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THE η-INVARIANTS OF CUSPED HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. In this paper, we define the η-invariant for a cusped hyperbolic 3-manifold and discuss some of its applications. Such an invariant detects the chirality of a hyperbolic knot or link and can be used to distinguish many links with homeomorphic complements.

0. **Introduction.** The η-invariant of the signature operator for a closed odd dimensional Riemannian manifold was introduced by Atiyah, Patodi and Singer in [1] in the 1970's. Since then much effort has been made toward generalizing the invariant to the cases of non-closed manifolds.

For a closed hyperbolic 3-manifold, the study of the η-invariant benefits from its complex-analytic relation with the hyperbolic volume of the manifold. Such a complex-analytic relation was first speculated by Thurston in [9]. A precise conjecture was formulated by Neumann and Zagier in [6] and later proved by Yoshida in [12]. By virtue of such a relation, Neumann and the first author in [3] obtained a Dehn-surgery formula of the η-invariant of a closed hyperbolic 3-manifold for sufficiently large surgery coefficients. Later in [7] the second author proved that such a formula is valid for all hyperbolic surgeries. The η-invariant is now almost as computable as the volume for a closed hyperbolic 3-manifold.

The purpose of this note is to define the η-invariant for a cusped hyperbolic 3-manifold and discuss some of its applications.

Our definition of the η-invariant depends on the choices of meridian-longitude pairs at the cusps. It differs by one-third of an integer when different choices are made. For a hyperbolic knot or link L in S^3, we define the η-invariant of L to be $\eta(S^3 - L)$ with the topologically standard choices of the meridians and longitudes for the components of L. It turns out that $\eta(L)$ is a well-defined link invariant for a hyperbolic knot or link. We show that such an invariant detects the chirality of a hyperbolic knot or link. It can also be used to distinguish many links with homeomorphic complements.

As for the case of a closed Riemannian 3-manifold, the η-invariant defined here can be thought of as a real-valued generalization of the Chern-Simons invariant of a cusped hyperbolic 3-manifold as defined by the first author in [2].

The rest of this note will be organized as follows. Section 1 presents some preliminary material necessary for our later discussions. In Section 2, we define the η-invariant for a

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cusped hyperbolic 3-manifold and then go on to explore some properties of the invariant. Section 3 exhibits a family of links with homeomorphic complements and different η-invariant. Finally in Section 4, we give a simplicial expression for the η-invariant of a cusped hyperbolic 3-manifold. As an example, we show how to calculate the invariants for the twisted Whitehead links.

1. **Preliminaries.** Let M be an oriented complete hyperbolic 3-manifold of finite volume with h cusps. Denote by $\mathcal{D}(M)$ the hyperbolic Dehn surgery space of M. For each $k = 1, \ldots, h$, fix a meridian-longitude basis $(\mathbf{m}_k, \mathbf{l}_k)$ for the homology of the torus T_k corresponding to a horospherical cross-section of cusp k. Let u_k (resp. v_k) be twice the logarithm of an eigenvalue of the holonomy of $\mathbf{m}_k$ (resp. $\mathbf{l}_k$). By [9] and [6], $\mathcal{D}(M)$ has complex dimension h and can be holomorphically parameterized by $\mathbf{u} = (u_1, \ldots, u_h) \in \mathbf{C}^h$ in a neighborhood of the origin in $\mathbf{C}^h$. Denote by $M_{\mathbf{u}}$ the manifold M with the hyperbolic structure parametrized by some $\mathbf{u} \in \mathcal{D}(M)$.

As shown in [12], there exist a link $\mathcal{L} \subset M$ and an orthonormal frame field $\mathcal{F} = (e_1, e_2, e_3)$ on $M - \mathcal{L}$ such that $\mathcal{F}$ has a special singularity at $\mathcal{L}$ and is homotopically linear in the cusps (we refer to [2] and [12] for the terminology). Let $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ be an orthonormal frame field defined on a neighborhood of each component of $\mathcal{L} \subset M$ such that κ_1 is tangent to $\mathcal{L}$ and has the same direction as e_1 near each component of $\mathcal{L}$. Both $\mathcal{F}$ and κ can be chosen so that they vary continuously with the analytic parameter $\mathbf{u}$, resulting in frame fields $\mathcal{F}_{\mathbf{u}}$ and $\kappa_{\mathbf{u}}$ in $M_{\mathbf{u}}$. Denote by $F(M_{\mathbf{u}})$ the oriented $SO(3)$ frame bundle of $M_{\mathbf{u}}$. Let (ω_i), (ω_{ij}), and (Ω_{ij}) be respectively the fundamental form, the connection form, and the curvature form of the hyperbolic metric connection on $F(M_{\mathbf{u}})$. Let C be the complex 3-form on $F(M_{\mathbf{u}})$ given by

$$C = \frac{1}{4\pi^2}\big(4\omega_1 \wedge \omega_2 \wedge \omega_3 - d(\omega_1 \wedge \omega_{23} + \omega_2 \wedge \omega_{31} + \omega_3 \wedge \omega_{12})\big) + \frac{i}{4\pi^2}(\omega_{12} \wedge \omega_{13} \wedge \omega_{23} + \omega_{12} \wedge \Omega_{12} + \omega_{13} \wedge \Omega_{13} + \omega_{23} \wedge \Omega_{23})$$

and define

$$f\big(\mathbf{u}; (\mathcal{L}, \mathcal{F}, \kappa)\big) = \int_{s(M_{\mathbf{u}} - \mathcal{L})} C - \frac{1}{2\pi} \sum_{K \subseteq \mathcal{L}} \int_{s'(K)} (\omega_1 - i\omega_{23}) \tag{1}$$

where $s \colon M_{\mathbf{u}} - \mathcal{L} \longrightarrow F(M_{\mathbf{u}})$ and $s' \colon \mathcal{L} \longrightarrow F(M_{\mathbf{u}})$ are the sections defined by $\mathcal{F}_{\mathbf{u}}$ and $\kappa_{\mathbf{u}}$ respectively. Then $f(\mathbf{u})$ defines an analytic function on $\mathcal{D}(M)$. When a different choice of the triple $(\mathcal{L}, \mathcal{F}, \kappa)$ is made, $f(\mathbf{u})$ differs only by an integral multiple of i. If $\mathbf{u} \in \mathcal{D}(M)$ represents a hyperbolic structure whose metric completion $\bar{M}_{\mathbf{u}}$ is the result of a hyperbolic Dehn-surgery on the cusps of M, then we have

$$\mathrm{Re}\big(f(\mathbf{u})\big) = \frac{1}{\pi^2}\ \mathrm{Volume}\ (\bar{M}_{\mathbf{u}}) + \frac{1}{2\pi} \sum_{k=1}^{h} \mathrm{Length}\ (\gamma_k)$$

and

$$\mathrm{Im}\ \big(f(\mathbf{u})\big) = 2\,\mathrm{CS}(\bar{M}_{\mathbf{u}}) + \frac{1}{2\pi} \sum_{k=1}^{h} \mathrm{Torsion}\ (\gamma_k) \pmod{\mathbf{Z}}$$

where γ_k $(1 \leq k \leq h)$ is the filled-in geodesic at the k-th cusp and $\mathrm{CS}(\bar{M}_{\mathbf{u}})$ is the Chern-Simons invariant of $\bar{M}_{\mathbf{u}}$.

Denote $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \ldots; p_h, q_h)$ where (p_k, q_k) is a pair of coprime integers for each $k = 1, \ldots, h$. Let $M(\mathbf{p}, \mathbf{q})$ be the result of (p_k, q_k)-Dehn surgery with respect to a fixed meridian-longitude pair $(\mathbf{m}_k, \mathbf{l}_k)$ at the k-th cusp of M. Then the η-invariant of $M(\mathbf{p}, \mathbf{q})$ is computed by the following formula

$$
\begin{aligned}
\eta\big(M(\mathbf{p}, \mathbf{q})\big) &= \frac{1}{3} \operatorname{Im} f\big(u(\mathbf{p}, \mathbf{q})\big) - \sum_{k=1}^{h} \frac{1}{6\pi p_k} \operatorname{Im}\big(v_k(\mathbf{p}, q)\big) \\
&\quad + \sum_{k=1}^{h} \Big(4s(q_k, p_k) - \frac{q_k}{3p_k}\Big) - \operatorname{sign}\big(Y(\mathbf{p}, q)\big).
\end{aligned} \tag{2}
$$

Here $f(\mathbf{u})$ depends on the choice of $(\mathbf{m}_k, \mathbf{l}_k)$ and can be calculated using (1) with some choice of the triple (L, F, κ). $s(q_k, p_k)$ is the classical Dedekind sum function defined by

$$
s(q, p) = \frac{1}{4p} \sum_{k=1}^{p-1} \cot\Big(\frac{k\pi}{p}\Big) \cot\Big(\frac{kq\pi}{p}\Big) \text{ if } p > 0,
$$
$$
s(q, p) = s(-q, -p) \text{ if } p < 0.
$$

$Y(\mathbf{p}, \mathbf{q})$ is the 4-manifold obtained by pasting (the lens space $L(p_k, q_k)$)$\times$ (the unit interval I) to $M(\mathbf{p}, \mathbf{q}) \times I$ along the copy of $(D^2 \times S^1)_k$ for $k = 1, \ldots, h$ as sketched in Figure 1 below. Its signature $\operatorname{sign}\big(Y(\mathbf{p}, \mathbf{q})\big)$ can be computed explicitly from Wall's nonadditivity formula ([11]) and depends only on $\operatorname{Ker}\big(H_1(\text{cusps}; \mathbf{Q}) \longrightarrow H_1(M; \mathbf{Q})\big)$ and the elements $\mathbf{l}_1, \ldots, \mathbf{l}_h, p_1\mathbf{m}_1 + q_1\mathbf{l}_1, \ldots, p_h\mathbf{m}_h + q_h\mathbf{l}_h$ of $H_1(\text{cusps})$.

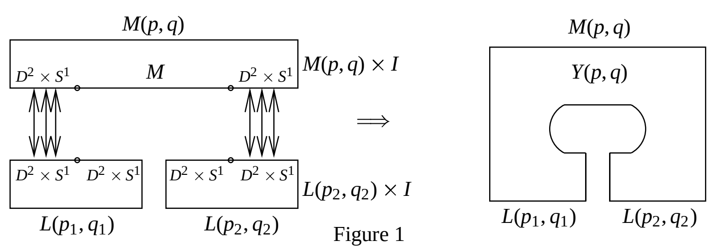

Figure 1

The above formula was first derived in [3] for each $p_k^2 + q_k^2 \longrightarrow \infty$ and was recently shown to be valid throughout hyperbolic Dehn-surgery space in [7].

2. **Definition and Properties.** In this section, we attempt to define the η-invariant for a cusped hyperbolic 3-manifold as a certain limit of the closed case. Note that as shown in [3], $\eta\big(M(\mathbf{p}, \mathbf{q})\big)$ takes on a dense set of values in $\mathbf{R}$ as each $p_k^2 + q_k^2 \longrightarrow \infty$. To deal with this situation, we "average" the η-invariant over larger and larger circles in hyperbolic Dehn surgery space.

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DEFINITION 2.1. Let M be an oriented complete hyperbolic 3-manifold of finite volume with h cusps. Fix a meridian-longitude pair $(\mathbf{m}_k, \mathbf{l}_k)$ at each cusp. Then

$$\eta(M) := \lim_{p_k^2+q_k^2\longrightarrow\infty, k=1,\ldots,h} \frac{1}{2}\Big(\eta\big(M(\mathbf{p},q)\big)+\operatorname{sign}\big(Y(\mathbf{p},\mathbf{q})\big)+\eta\big(M(\mathbf{p},-\mathbf{q})\big)+\operatorname{sign}\big(Y(\mathbf{p},-\mathbf{q})\big)\Big).$$

Note that in denoting $\eta(M)$ we have suppressed mention of the choice of the meridian-longitude pairs.

We need to check that the above limit exists.

From (2) we get

$$\begin{aligned}\eta\big(M(\mathbf{p},\mathbf{q})\big) + \operatorname{sign}\big(Y(\mathbf{p},\mathbf{q})\big) = \frac{1}{3}\operatorname{Im} f\big(u(\mathbf{p},\mathbf{q})\big)\\ -\sum_{k=1}^{h}\frac{1}{6\pi p_k}\operatorname{Im}\big(v_k(\mathbf{p},\mathbf{q})\big)\\ +\sum_{k=1}^{h}\Big(4s(q_k,p_k)-\frac{q_k}{3p_k}\Big)\end{aligned}$$

and

$$\begin{aligned}\eta\big(M(\mathbf{p},-\mathbf{q})\big) + \operatorname{sign}\big(Y(\mathbf{p},-\mathbf{q})\big) &= \frac{1}{3}\operatorname{Im} f\big(u(\mathbf{p},-\mathbf{q})\big)\\ &\quad -\sum_{k=1}^{h}\frac{1}{6\pi p_k}\operatorname{Im}\big(v_k(\mathbf{p},-\mathbf{q})\big)\\ &\quad +\sum_{k=1}^{h}\Big(4s(-q_k,p_k)-\frac{-q_k}{3p_k}\Big)\\ &= \frac{1}{3}\operatorname{Im} f\big(u(\mathbf{p},-\mathbf{q})\big)-\sum_{k=1}^{h}\frac{1}{6\pi p_k}\operatorname{Im}\big(v_k(\mathbf{p},-\mathbf{q})\big)\\ &\quad -\sum_{k=1}^{h}\Big(4s(q_k,p_k)-\frac{q_k}{3p_k}\Big).\end{aligned}$$

For each k, if $p_k \longrightarrow \infty$, then we have

$$\frac{1}{p_k}\operatorname{Im}\big(v_k(\mathbf{p},\mathbf{q})\big)\longrightarrow 0 \text{ as } p_k^2+q_k^2\longrightarrow\infty;$$

If $q_k \longrightarrow \infty$, then since

$$p_k u_k(\mathbf{p},\mathbf{q}) + q_k v_k(\mathbf{p},\mathbf{q}) = 2\pi i,$$

we have also

$$\frac{1}{p_k}\operatorname{Im}\big(v_k(\mathbf{p},\mathbf{q})\big) = \frac{2\pi}{p_k q_k}-\frac{1}{q_k}\operatorname{Im}\big(u_k(\mathbf{p},\mathbf{q})\big)\longrightarrow 0 \text{ as } p_k^2+q_k^2\longrightarrow\infty.$$

Thus it follows that

$$\begin{aligned}\eta(M) &= \lim_{p_k^2+q_k^2\longrightarrow\infty, k=1,\ldots,h}\frac{1}{2}\Big(\eta\big(M(\mathbf{p},\mathbf{q})\big)+\operatorname{sign}\big(Y(\mathbf{p},\mathbf{q})\big)+\eta\big(M(\mathbf{p},-\mathbf{q})\big)\\ &\qquad +\operatorname{sign}\big(Y(\mathbf{p},-\mathbf{q})\big)\Big)\\ &= \lim_{p_k^2+q_k^2\longrightarrow\infty, k=1,\ldots,h}\frac{1}{2}\Big(\frac{1}{3}\operatorname{Im} f\big(u(\mathbf{p},\mathbf{q})\big)+\frac{1}{3}\operatorname{Im} f\big(u(\mathbf{p},-\mathbf{q})\big)=\frac{1}{3}\operatorname{Im} f(\mathbf{u}^0)\Big)\end{aligned} \tag{3}$$

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where $\mathbf{u}^0 \in D(M)$ represents the complete hyperbolic structure on M.

REMARK. $\eta(M) = \frac{1}{3}\operatorname{Im} f(\mathbf{u}^0)$ might be taken as another (equivalent) definition of $\eta(M)$. As we shall see later, defining the η-invariant as a limit of the closed case has its advantage when we come to compare the η-invariants of two manifolds.

By combining (2) and (3), we obtain

$$\eta(M) = \lim_{p_k^2+q_k^2\to\infty, k=1,\ldots,h}\left(\eta\big(M(\mathbf{p},q)\big) + \operatorname{sign}\big(Y(\mathbf{p},\mathbf{q})\big) - \sum_{k=1}^{h}\Big(4s(q_k,p_k) - \frac{q_k}{3p_k}\Big)\right). \tag{4}$$

Let M be a complete, cusped hyperbolic 3-manifold of finite volume. Let $L \subset M$ be a link in M and F an orthonormal frame field which is homotopically linear in the cusps and has a special singularity at L. The Chern-Simons invariant of M was defined by the first author in [2] as

$$\operatorname{CS}(M) \equiv \frac{1}{8\pi^2}\int_{s(M-L)} Q - \frac{1}{4\pi}\sum_{K\subseteq L}\tau(K) \quad \Big(\operatorname{mod}\frac{1}{2}\Big) \tag{5}$$

where $s\colon M - L \longrightarrow F(M)$ is the section defined by F and Q is the Chern-Simons form

$$\omega_{12}\wedge\omega_{13}\wedge\omega_{23} + \omega_{12}\wedge\Omega_{12} + \omega_{13}\wedge\Omega_{13} + \omega_{23}\wedge\Omega_{23}.$$

$\tau(K)$ is the torsion of $K \subseteq L$ which is well defined modulo 2π.

Since $Q = 4\pi^2\operatorname{Im}(C)$, we have

$$\begin{aligned}
\operatorname{Im}\Big(f\big(\mathbf{u}^0;(L,F,\kappa)\big)\Big) &= \lim_{\mathbf{u}\to\mathbf{u}^0}\operatorname{Im}\Big(f\big(\mathbf{u};(L,F_{\mathbf{u}},\kappa_{\mathbf{u}})\big)\Big)\\
&= \lim_{\mathbf{u}\to\mathbf{u}^0}\operatorname{Im}\Big(\int_{s(M_{\mathbf{u}}-L)} C - \frac{1}{2\pi}\sum_{K\subseteq L}\int_{s'(K)}(\omega_1 - i\omega_{23})\Big)\\
&= \lim_{\mathbf{u}\to\mathbf{u}^0}\Big(\frac{1}{4\pi^2}\int_{s(M_{\mathbf{u}}-L)} Q + \frac{1}{2\pi}\sum_{K\subseteq L}\int_{s'(K)}\omega_{23}\Big)\\
&= \frac{1}{4\pi^2}\int_{s(M-L)} Q + \frac{1}{2\pi}\sum_{K\subseteq L}\int_{s'(K)}\omega_{23}.
\end{aligned}$$

Notice that $-\int_{s'(K)}\omega_{23}$ is equal to $\tau(K)$ modulo 2π. Thus it follows from (3) and (5) that

$$\operatorname{CS}(M) \equiv \frac{1}{2}\operatorname{Im}\big(f(\mathbf{u}^0)\big) \equiv \frac{3}{2}\eta(M) \quad \Big(\operatorname{mod}\frac{1}{2}\Big). \tag{6}$$

Thus the Chern-Simons invariant is completely determined by the η-invariant for a cusped hyperbolic 3-manifold.

Since the Chern-Simons invariant does not depend on the choice of the meridian-longitude pairs, we have from (6)

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PROPOSITION 2.2. *$\eta(M)$ differs only by one-third of an integer when different choices of the meridian-longitude pairs are made.*

Let L be a knot or link in S^3. Then for each component S_k of L, we have the topologically standard meridian and longitude $(\mathbf{m}_k, \mathbf{l}_k)$ which are a pair of simple closed curves on $\partial N(S_k)$, the boundary of a tubular neighborhood of S_k, determined up to isometry by the homology $\mathbf{m}_k = 0 \in H_1(N_k)$ and $\mathbf{l}_k = 0 \in H_1(S^3 - N_k)$. If L is hyperbolic, we define the η-invariant of L to be $\eta(S^3 - L)$ with the choice of the topologically standard meridian and longitude for each component of L.

PROPOSITION 2.3. *The η-invariant is a well-defined link invariant for a hyperbolic knot or link.*

PROOF. Suppose that L and L' are two equivalent hyperbolic knots or links in S^3. Let $(\mathbf{m}_k, \mathbf{l}_k)$ and $(\mathbf{m}'_k, \mathbf{l}'_k)$ be the topological meridian and longitude for each component of L and L' respectively. Then there exists an orientation-preserving homeomorphism $h: S^3 \longrightarrow S^3$ sending L to L'. Thus we have an induced orientation-preserving homeomorphism $h': S^3 - L \longrightarrow S^3 - L'$ sending each $(\mathbf{m}_k, \mathbf{l}_k)$ to $(\mathbf{m}'_k, \mathbf{l}'_k)$. Since both $S^3 - L$ and $S^3 - L'$ are complete hyperbolic 3-manifold of finite volume, by the Mostow-Prasad Rigidity Theorem, h' is homotopic to an orientation-preserving hyperbolic isometry $\bar{h}: S^3 - L \longrightarrow S^3 - L'$. Clearly, $\bar{h}$ also sends each $(\mathbf{m}_k, \mathbf{l}_k)$ to $(\mathbf{m}'_k, \mathbf{l}'_k)$. ∎

PROPOSITION 2.4. *Let L be a hyperbolic knot or link in S^3. Denote by L^* its mirror image. Then we have $\eta(L^*) = -\eta(L)$.*

PROOF. Let $(\mathbf{m}_k, \mathbf{l}_k)$ be the topological meridian-longitude pair for each component of L. Let $\iota: M = (S^3 - L) \longrightarrow M^* = (S^3 - L^*)$ be an orientation-reversing isometry. Then ι sends each $(\mathbf{m}_k, \mathbf{l}_k)$ to either $(\mathbf{m}_k, -\mathbf{l}_k)$ or $(-\mathbf{m}_k, \mathbf{l}_k)$.

Let each $p_k^2 + q_k^2$ be sufficiently large so that $M(\mathbf{p}, \mathbf{q})$ has a hyperbolic structure. Then the unique hyperbolic structure on $M(\mathbf{p}, \mathbf{q})$ determines an incomplete hyperbolic structure on M. Completing this structure kills off each curve $p_k\mathbf{m}_k + q_k\mathbf{l}_k$ and produces $M(\mathbf{p}, \mathbf{q})$. The orientation-reversing isometry ι sends each $p_k\mathbf{m}_k + q_k\mathbf{l}_k$ to $p_k\mathbf{m}_k - q_k\mathbf{l}_k$ or $-p_k\mathbf{m}_k + q_k\mathbf{l}_k$. Thus after completion we have an orientation-reversing isometry between $M(\mathbf{p}, \mathbf{q})$ and $M^*(\mathbf{p}, -\mathbf{q}) = M^*(-\mathbf{p}, \mathbf{q})$. Therefore $\eta\big(M(\mathbf{p}, \mathbf{q})\big) = -\eta\big(M^*(\mathbf{p}, -\mathbf{q})\big)$.

Denote by $Y^*(\mathbf{p}, -\mathbf{q})$ the 4-manifold obtained by pasting $L(p_k, -q_k) \times I$ to $M^*(\mathbf{p}, -\mathbf{q}) \times I$ along each copy of the solid torus $(D^2 \times S^1)_k$ in $M^*(\mathbf{p}, -\mathbf{q})$. Since there is an orientation-reversing homeomorphism between lens spaces $L(p_k, q_k)$ and $L(p_k, -q_k)$, we get an orientation-reversing homeomorphism between $Y(\mathbf{p}, q)$ and $Y^*(\mathbf{p}, -\mathbf{q})$. Hence $\operatorname{sign}\big(Y^*(\mathbf{p}, -\mathbf{q})\big) = -\operatorname{sign}\big(Y(\mathbf{p}, q)\big)$. It follows from (4) that

$$
\begin{aligned}
\eta(L^*) &= \lim_{p_k^2+q_k^2 \to \infty, k=1,\ldots,h} \left(\eta\big(M^*(\mathbf{p}, -\mathbf{q})\big) + \operatorname{sign}\big(Y^*(\mathbf{p}, -\mathbf{q})\big) - \sum_{k=1}^{h} \left(4s(-q_k, p_k) - \frac{-q_k}{3p_k} \right) \right) \\
&= - \lim_{p_k^2+q_k^2 \to \infty, k=1,\ldots,h} \left(\eta\big(M(\mathbf{p}, \mathbf{q})\big) + sign\big(Y(\mathbf{p}, \mathbf{q})\big) - \sum_{k=1}^{h} \left(4s(q_k, p_k) - \frac{q_k}{3p_k} \right) \right) \\
&= -\eta(L).
\end{aligned}
$$

∎

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COROLLARY 2.5. *If L is an amphicheiral hyperbolic knot or link, then $\eta(L) = 0$.*

QUESTION. Does the vanishing of the η-invariant imply that L is amphicheiral?

The answer to the above question is unlikely to be affirmative although no example has been found.

EXAMPLE. The fact that the Borromean rings is amphicheiral can be seen from its projection as shown in Figure 2. Thus the η-invariant (hence the Chern-Simons invariant) of the Borromean rings is zero.

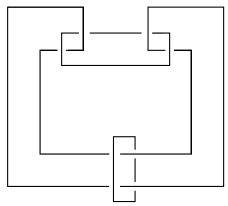

Figure 2

REMARK. The vanishing of the Chern-Simons invariant for the complement of the Borromean rings was shown by Ruberman and the first author in [4] by viewing the Borromean rings as the result of mutating a two-fold cover of the complement of the Whitehead link along a twice-punctured disk.

3. **Some links with homeomorphic complements and different η-invariants.** Let L be a two-component link as shown in Figure 3(a). Suppose that L is hyperbolic with zero linking number (*e.g.*, the Whitehead link). Let L_k denote the link obtained from L by adding k full twists to one of its components as in Figure 3(b). Then L_k and L have homeomorphic complements.

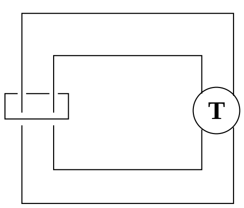

Figure 3(a)

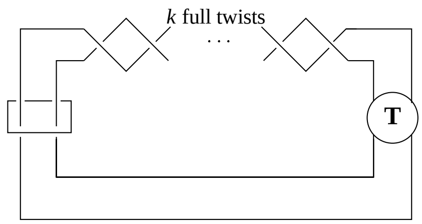

Figure 3(b)

Denote by $M_k(p_1, q_1;\ p_2, q_2)$ the result of the $(p_1, q_1;\ p_2, q_2)$-surgery along L_k with respect to its standard meridian-longitude pairs. Let $Y_k(p_1, q_1; p_2, q_2)$ be the 4-manifold obtained by pasting $L(p_1, q_1) \times I$ and $L(p_2, q_2) \times I$ to $M_k(p_1, q_1; p_2, q_2) \times I$ along two copies of solid tori as shown in Figure 1 of Section 2. Since the linking number of L_k is

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0, using Wall's nonadditivity formula [W], we get $\operatorname{sign}\big(Y_k(p_1,q_1;p_2,q_2)\big)=0$. Thus (4) of Section 2 implies

(7)
$$\eta(L_k)=\lim_{p_i^2+q_i^2\to\infty,i=1,2}\Big(\eta\big(M_k(p_1,q_1;p_2,q_2)\big)-4s(q_1,p_1)-4s(q_2,p_2)+\frac{q_1}{3p_1}+\frac{q_2}{3p_2}\Big).$$

Since the linking number of L_k is zero, $M_k(p_1,q_1;p_2,q_2)$ and $M_{k+1}(p_1,q_1+p_1;p_2,q_2)$ represent two surgery descriptions of the same manifold (see Rolfsen's book [8] for instance).

It follows from (7) that
$$\eta(L_{k+1})=\eta(L_k)+\frac{1}{3}$$
and thus
$$\eta(L_k)=\eta(L)+\frac{k}{3}.$$

Since $\eta(L_{k_1})\neq\eta(L_{k_2})$ for $k_1\neq k_2$, we have the following

COROLLARY 3.1. *L_{k_1} is equivalent to L_{k_2} if and only if $k_1=k_2$.*

4. **A simplicial formula for the η-invariant of a cusped hyperbolic 3-manifold.** Let M be an oriented complete finite-volume hyperbolic 3-manifold with h cusps. Suppose M has an ideal triangulation $M=\Delta(z_1^0)\cup\cdots\cup\Delta(z_n^0)$ where each $\Delta(z_j^0)$ is an ideal tetrahedron described by a complex parameter z_j^o after choosing an edge for each $\Delta(z_j^0)$. Denote
$$\mathbf{z}^0=\big(\log z_1^0,\dots,\log z_n^0,\log(1-z_1^0),\dots,\log(1-z_n^0)\big).$$
Then $\mathbf{z}^0$ is determined by the consistency and cusp relations in the form
$$\mathbf{z}^0U=\pi i\mathbf{d}$$
where U is an integer $2n\times(n+2h)$-matrix and $\mathbf{d}=(d_1,\dots,d_{n+2h})$ is some integer vector (see [5]). Let $\mathbf{c}=(c_1',\dots,c_n',c_1'',\dots,c_n'')$ be a solution to the equation
$$\mathbf{c}U=\mathbf{d}.$$

Denote by $M_{\mathbf{u}}$ the hyperbolic structure on M obtained by deforming the parameter $\mathbf{u}^0=(z_1^0,\dots,z_n^0)$ to $\mathbf{u}=(z_1,\dots,z_n)$. Suppose that the metric completion of $M_{\mathbf{u}}$ is $M(\mathbf{p},\mathbf{q})$ for some $(\mathbf{p},\mathbf{q})=(p_1,q_1;\dots;p_h,q_h)$. Then by combining Neumann's simplicial formula for the analytic function and (1) of Section 2 we get
$$\eta\big(M(\mathbf{p},\mathbf{q})\big)=\alpha-\frac{1}{3\pi^2}\operatorname{Im}\sum_{j=1}^{n}\Big(iR(z_j)+\frac{\pi}{2}\big(\overline{c_j'}\log(1-z_j)-\overline{c_j''}\log z_j\big)\Big)$$
$$-\sum_{k=1}^{h}\frac{1}{6\pi p_k}\operatorname{Im}(v_k)+\sum_{k=1}^{h}\Big(4s(q_k,p_k)-\frac{q_k}{3p_k}\Big)-\operatorname{sign}\big(Y(\mathbf{p},\mathbf{q})\big)$$

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where α is a constant which can be determined via a "boot-strapping" procedure, using surgery relations discovered between hyperbolic 3-manifolds. $R(z)$ is the Rogers dilogarithm function defined by

$$R(z) = \frac{1}{2}\log(z)\log(1-z) + Li_2(z) = \frac{1}{2}\log(z)\log(1-z) - \int_0^z \log(1-t)\, d\log t$$

with log the standard branch on $\mathbf{C} - (-\infty, 0]$ and $\bar{x}$ denotes the complex conjugate of x.

Thus it follows from (3) and (4) of Section 2 that

$$\begin{aligned} \eta(M) &= \frac{1}{3}\operatorname{Im} f(\mathbf{u}^0) \\ &= \alpha - \frac{1}{3\pi^2}\operatorname{Im}\Big(\sum_{j=1}^{n}\Big(iR(z_j^0) + \frac{\pi}{2}\big(\overline{c_j'}\log(1-z_j^0) - \overline{c_j''}\log z_j^0\big)\Big)\Big). \end{aligned} \tag{8}$$

To finish up the section, we calculate the η-invariants of twisted versions of the Whitehead link. We first do it for the Whitehead link W as in Figure 4.

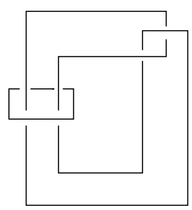

Figure 4

As described in [10], $S^3 - W$ can be obtained by gluing faces of an ideal octahedron in pairs. By subdividing the octahedron, we get an ideal triangulation of the complement with four simplices. We refer to [3] for the pictures and gluing equations.

The four ideal tedrahedra are all described by $e^{\frac{i\pi}{2}} = i$ with a choice of an edge for each tedrahedron. Namely, we have $z_1^0 = z_2^0 = z_3^0 = z_4^0 = i$.

As computed in [3],

$$f(\mathbf{u}) = \frac{1}{\pi^2 i}\Big(R(z_1) + R(z_2) + R\Big(\frac{1}{1-z_3}\Big) + R\Big(\frac{1}{1-z_4}\Big)\Big) + 2i \tag{9}$$

where the constant $2i$ was obtained by using the fact that $(1,1;p,q)$-surgery on W is the (p,q)-surgery on the figure-eight knot for which the constant had been determined in [12].

It follows from (8) and (9) that

$$\eta(W) = \frac{1}{3}\operatorname{Im}\Big(\frac{1}{\pi^2 i}\Big(R(i) + R(i) + R\Big(\frac{1}{1-i}\Big) + R\Big(\frac{1}{1-i}\Big)\Big) + 2i\Big) = \frac{5}{12}.$$

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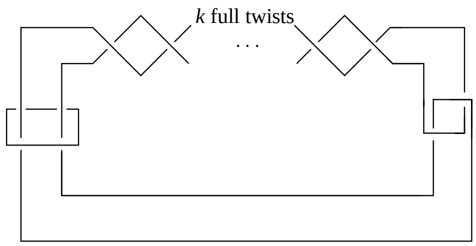

Figure 5

Denote by W_k the twisted Whitehead link as shown in Figure 5.

Then as in Section 3, we have

$$\eta(W_k) = \frac{5}{12} + \frac{k}{3}.$$

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