Dividing these and rearranging gives \( k \) (and hence \( a \)) as

\[
k = \frac{2\pi(\pi - 2)}{\pi^2 - 8} - 1 \approx 2.8366 \quad a = \left(1 - \frac{\pi}{2}\right) \left(\frac{2}{\pi}\right)^k \approx 0.15855.
\]

So was our original assumption that \( \sin x \approx x + ax^k \) a reasonable one? The following table shows that the expression \( x - 0.15855x^{2.8366} \) is within 0.5% of \( \sin x \) over the whole range \( 0 \leq x \leq \pi/2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin x )</th>
<th>( x - 0.15855x^{2.8366} )</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1987</td>
<td>0.1984</td>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3894</td>
<td>0.3882</td>
<td>0.3</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5646</td>
<td>0.5628</td>
<td>0.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7174</td>
<td>0.7158</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.8415</td>
<td>0.8415</td>
<td>0.002</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>0.8660</td>
<td>0.8665</td>
<td>0.1</td>
</tr>
<tr>
<td>1.2</td>
<td>0.9320</td>
<td>0.9341</td>
<td>0.2</td>
</tr>
<tr>
<td>1.4</td>
<td>0.9854</td>
<td>0.9882</td>
<td>0.3</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

The first two terms of the Maclaurin’s expansion, \( x - x^3/6 \), as much as 7.5% in error for \( x = \pi/2 \).

The interested reader should be able to find a two-term expression for \( \log_e(1 + x) \) between \( x = 0 \) and \( x = 1 \) accurate to within 2%.

If the accuracy of the above expressions is less than that desired, then additional terms of the Maclaurin expression can be introduced initially. For example

\[
\sin x \approx x - \frac{x^3}{6} + 0.0081759x^{4.9128}
\]
gives values accurate to within 0.01% throughout \( 0 \leq x \leq \pi/2 \).

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69.20 A problem associated with fitting straight lines to data

In an examination question, set a few years ago by The Royal Institute of Chemistry, candidates were invited to confirm that the following data for the pressure \( p \) and the volume adsorbed \( v \) of a gas are expressed by Langmuir isotherm. The data is for adsorption of hydrogen gas on a copper powder surface at 25°C (298K).
The equation of a Langmuir isotherm may be written in the form

\[ v = \frac{p}{a + bp} \]

where \(a\) and \(b\) are constants. Hence the question is asking for the validity of this equation to be established for the given data and inviting estimates for \(a\) and \(b\).

On the face of it this should be a straightforward problem. However, difficulties experienced by some of our students, who have used this examination question for revision purposes, have thrown up ideas on fitting straight lines which we feel may be of general interest.

The difficulties stem from the fact that the isotherm equation may be rearranged in either of the straight line forms

\[ \frac{p}{v} = bp + a \]  \(\text{(1)}\)

or

\[ \frac{1}{v} = a \left( \frac{1}{p} \right) + b. \]  \(\text{(2)}\)

There are, therefore, two possible methods readily available for estimating \(a\) and \(b\). We could plot a graph of \(p/v\) against \(p\) or we could plot a graph of \(1/v\) against \(1/p\). In either case the resulting graph should be a straight line if the data can be expressed as a Langmuir isotherm. Further, from the gradient and position of such lines we can estimate \(a\) and \(b\).

### Diverging lines

"If a railroad rail a mile long is raised 200 feet in the centre, how much closer would it bring the two ends? Answer: less than 6 inches." From Ripley’s *Mammoth believe it or not.*

"If a railway line a mile long were raised 200 feet in the centre, how much closer would it bring the two ends? Answer: Approximately 15 feet." From *More puzzles to puzzle you* by Jonathan Always.

Both sent in by David Singmaster.
The following diagrams illustrate the data points obtained when each of these methods of solution are initiated.

In (1) we may readily fit a straight line to the data points. Hence the candidate who rearranged the equation into the form given by equation (1) may now readily proceed to estimate \( a \) and \( b \).

The candidate who chose to rearrange the equation into the form given by equation (2) is not in such a fortunate position, since, as illustrated, it is not at all clear that his data points satisfy a linear relationship. Indeed to the eye a curve looks more likely. Hence a candidate adopting this approach would now be quite probably entertaining serious doubts about his method for answering the examination question. What has gone wrong with his theoretically reasonable approach?

The answer may be found in the quality of the data. If the data is accurate, the data points in both diagrams would lie on straight lines and there would be no problem answering the question by either of the two approaches we have suggested. However, if the data contains errors a detailed examination of their effect will reveal why choosing equation (2) to answer the question was an unfortunate choice. Such errors may arise from the experimental methods used to study the variation of adsorption with gas pressure, or they may reflect a deviation from the true Langmuir adsorption behaviour in this particular system.
To take account of errors in the data suppose we write equation (1) in the form
\[
\frac{p}{v} = bp + a + \epsilon_p
\]
where \( \epsilon_p \) is the error in the value of \( p/v \) at a given pressure \( p \). Then on the graph of \( p/v \) against \( p \) the influence of the error \( \epsilon_p \) will be affected by the scales used. If \( n \) units of \( p/v \) are represented by one unit of the graph paper, then the error \( \epsilon_p \) is represented by \( \epsilon_p/n \) units of the graph paper.

Rearranging equation (1') to match the form of equation (2), we have
\[
\frac{1}{v} = a \left( \frac{1}{p} \right) + b + \frac{\epsilon_p}{p}
\]
and thus, for a given value of \( p \), the error in the value of \( 1/v \) is \( \epsilon_p/p \). Then assuming that \( m \) units of \( 1/v \) are represented by one unit of the graph paper, the error in the value of \( 1/v \) is represented by \( \epsilon_p/mp \) units of the graph paper.

When comparing the graphs obtained by the two approaches we used the same size of graph paper in each case. Hence the relative magnitudes of \( n \) and \( mp \) will clearly influence any discussion of the goodness of fit of straight lines to the data points in each graph. For our graphs we chose \( n = 50 \) and \( m = 0.5 \). Hence at all the data points, except the point corresponding to \( p = 204.8 \), the value of \( mp \) is considerably smaller than \( n \). Thus the points plotted in the \( p/v \) against \( p \) graph may be expected to give a better straight line fit than the points in the \( 1/v \) against \( 1/p \) graph. This is indeed borne out by the graphs.

To conclude we should like to pose a question. How does one decide initially which rearrangement of a given formula is likely to give the best straight line fit?

D. J. COLWELL, J. R. GILLETT and S. I. E. GREEN

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69.21 An illustration of rounding error on computers

Rounding error is a nuisance to which any user of a computer should pay close attention. We have come across a striking example in the statistical context of seed sampling. (We are grateful to Mr G. Chadwick and Mr A. Curtis of the Department of Agriculture and Fisheries for Scotland, East Craigs, Edinburgh for bringing this sampling problem to us.)

Batches each of around 30 000 seeds were to be checked for quality; a batch was to be rejected if it had more than 7% impure seeds. Since testing for impurities is expensive, and destructive of the seed, the whole batch could not be checked; instead a sample of size \( N \) would be taken at random...