i \) and \( u \) instead of \( l \), we obtain the same result.

For the proof of Theorem 1, we also need the following lemma (see [6]).

**Lemma.** Let \( X, Y \) be two metric space with a map \( f : X \mapsto Y \), and \( \delta, c \) be positive constants. Then we have

(a) if \( d(f(x), f(y)) \geq c d(x, y)^\delta \), then

\[
\dim(Y) \geq \frac{1}{\delta} \dim(X) ;
\]
(b) if $f(X) = Y$ and $d(f(x), f(y)) \leq cd(x, y)\delta$, then

$$\dim(Y) \leq \frac{1}{\delta} \dim(X).$$

Now we can prove Theorem 1.

**Proof of Theorem 1.** Define $f : \Sigma^+ \mapsto A_M$ by

$$f(i_1, i_2, \ldots) = \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x \quad \forall x \in X.$$  

Since the limit on the right-hand side is independent of $x$, $f$ is well defined. By disjointness we have

$$c = \inf \{ d(x, y) \mid x \in T_i(A), y \in T_j(A), i \neq j \} > 0.$$

For any $i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots) \in \Sigma^+$, we estimate the distance between $a = \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x$ and $b = \lim_{m \to \infty} T_{j_1} \circ T_{j_2} \circ \ldots \circ T_{j_m} x$. Suppose

$$d'(i, j) = (s_{i_1} s_{i_2} \ldots s_{i_k}),'$$

that is

$$i_t = j_t, \quad 1 \leq t \leq k; \quad i_{k+1} \neq j_{k+1}.$$

Let

$$a' = \lim_{m \to \infty} T_{i_1} \circ \ldots \circ T_{i_k} x \quad \text{and} \quad b' = \lim_{m \to \infty} T_{j_1} \circ \ldots \circ T_{j_m} x.$$

We have $a' \in T_{i_{k+1}}(A), b' \in T_{j_{k+1}}(A)$, so that

$$d(a', b') \geq c.$$

Moreover,

$$a = T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_k} a', \quad b = T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_k} b'.$$

Hence

$$d(a, b) \geq s_{i_1} s_{i_2} \ldots s_{i_k} \quad d'(i, j) \geq s_{i_1} s_{i_2} \ldots s_{i_k} c = c d'(i, j)^{-1}.$$  

Applying the first part of the lemma ($\delta = l^{-1}$) we know

$$\dim(A_M) \geq \frac{1}{l-1} \dim(\Sigma^+, d') = l.$$

On the other hand, using the second part of the lemma ($\delta = u^{-1}$) and the result of Proposition 1 when $\tilde{s}_i$ is instead of $s_i$ and $u$ instead of $l$, we obtain

$$\dim(A_M) \leq u.$$  

The proof is completed.
4. Generalization

Now we assume that $M$ is reducible. At first we suppose that

$$M = \begin{pmatrix} M_1 & M_{12} \\ 0 & M_2 \end{pmatrix}$$

where $M_1$ and $M_2$ are $m \times m$ and $(n - m) \times (n - m)$ irreducible matrices respectively.

If $M_{12} = 0$, we consider $(X; T_1, \ldots, T_m)$ and $(X; T_{m+1}, \ldots, T_n)$, and get

$$l_1 \leq \dim(A_{M_1}) \leq u_1 \quad \text{and} \quad l_2 \leq \dim(A_{M_2}) \leq u_2$$

where

$$\|M_1 S'_1\| = \|M_1 S'_1 \mid S'_1\| = 1 \quad \text{and} \quad \|M_2 S'_2\| = \|M_2 S'_2 \mid S'_2\| = 1$$

and

$$S_1 = \begin{pmatrix} s_1 & 0 \\ \vdots & \ddots & 0 \\ 0 & \ldots & s_m \end{pmatrix}, \quad S_2 = \begin{pmatrix} s_{m+1} & 0 \\ \vdots & \ddots \\ 0 & \ldots & s_n \end{pmatrix},$$

with similar definitions for $\tilde{S}_1$ and $\tilde{S}_2$.

Clearly,

$$A_M = A_{M_1} \cup A_{M_2} \quad \text{and} \quad l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, l_2\}, \quad u = \max\{u_1, u_2\}.$$ 

And we have

$$\|M S'_1\| = \max\{\|M_1 S'_1\|, \|M_2 S'_2\|\} = 1,$$

and

$$\|M \tilde{S}^u\| = \max\{\|M_1 \tilde{S}^u\|, \|M_2 \tilde{S}^u\|\} = 1.$$

If $M_{12} \neq 0$, the $M$-admissible sequence related to $M_{12}$ is

$$i_1, i_2, \ldots, i_k, j_{k+1}, j_{k+2}, \ldots$$

where $(i_1, \ldots, i_k)$ and $(j_{k+1}, j_{k+2}, \ldots)$ are $M_1$ and $M_2$ admissible respectively with $M_{i_kj_{k+1}} = 1$. Since the set

$$\{(i_1, i_2, \ldots, i_k) \mid (i_1, i_2, \ldots, i_k) \quad \text{is M-admissible, } k = 1, 2, \ldots\}$$
is countable, we denote the related composite maps as \( T(1), T(2), \ldots \). Using \( A_{M,12} \) to denote the set of all the \( M \)-attractive points related with \( M_{12} \), we have

\[
A_{M,12} = \bigcup_{i=1}^{\infty} T(i)(A_{M,2}).
\]

So

\[
dim(A_{M,12}) = \dim \left( \bigcup_{i=1}^{\infty} T(i)(A_{M,2}) \right)
\]

\[
= \sup_{1 \leq i < \infty} \{ \dim (T(i)(A_{M,2})) \} = \dim(A_{M,2}).
\]

But

\[
A_M = A_{M,1} \bigcup A_{M,2} \bigcup A_{M,12},
\]

hence

\[
dim(A_M) = \max \{ \dim(A_{M,1}), \dim(A_{M,2}) \}
\]

and

\[
1 \leq \dim(A_M) \leq u.
\]

Certainly

\[
\|MS^l\| = 1 \quad \text{and} \quad \|MS^u\| = 1.
\]

In general, a Markov transition matrix \( M \) with \( \|M\| > 0 \), can acquire the form

\[
\tilde{M} = \begin{pmatrix}
M_1 & \cdots & * \\
\vdots & \ddots & \vdots \\
0 & \cdots & M_k
\end{pmatrix}
\]

through permutation, where \( M_1, \ldots, M_{k-1} \) are irreducible, and \( M_k \) irreducible or of the form

\[
M_k = \begin{pmatrix}
0 & * \\
\vdots & \ddots \\
0 & 0
\end{pmatrix}.
\]

Without loss of generality, we can assume

\[
M = \begin{pmatrix}
M_1 & \cdots & * \\
\vdots & \ddots & \vdots \\
0 & \cdots & M_{k-1}
\end{pmatrix}.
\]
If $M_k$ is irreducible, from the above we know that

$$\dim(A_M) = \max\{\dim(A_{M_1}), \ldots, \dim(A_{M_k})\}.$$ 

Hence

$$l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, \ldots, l_k\} \quad \text{and} \quad u = \max\{u_1, \ldots, u_k\}$$

and $l_i, u_i$ have the same meaning as in the case $k = 2$.

In the case

$$M_k = \begin{pmatrix} 0 & \ast \\ \vdots & \ddots \\ 0 & 0 \end{pmatrix},$$

there is no infinite $M$-admissible related to $M_k$, so $A_{M_k} = \emptyset$. Since $\|M\| > 0$, we must have $k > 1$. Again we have

$$\dim(A_M) = \max\{\dim(A_{M_1}), \ldots, \dim(A_{M_{k-1}}), \dim(A_{M_k})\}$$

and

$$l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, \ldots, l_{k-1}\} \quad \text{and} \quad u = \max\{u_1, \ldots, u_{k-1}\}.$$ 

In both cases we can easily see

$$\|MS^l\| = 1 \quad \text{and} \quad \|MS^u\| = 1.$$

Now we have proved

**Theorem 2.** Suppose that $(X; T_1, \ldots, T_n)$ is a disjoint hyperbolic iterated function system, and $M$ is a Markov transition matrix with at least one non-zero eigenvalue. Then we have

$$l \leq \dim(A_M) \leq u$$

where

$$\|MS^l\| = 1 \quad \text{and} \quad \|MS^u\| = 1.$$
5. Not all $T_i$'s need to be contractions

Feiste showed in [3] that for an iterated function system $(X; T_1, \ldots, T_n)$, if it is cyclically contracting with respect to an irreducible Markov transition matrix $M$, the Markov attractor $A_M$ exists. We will show that the result of Theorem 2 holds in this case.

**Definition 6.** Suppose a Markov transition matrix $M$ is given. A *path* from $i_1$ to $i_k$ is a finite $M$-admissible sequence $i_1, i_2, \ldots, i_k$. A *cycle* is a path with $M_{i_ki_1} = 1$. By *elementary path* or *elementary cycle* we mean a path or a cycle for which $i_s \neq i_t$, when $s \neq t$.

**Definition 7.** Let $(X; T_1, \ldots, T_n)$ be an iterated function system, where $T_i$'s are Lipschitz maps with $\text{Lip}(T_i) = r_i$, and $M$ be a Markov transition matrix. $(X; T_1, \ldots, T_n)$ is called cyclically contracting if for any elementary cycle $i_1, \ldots, i_k$ we have $r_{i_1}r_{i_2}\ldots r_{i_k} < 1$.

In [3] we find the following theorem.

**Theorem.** (Feiste) Let $(X; T_1, \ldots, T_n)$ be an iterated function system, where $T_i$'s are Lipschitz maps. If $(X; T_1, \ldots, T_n)$ is cyclically contracting with respect to an irreducible Markov transition matrix $M$ then there is a unique $m$-tuple $B = (B_1, \ldots, B_n)$ of compact subsets $B_i \subseteq X$ with

$$B_i = \bigcup_{M_{ij}=1} T_j(B_j)$$

for all $i \in \{1, \ldots, n\}$.

Bandt proved in [1] that

$$B_i = \{a | a = \lim_{m \to \infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x; (i_1, i_2, \ldots, i_m, \ldots) \in \Sigma^+_M, x \in X\} .$$

Let $A_M = \bigcup_{i=1}^n B_i$. We also call it the Markov attractor of $(X; T_1, \ldots, T_n)$, though the attractor $A$ may not exist in this situation. If $B_i \cap B_j = \emptyset$ we also call $(X; T_1, \ldots, T_n)$ disjoint iterated function system.

Now we generalize Theorem 1 to the case $(X; T_1, \ldots, T_n)$ is cyclically contracting.

Suppose that $M$ is an irreducible Markov transition matrix, $\{s_1, s_2, \ldots, s_n\}$ is a group of positive constants such that for any elementary cycle $i_1, i_2, \ldots, i_k$ we
have $s_i, s_{i_2}, \ldots, s_{i_k} \leq 1$, and $l$ is a constant satisfying $\|MS^l\| = 1$. As in Section 2, we define a metric $d''$ on $\Sigma^+_M$ by

$$d''(i, j) = \begin{cases} (s_i \ldots s_{i_k})'v_i, & i_1 = j_1, \ldots, i_k = j_k; \quad i_{k+1} \neq j_{k+1} \\ \max\{s_i\} & i_1 \neq j_1, \end{cases}$$

where $i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots)$ are $M$-admissible sequences, and $v = (v_1, v_2, \ldots, v_n)'$ has the same meaning as in the proof of Proposition 1.

Suppose $i = (i_1, \ldots, i_k, i_{k+1}, \ldots), j = (i_1, \ldots, i_k, j_{k+1}, \ldots)$ and $t = (i_1, \ldots, i_r, t_{r+1}, \ldots)$ are $M$-admissible sequences with $i_{r+1} \neq t_{r+1}, i_{k+1} \neq j_{k+1}$ and $r < k$. We have

$$d''(i, t) = (s_i \ldots s_{i_k})'v_i = (s_i \ldots s_{i_k})' \sum_{v=1}^{n} (MS^l)'_{i,v} v_v \geq (s_i \ldots s_{i_k})'v_i = d''(i, j).$$

Hence $d''(i, j) \leq d''(i, t) + d''(t, j)$. Thus $d''$ is really a metric on $\Sigma^+_M$. But this time $d''$ is not a metric on $\Sigma^+_M$. In [1] it is shown that $(s_i \ldots s_{i_k})'v_i \to 0$, as $k \to \infty$. In the same way as in Section 2 we can also prove

**Proposition 2.** $\dim(\Sigma^+_M, d'') = 1$.

Using Lemma 2, with $\delta = l^{-1}$ for the first part, and $\delta = u^{-1}$ for the second, we obtain

**Theorem 3.** Let $M$ be an irreducible Markov transition matrix, and let $\{X; T_1, \ldots, T_n\}$ be a disjoint iterated function system cyclically contracting with respect to $M$ satisfying

$$s_i d(x, y) \leq d(T_i x, T_i y) \leq \tilde{s}_i d(x, y).$$

Then

$$l \leq \dim(A_M) \leq u$$

where $\|MS^l\| = 1$ and $\|M^{\tilde{s}_u}\| = 1$.

As in Section 3, we can generalise Theorem 3 to the case $\|M\| > 0$, where $M$ need not be irreducible. Finally we obtain
THEOREM 4. Suppose that $M$ is a Markov transition matrix with at least one non-zero eigenvalue, and that $(X; T_1, \ldots, T_n)$ is a disjoint iterated function system cyclically contracting with respect to $M$ satisfying
\[ s_i d(x, y) \leq d(T_i x, T_i y) \leq \tilde{s}_i d(x, y). \]
Then we have
\[ l \leq \dim(A_M) \leq u \]
where $\|M S^l\| = 1$ and $\|M S^u\| = 1$.

NOTE ADDED IN PROOF. The technique used in the proofs of Theorem 1 and 3 can also be applied to fractals constructed with sofic systems (see [1]). Let $(X; T_1, \ldots, T_n)$ be an iterated function system. Let $Q_i (i = 1, \ldots, m)$ be a non-empty subset of $\{1, \ldots, n\} \times \{1, \ldots, m\}$ and $F = \{(k, k_2, \ldots)\}$ there are $i_0, i_1, \ldots, \in \{1, \ldots, m\}$, with $(k, i_2) \in Q_{i_1 - 1}$. If $(X; T_1, \ldots, T_n)$ is cyclically contracting with $F$, there exist non-empty compact subsets $C_1, \ldots, C_n$ such that $C_i = \bigcup_{(k, j) \in Q_i} T_k(C_j)$ is a disjoint union. Construct an $m \times m$ matrix such that $M(r, \alpha)$ by letting $M_{ij} = \sum r_k \delta_{(k, j) \in Q_i}$, for $r = (r_1, \ldots, r_N)$ with $r_i > 0$. Assume $M(r, \alpha)$ is irreducible. Then there is a unique $\alpha$, such that $\|M(r, \alpha)\| = 1$, if $r_i = Lip(T_i)$.

THEOREM. Suppose $(X; T_1, \ldots, T_N)$ is cyclically contracting with $F$ and satisfies $s_i d(x, y) \leq d(T_i x, T_i y) \leq \tilde{s}_i d(x, y)$. Suppose for each $i$, $C_i = \bigcup_{(k, j) \in Q_i} T_k(C_j)$ is a disjoint union. Then $l \leq \dim(C_i) \leq u$, where $l$ and $u$ are determined by $\|M(s, l)\| = 1$ and $\|M(s, u)\| = 1$, and where $s = (s_1, s_2, \ldots, s_N)$, $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_N$.

Details of the proof of the theorem will appear in my Ph.D thesis.

References


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